Phys101 Lectures 30, 31

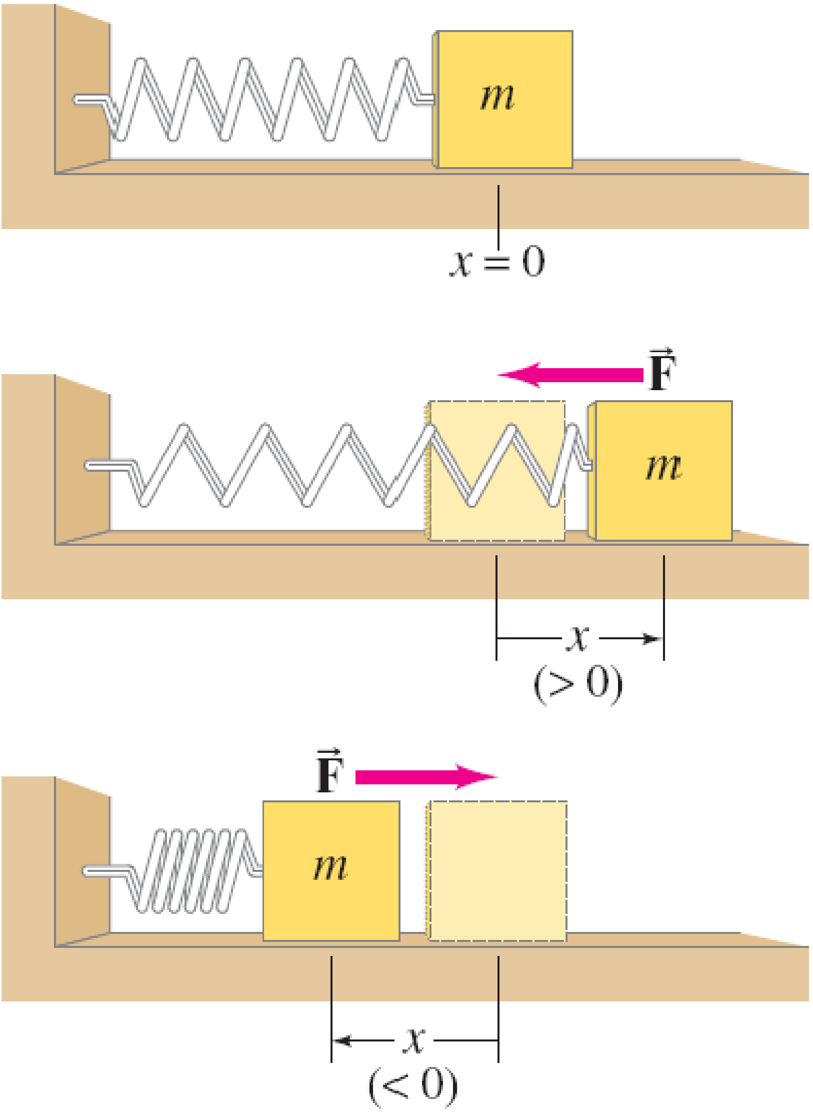
Oscillations

Key points:

- **Simple Harmonic Motion (SHM)**
- **SHM Related to Uniform Circular Motion**
- **The Simple Pendulum**

Ref: 14-1,2,3,4,5.

14-1 Oscillations of a Spring



If an object **oscillates** back and forth over the same path, each cycle taking the same amount of time, the motion is called **periodic**. The **mass** and **spring** system is a useful model for a periodic system.

14-1 Oscillations of a Spring

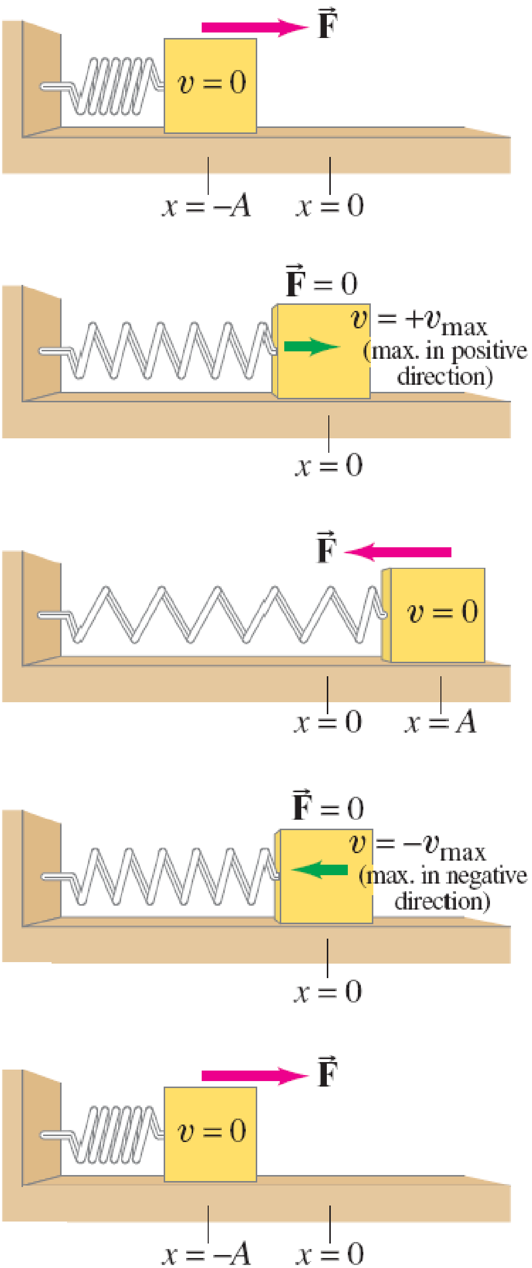
We assume that the system is **frictionless**. There is a point where the spring is neither stretched nor compressed; this is the **equilibrium** position. We measure **displacement** from that point (x = 0 on the previous figure).

The **force** exerted by the spring depends on the **displacement**:

$$F = -kx.$$

- The **minus** sign on the force indicates that it is a **restoring** force—it is directed to restore the mass to its **equilibrium** position.
- k is the **spring constant.**
- The force is **not constant**, so the acceleration is not constant either.

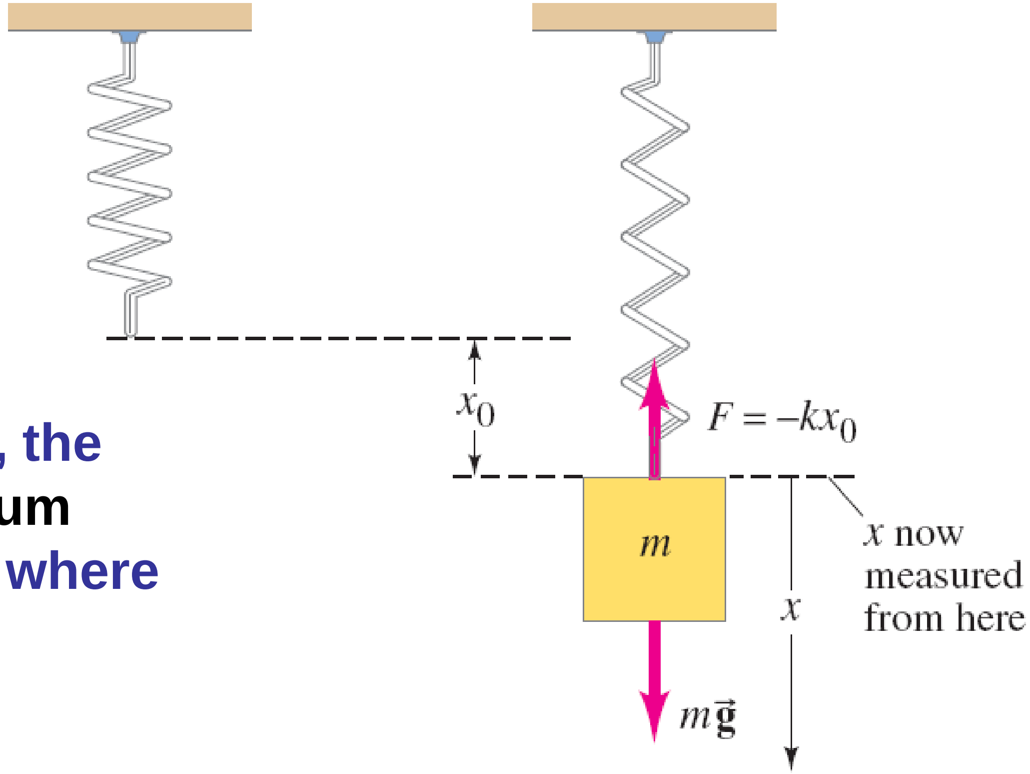
14-1 Oscillations of a Spring



- **Displacement** is measured from the **equilibrium** point.
- **Amplitude** is the **maximum** displacement.
- A **cycle** is a full to-and-fro motion.
- **Period,** T, is the **time** required to complete one cycle.
- **Frequency, f,** is the **number** of cycles completed per second. The unit of frequency is Hz (cycles per second).

$$f = \frac{1}{T}$$

14-1 Oscillations of a Spring



If the spring is hung vertically, the only change is in the equilibrium position, which is at the point where the spring force equals the gravitational force.

Demo

14-2 Simple Harmonic Motion

Any vibrating system where the restoring force is proportional to the negative of the displacement is in simple harmonic motion (SHM), and is often called a simple harmonic oscillator (SHO).

Substituting $F = kx$ into Newton's second law gives the equation of motion:

$$-kx = m\frac{d^2x}{dt^2}, \qquad i.e. \qquad \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$\text{Or,} \qquad \frac{d^2x}{dt^2} = -\omega^2 x \qquad \text{where} \qquad \omega^2 = \frac{k}{m}$$

The solution has the form:

$$x = A\cos(\omega t + \phi).$$

How do you know?

We can guess

and then verify:

$$\frac{d}{dt}\left[A\cos(\omega t + \phi)\right] = -\omega A\sin(\omega t + \phi)$$ ← Velocity

$$\frac{d^2}{dt^2}\left[A\cos(\omega t + \phi)\right] = -\omega^2 A\cos(\omega t + \phi)$$ ← Acceleration

14-4 Simple Harmonic Motion Related to Uniform Circular Motion

If we look at the projection onto the *x* axis of an object moving in a circle of radius *A* at a constant angular velocity ω, we find that the *x* component of the circular motion is in fact a SHM.

Demo

http://www.surendranath.org/Applets/Oscillations/SHM/SHMApplet.html

$$x = \mathrm{A}\cos(\theta)$$

$$\theta = \omega \mathrm{t} + \phi$$

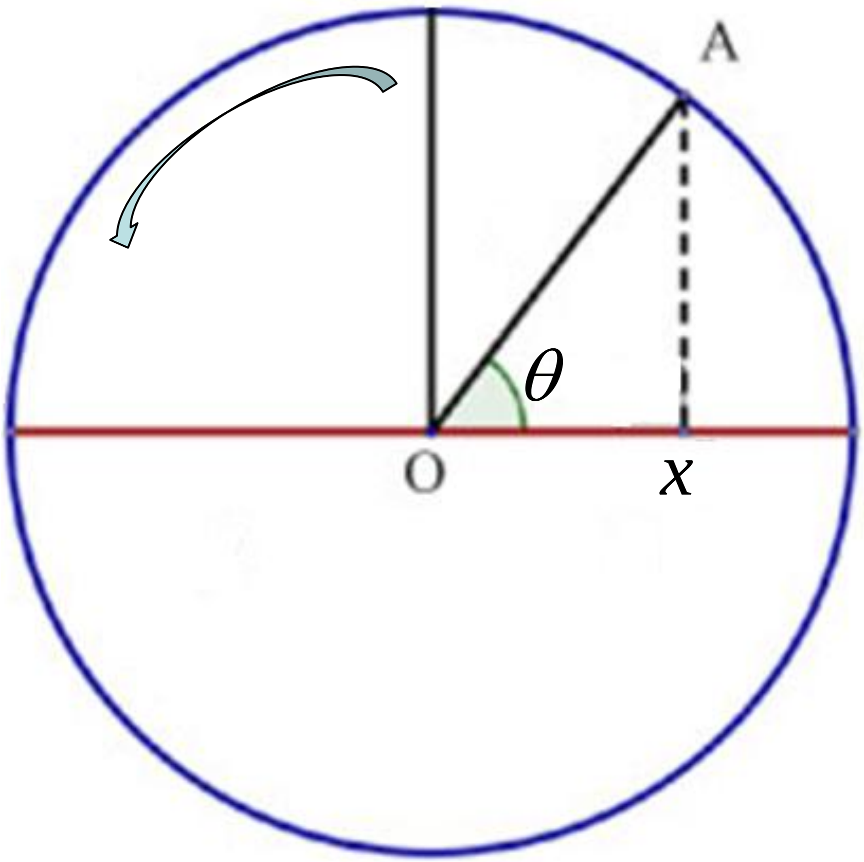
$$\phi \text{ - initial angular position}$$

$$x = \mathrm{A}\cos(\omega \mathrm{t} + \phi)$$

A - Amplitude

ω - Angular frequency

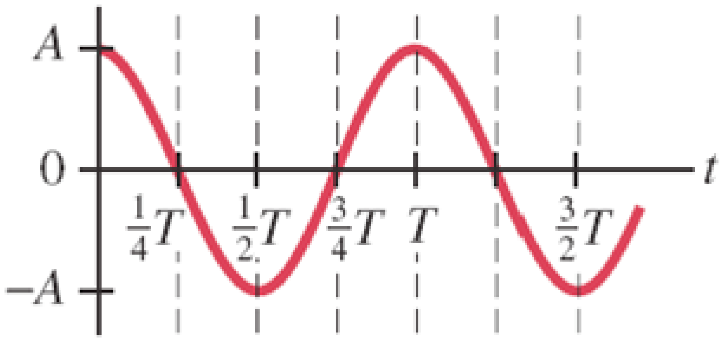
ϕ - initial phase



These figures illustrate the meaning of ϕ:

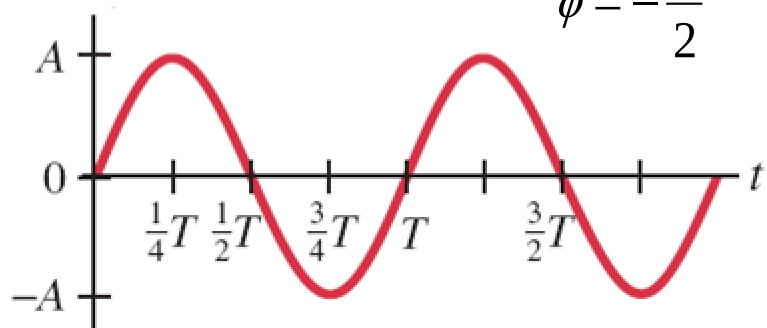
$$x = A\cos(\omega t)$$

$$\phi = 0$$

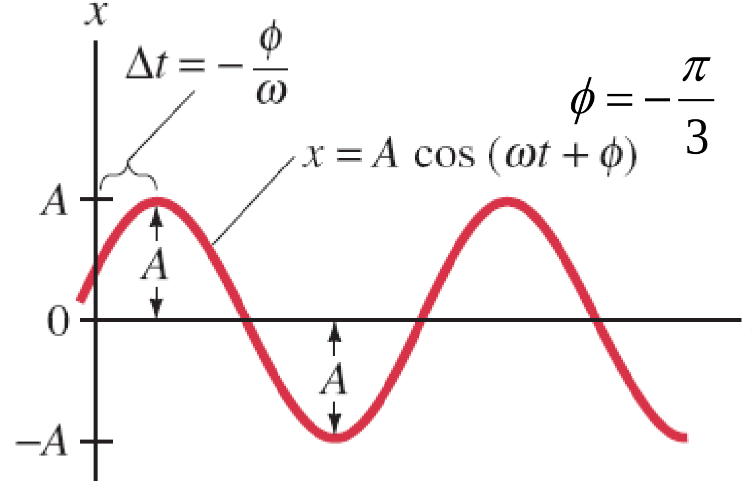


$$x = A\cos\left(\omega t - \frac{\pi}{2}\right)$$

$$\phi = -\frac{\pi}{2}$$



$$x = A\cos\left(\omega t - \frac{\pi}{3}\right)$$



$$\text{Since } \cos 0 = 1,$$

$$x_{max} \text{ occurs when } \omega t + \phi = 0,$$

$$i.e., \quad t = -\frac{\phi}{\omega}$$

ϕ < 0 if the nearest max is on the right.

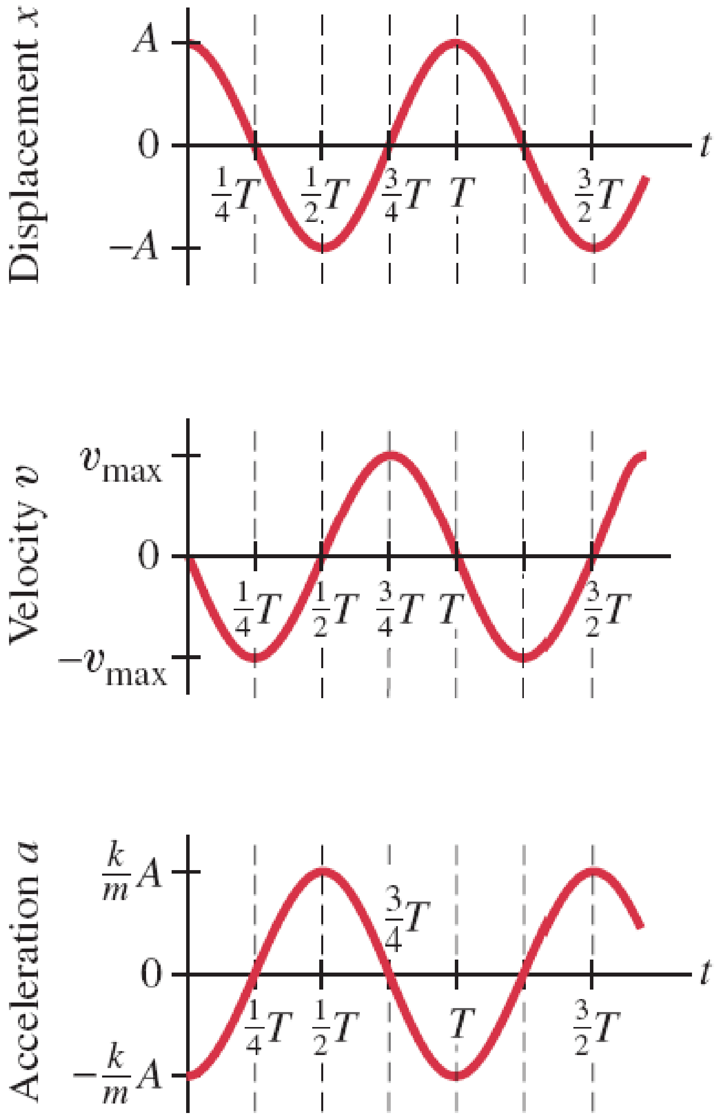
14-2 Simple Harmonic Motion

Because $\omega = 2\pi f = \sqrt{k/m}$, **then**

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}},$$

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

14-2 Simple Harmonic Motion



The velocity and acceleration for simple harmonic motion can be found by differentiating the displacement:

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \phi).$$

14-2 Simple Harmonic Motion

Example 14-3: A vibrating floor.

A large motor in a factory causes the floor to vibrate at a frequency of 10 Hz. The amplitude of the floor's motion near the motor is about 3.0 mm. Estimate the maximum acceleration of the floor near the motor.

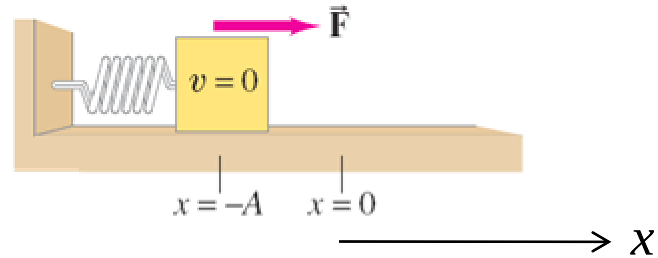
[Solution] f=10Hz, ω=2πf=20π; 3.0 mm=3.0x10^{-3}m

$$a = -\omega^2 \mathrm{A}\cos(\omega t + \phi) \quad (\text{Here } a \text{ means } a_x)$$

$$a_{max} = \omega^2 \mathrm{A}$$

$$a_{max} = (20\pi)^2 \times 3.0 \times 10^{-3} = 12\mathrm{m/s}^2$$

Example 14-5: Spring calculations.

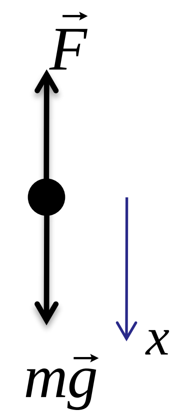


A spring stretches 0.150 m when a 0.300-kg mass is gently attached to it. The spring is then set up horizontally with the 0.300-kg mass resting on a frictionless table. The mass is pushed so that the spring is compressed 0.100 m from the equilibrium point, and released from rest. Determine: (a) the spring stiffness constant k and angular frequency ω; (b) the amplitude of the horizontal oscillation A; (c) the magnitude of the maximum velocity v_{max}; (d) the magnitude of the maximum acceleration a_{max} of the mass; (e) the period T and frequency f; (f) the displacement x as a function of time; and (g) the velocity at t = 0.150 s.

(a)

$$\vec{F} + m\vec{g} = 0, \quad mg + F_x = 0, \quad mg - kx = 0$$

$$k = \frac{mg}{x} = \frac{0.3 \times 9.8}{0.15} = 19.6\, N/m$$



(b) A=0.100m

(c) $v = -\omega A\sin(\omega t + \phi)$ $\quad v_{max} = \omega A = \sqrt{\frac{k}{m}}A = 0.100\sqrt{\frac{19.6}{0.300}} = 0.808 m/s$

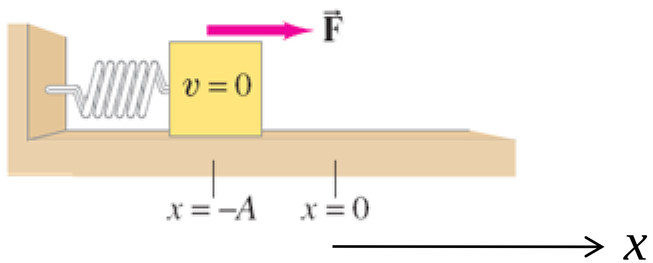
(d) $$a = -\omega^2 A\cos(\omega t + \phi) \quad (\text{Here } a \text{ means } a_x)$$

$$a_{max} = \omega^2 A = \frac{k}{m} A = \frac{19.6}{0.300} \times 0.100 = 6.53 m/s^2$$

(e) $$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{19.6}{0.300}} = 8.08\ rad/s$$

$$f = \frac{\omega}{2\pi} = 1.29\ Hz \quad \text{(cycles/second)}$$

$$T = \frac{1}{f} = 0.777\ s$$



(f) In general, $x = A\cos(\omega t + \phi)$

From a specific given point, here for example, when t=0, x= -0.100;

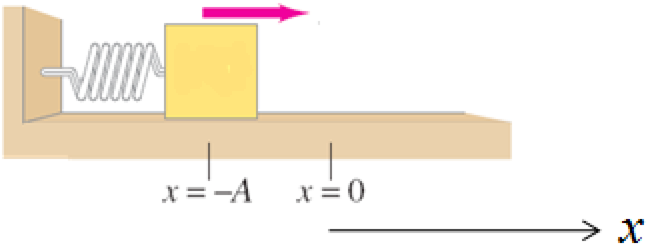
We can determine ϕ: $0.100 = -0.100\cos(\phi) \Rightarrow \phi = -\pi$ or π

$$\therefore \quad x = A\cos(\omega t + \phi) = 0.100\cos(8.08t - \pi) = -0.100\cos(8.08t)$$

(g) $$v = \frac{dx}{dt} = (8.80)0.100\,sin(8.08t) = 0.808\,sin(8.08 \times 0.15) = 0.756 m/s$$

Example 14-6: Spring is started with a push.

Suppose the spring of Example 14–5 (where $\omega = 8.08\ \mathrm{s}^{-1}$) is compressed 0.100 m from equilibrium ($x_0 = -0.100$ m) but is given a shove to create a velocity in the $+x$ direction of $v_0 = 0.400$ m/s. Determine (a) the phase angle ϕ, (b) the amplitude A, and (c) the displacement x as a function of time, $x(t)$.



[Solution] $x = A\cos(\omega t+\phi), \quad v = -\omega A \sin(\omega t + \phi)$

Use the initial conditions to determine A and ϕ:

$$\text{when } t = 0, x = -0.100\,m,\ v = 0.400\,m/s$$

$$i.e.,\quad -0.1 = A\cos\phi \qquad (1)$$

$$0.4 = -\omega A\sin\phi \qquad (2)$$

$$\frac{(2)}{(1)}:\quad -4 = -\omega\tan\phi$$

$$\tan\phi = \frac{4}{\omega} = \frac{4}{8.08} = 0.495$$

$$\phi = 26.3°, \quad \text{or} \quad 206.3°$$

$$\text{Since } \sin\phi < 0, \text{and } \cos\phi < 0,$$

$$\phi = 206.3° = 3.60 \text{ rad}$$

$$\text{From (1)},\quad A = \frac{-0.1}{\cos\phi} = \frac{-0.1}{\cos 206.3°} = 0.112m$$

$$x = 0.112\cos(8.08t + 3.60)$$

14-3 Energy in the Simple Harmonic Oscillator

The **mechanical energy** of an object in simple harmonic motion is:

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2.$$

i-clicker question 30-1:

Is the mechanical energy of a simple harmonic oscillator conserved?

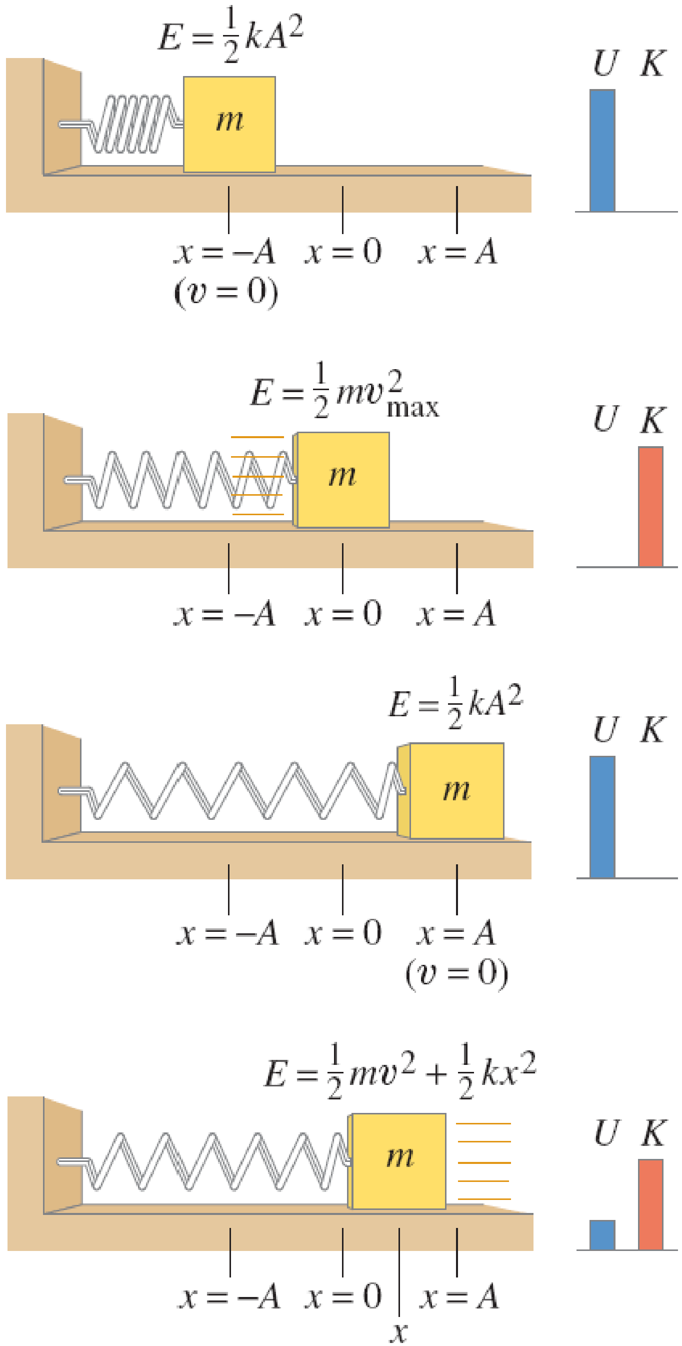
(A) Yes.

(B) No.

Why?

By definition, **simple** harmonic motion means no friction, more generally no energy loss.

14-3 Energy in the Simple Harmonic Oscillator



If the mass is at the limits of its motion, the energy is all potential.

If the mass is at the equilibrium point, the energy is all kinetic.

We know what the potential energy is at the turning points:

$$E = \frac{1}{2}kA^2.$$

Which is equal to the total mechanical energy.

14-3 Energy in the Simple Harmonic Oscillator

The **total energy** is, therefore, $\frac{1}{2}kA^2$.

And we can write:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2.$$

This can be solved for the **velocity** as a function of **position**:

$$v = \pm v_{\mathrm{max}}\sqrt{1 - \frac{x^2}{A^2}},$$

where $$v_{\mathrm{max}}^2 = (k/m)A^2.$$

14-3 Energy in the Simple Harmonic Oscillator

Example 14-7: Energy calculations.

For the simple harmonic oscillation of Example 14–5 (where k = 19.6 N/m, A = 0.100 m, x = -(0.100 m) cos 8.08t, and v = (0.808 m/s) sin 8.08t), determine (a) the total energy, (b) the kinetic and potential energies as a function of time, (c) the velocity when the mass is 0.050 m from equilibrium, (d) the kinetic and potential energies at half amplitude ($x = \pm A/2$).

(a)

$$E = \frac{1}{2}kA^2 = \frac{1}{2}(19.6)(0.1)^2 = 0.098J$$

(b)

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m[\omega A\,sin(\omega t+\phi)]^2 = \frac{1}{2}(0.3)[(0.808)sin(8.08t)]^2 = 0.098\,sin^2(8.08t)$$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}k[A\,cos(\omega t+\phi)]^2 = \frac{1}{2}(19.6)[(0.808)cos(8.08t)]^2 = 0.098\,cos^2(8.08t)$$

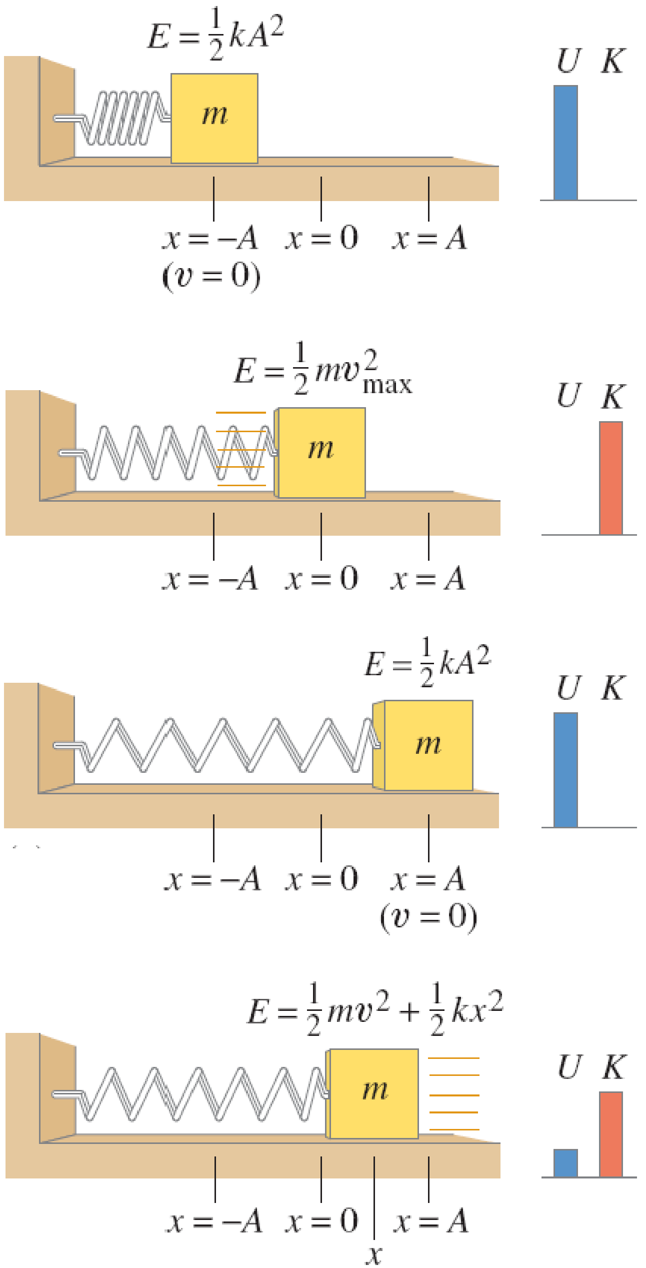
(c)

$$v = v_{max}\sqrt{1-\frac{x^2}{A^2}} = 0.808\sqrt{1-\left(\frac{1}{2}\right)^2} = 0.70\,m/s$$

(d)

$$U = \frac{1}{2}(19.6)(0.05)^2 = 0.0245\,J; \quad K = E-U = 0.098-0.0245 = 0.0735\,J.$$

14-3 Energy in the Simple Harmonic Oscillator



Conceptual Example 14-8: Doubling the amplitude.

Suppose this spring is stretched twice as far (to x = 2A).What happens to (a) the energy of the system, (b) the maximum velocity of the oscillating mass, (c) the maximum acceleration of the mass?

(a) The energy quadruples since

$$E \propto A^2$$

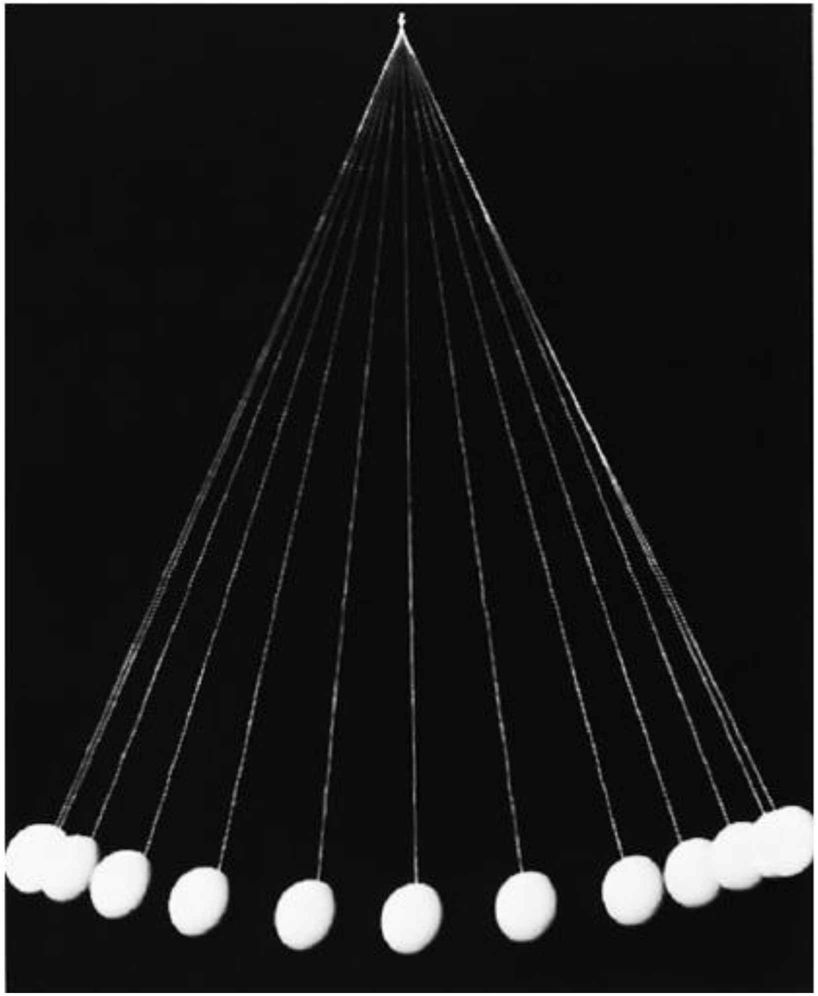
(b) The maximum velocity doubles

since $v_{max} \propto \sqrt{E}$ $\left(E = \frac{1}{2} m v_{max}^2 \right)$

(c) The maximum acceleration doubles since

$$a_{max} = \omega^2 A$$

14-5 The Simple Pendulum



A **simple pendulum** consists of a mass at the end of a lightweight cord. We assume that the cord does not **stretch**, and that its mass is **negligible**.

14-5 The Simple Pendulum

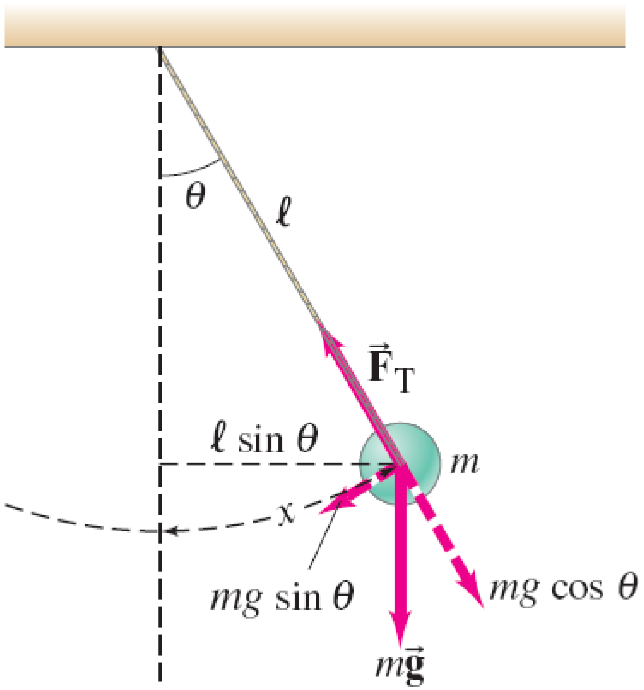
In order to be in SHM, the restoring force must be **proportional** to the negative of the **displacement**. Here we have:

$$F = -mg\sin\theta,$$

which is proportional to **sin *θ*** and not to ***θ*** itself.

However, if the angle is **small**, **sin θ ≈ θ**.

Therefore, for small angles, we have



$$F \approx -\frac{mg}{l}x = m\frac{d^2x}{dt^2}$$

where $x = l\theta$

Then, $\dfrac{d^2x}{dt^2} = -\dfrac{g}{l}x$

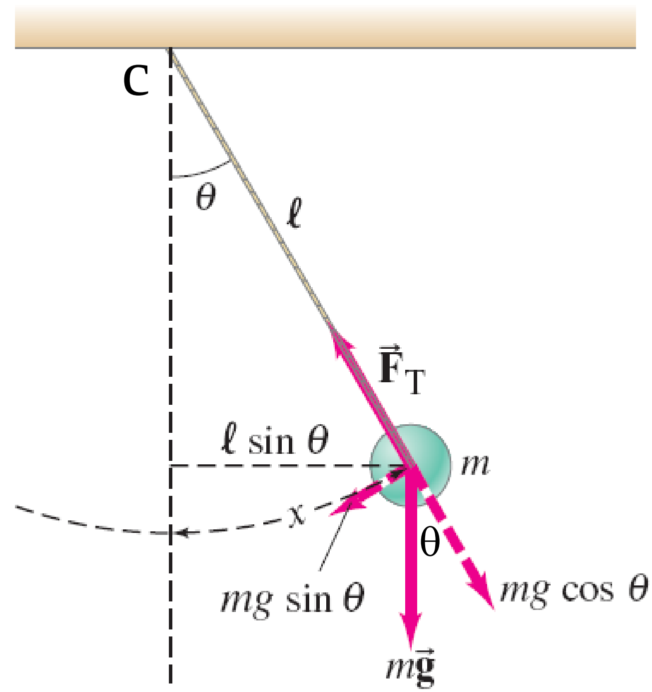
Therefore, it is a SHM with

$$\omega = \sqrt{\frac{g}{l}}$$

Another approach: from the angular point of view

The Simple Pendulum

From the angular point of view



$$\tau = I\alpha$$

$$-mgl\sin\theta = ml^2\frac{d^2\theta}{dt^2}$$

For small θ, $\sin\theta \approx \theta \quad (rad)$

$$-g\theta = l\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

"-" sign: when the angular displacement is positive, the torque is positive; when the angular displacement is negative, the torque is positive.

The angular acceleration is proportional to the negative angular displacement. Therefore, it's SHM.

Compared to, $\frac{d^2x}{dt^2} = -\omega^2 x$ We have $\omega = \sqrt{\frac{g}{l}}$ **We can measure g!**

14-5 The Simple Pendulum

Example 14-9: Measuring *g*.

A geologist uses a simple pendulum that has a length of 37.10 cm and a frequency of 0.8190 Hz at a particular location on the Earth. What is the acceleration of gravity at this location?

$$\omega = \sqrt{\frac{g}{l}}$$

$$\mathrm{g} = \omega^2 l = (2\pi f)^2 l = (2\pi \times 0.8190)^2 (0.371) = 9.824\, m/s^2$$

Summary of Chapter 14

• For SHM, the restoring force is proportional to the displacement.

•Period for a mass on a spring: $$T = 2\pi\sqrt{\frac{m}{k}}.$$

• SHM is sinusoidal.

• During SHM, the total energy is continually changing from kinetic to potential and back.

• A simple pendulum approximates SHM if its amplitude is not large. Its period in that case is:

$$T = 2\pi\sqrt{\frac{L}{g}}.$$