Phys101 Lectures 16, 17, 18
Rotational Motion

Key points:

- **Rotational Kinematics**
- **Rotational Dynamics; Torque and Moment of Inertia**
- **Rotational Kinetic Energy**
- **Angular Momentum and Its Conservation**

$$\tau = I\alpha$$

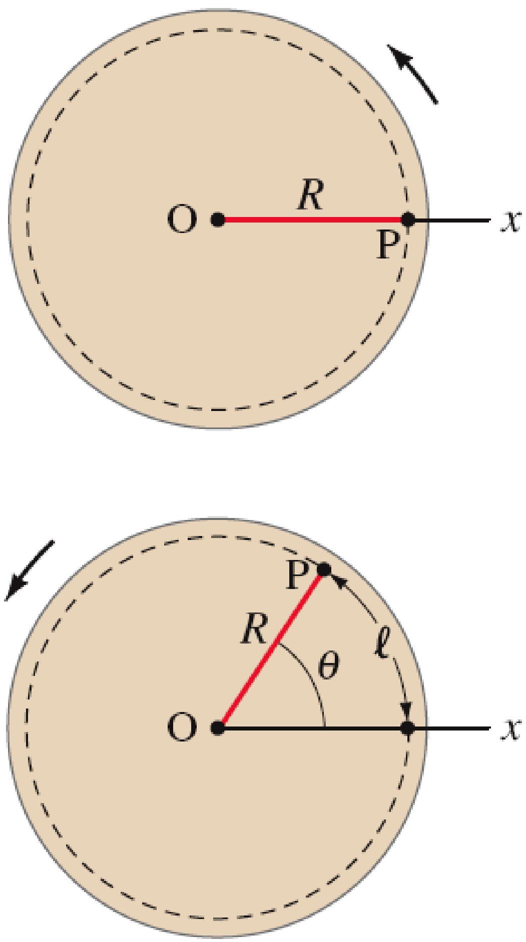
$$K_R = \frac{1}{2} I\omega^2$$

$$L = I\omega$$

$$\tau = \frac{dL}{dt}$$

Ref: 8-1,2,3,4,5,6,8,9.

Angular Quantities



In purely **rotational** motion, all points on the object move in **circles** around the **axis** of rotation ("*O*"). The **radius** of the circle is *R*. All points on a straight line drawn through the axis move through the same **angle** in the same **time**. The angle θ in **radians** is defined:

$$\theta = \frac{l}{R},$$

where *l* is the **arc length**.

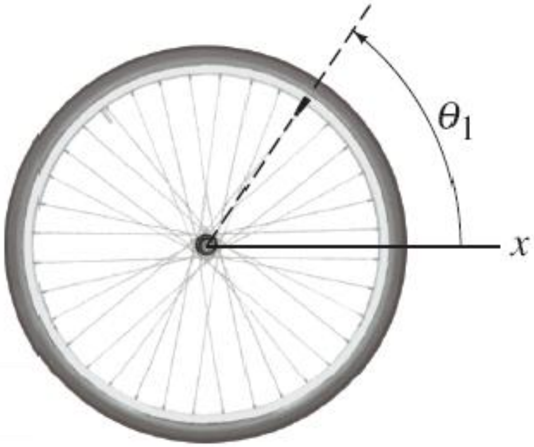
Sign convention: + CCW

\- CW

$$2\pi(rad) = 360^\circ \quad (1\,\mathrm{rev})$$

$$\pi(rad) = 180^\circ$$

Angular Quantities

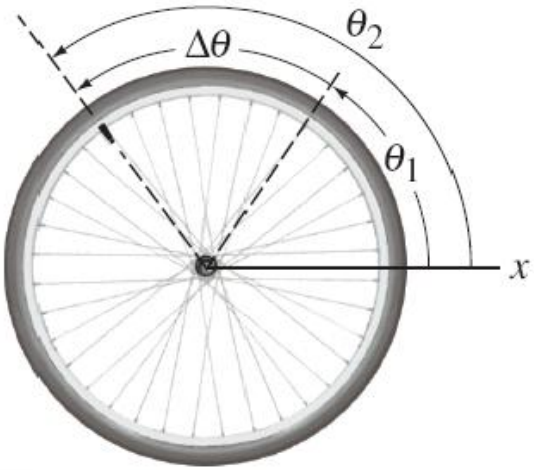


Angular displacement:

$$\Delta\theta = \theta_2 - \theta_1 .$$

The average angular velocity is defined as the total angular displacement divided by time:

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t} .$$



The instantaneous angular velocity:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} .$$

Angular Quantities

The angular acceleration is the rate at which the angular velocity changes with time:

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{\Delta t} = \frac{\Delta \omega}{\Delta t}.$$

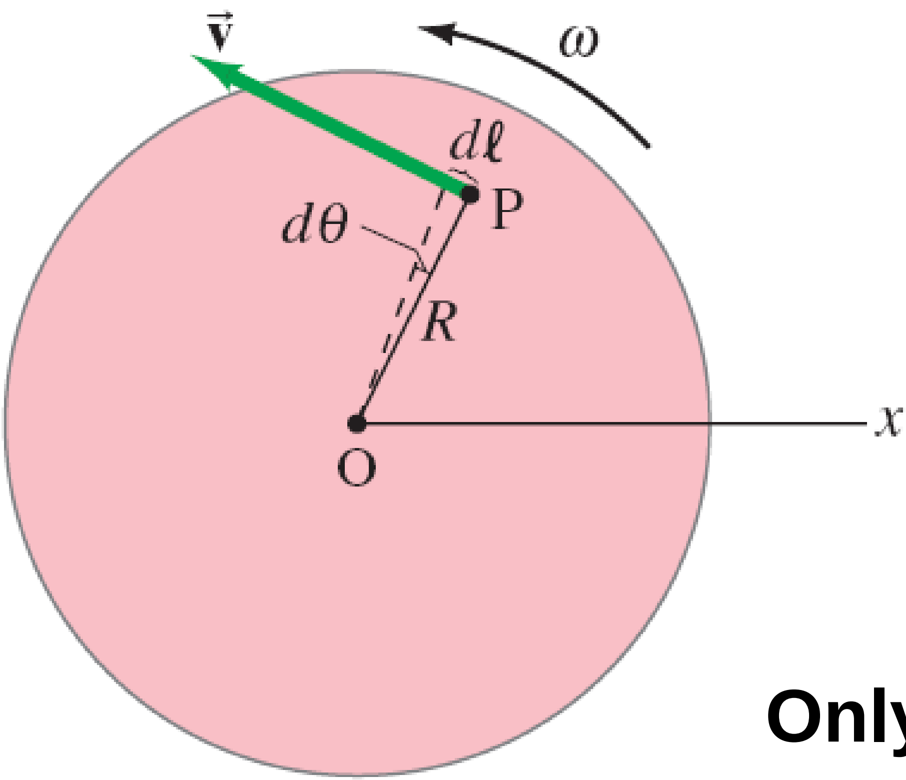
The instantaneous acceleration:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}.$$

Angular Quantities

Every point on a rotating body has an **angular velocity** ω and a **linear velocity** v.

They are related: $v = R\omega.$



$$d\theta = \frac{dl}{R}$$

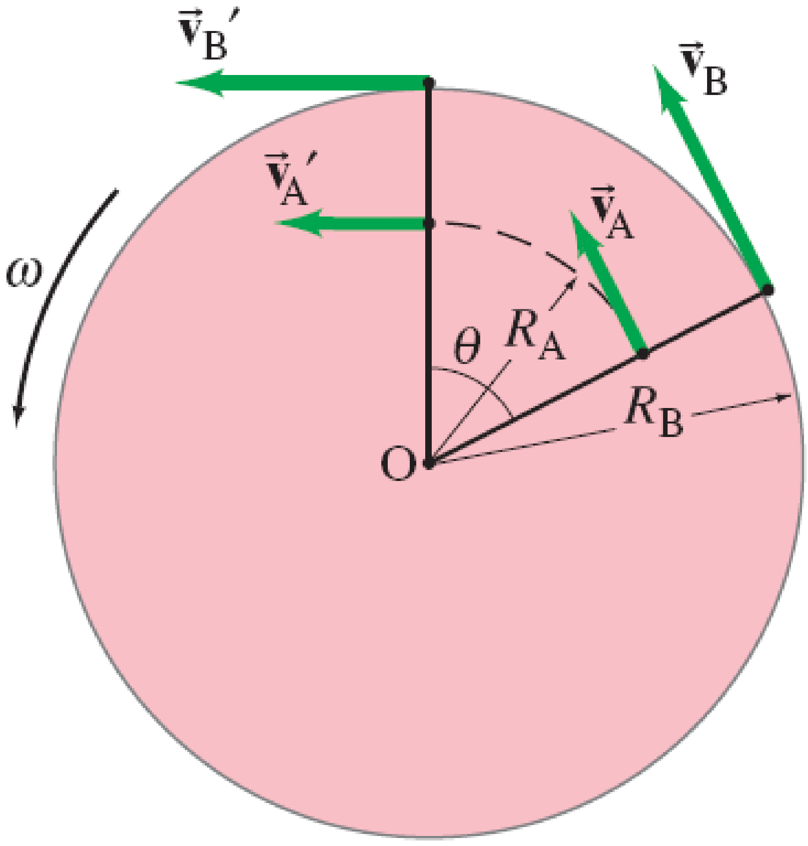
$$dl = R\,d\theta$$

$$\frac{dl}{dt} = R\frac{d\theta}{dt}$$

$$v = R\omega$$

Only if we use radians!

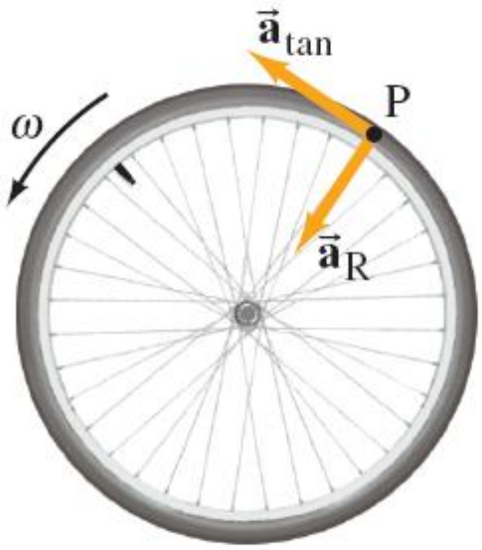
Angular Quantities



Objects farther from the axis of rotation will move faster.

$$v = R\omega.$$

Angular Quantities



If the angular velocity of a rotating object changes, it has a tangential acceleration:

$$a_{\tan} = \frac{dv}{dt} = R\frac{d\omega}{dt} = R\alpha.$$

Even if the angular velocity is constant, each point on the object has a centripetal acceleration:

$$a_{R} = \frac{v^2}{R} = \frac{(R\omega)^2}{R} = \omega^2 R.$$

Angular Quantities

Here is the correspondence between linear and rotational quantities:

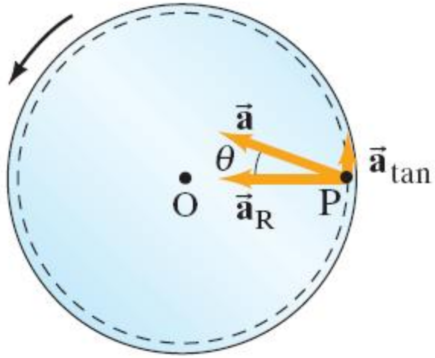
TABLE 8–1 Linear and Rotational Quantities

Linear	Type	Rotational	Relation
x	displacement	θ	$x = r\theta$
v	velocity	ω	$v = r\omega$
$a_{\tan}$	acceleration	α	$a_{\tan} = r\alpha$

x **represents arc length**

$$a_R = \frac{v^2}{R} = \frac{(R\omega)^2}{R} = \omega^2 R.$$

(a)



Example: Angular and linear velocities and accelerations.

A carousel is initially at rest. At $t = 0$ it is given a constant angular acceleration $\alpha = 0.060$ rad/s^2, which increases its angular velocity for 8.0 s. At $t = 8.0$ s, determine the magnitude of the following quantities: (a) the angular velocity of the carousel; (b) the linear velocity of a child located 2.5 m from the center; (c) the tangential (linear) acceleration of that child; (d) the centripetal acceleration of the child; and (e) the total linear acceleration of the child.

(a) $\omega = \omega_0 + \alpha\,\Delta t = 0.060 \times 8.0 = 0.48\, rad/s$

(b) $v = \omega R = 0.48 \times 2.5 = 1.2\, m/s$

(c) $a_{tan} = \alpha R = 0.060 \times 2.5 = 0.15\, m/s^2$

(d) $a_R = \dfrac{v^2}{R} = \dfrac{1.2^2}{2.5} = 0.58 m/s^2$

(e) $a = \sqrt{a_R^2 + a_t^2} = \sqrt{0.58^2 + 0.15^2} = 0.60\, m/s^2$

Angular Quantities

The **frequency** is the number of complete **revolutions** per second:

$$f = \frac{\omega}{2\pi}.$$

Frequencies are measured in **hertz**:

$$1 \text{ Hz} = 1 \text{ s}^{-1}.$$

The **period** is the time one revolution takes:

$$T = \frac{1}{f}.$$

Constant Angular Acceleration

The equations of motion for constant angular acceleration are the same as those for linear motion, with the substitution of the angular quantities for the linear ones.

Angular	Linear
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$x = v_0 t + \frac{1}{2}at^2$
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v^2 = v_0^2 + 2ax$
$\overline{\omega} = \dfrac{\omega + \omega_0}{2}$	$\overline{v} = \dfrac{v + v_0}{2}$

Example: Centrifuge acceleration.

A centrifuge rotor is accelerated from rest to 20,000 rpm in 30 s. (a) What is its average angular acceleration? (b) Through how many revolutions has the centrifuge rotor turned during its acceleration period, assuming constant angular acceleration?

(a)

$$\omega = 20000 \frac{rev}{min} \frac{2\pi\ rad}{1\, rev} \frac{1\, min}{60\ s} = 2100\ rad/s$$

$$\overline{\alpha} = \frac{\omega - \omega_0}{\Delta t} = \frac{2100}{30} = 70\ rad/s^2$$

(b)

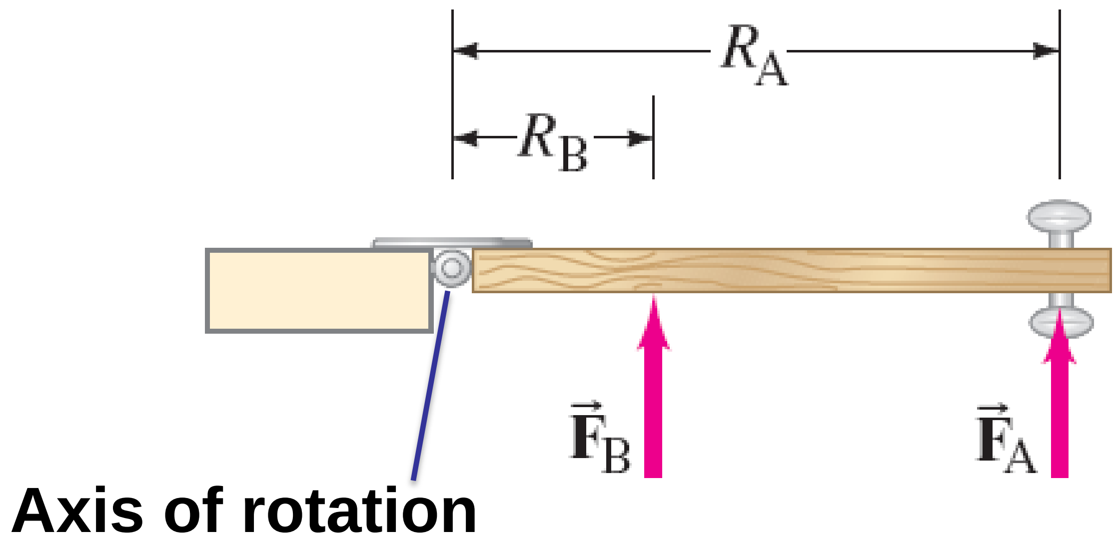
$$\theta = \theta_0 + \omega_0\, t + \frac{1}{2}\alpha\, t^2 = \frac{1}{2}\alpha\, t^2 = \frac{1}{2}(70)\, 30^2 = 31500\ rad$$

$$\theta = 31500\ rad\ \frac{1\, rev}{2\pi\ rad} = 5000\ rev$$

Torque

To make an object start rotating, a force is needed; both the position and direction of the force matter.

The perpendicular distance from the axis of rotation to the line along which the force acts is called the lever arm.



In a FBD, don't shift the forces sideways!

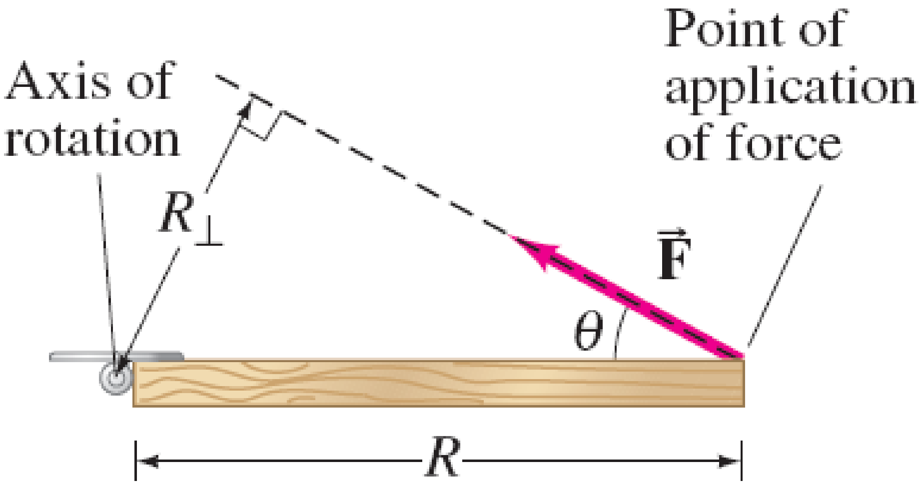
Torque

A longer lever arm is very helpful in rotating objects.

Torque

Here, the lever arm for F_A is the distance from the knob to the hinge; the lever arm for F_D is zero; and the lever arm for F_C is as shown.

Torque

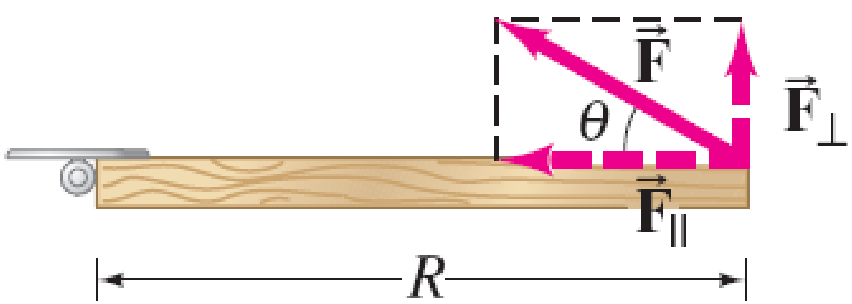


The **torque** is defined as:

$$\tau = R_\perp F.$$

$$R_\perp = R\sin\theta$$

$$\tau = RF\sin\theta$$



Also: $\tau = RF_\perp$

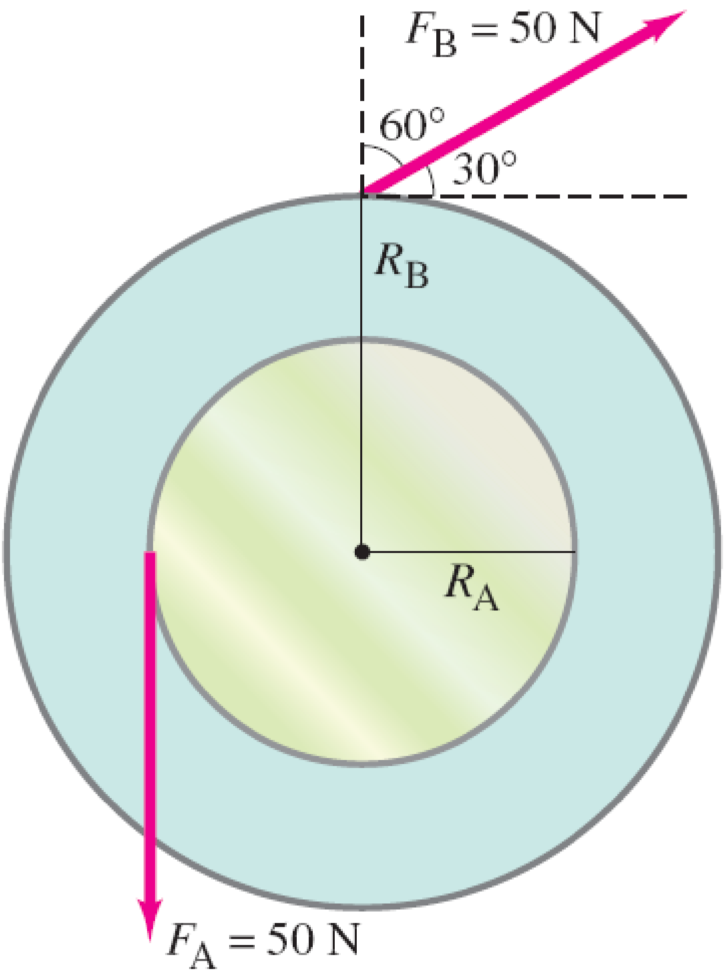
That is, $F_\perp = F\sin\theta$

$$\tau = RF\sin\theta$$

θ is the angle between $\vec{F}$ and $\vec{R}$.

Example: Torque on a compound wheel.

Two thin disk-shaped wheels, of radii R_A = 30 cm and R_B = 50 cm, are attached to each other on an axle that passes through the center of each, as shown. Calculate the net torque on this compound wheel due to the two forces shown, each of magnitude 50 N.



[Solution]

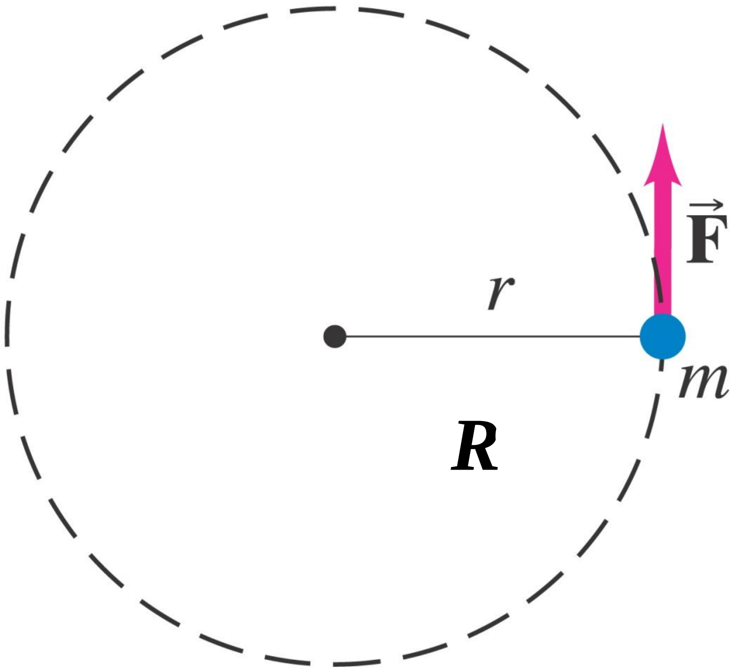
$$\tau = F_A R_A - F_B R_B \sin 60°$$
$$= 50(0.30) - 50(0.50)\sin 60°$$
$$= -6.7\ m \cdot N$$

CCW: +

CW: -

Rotational Dynamics; Torque and Rotational Inertia

Knowing that $F = ma$, we see that $\tau = mR^2\alpha$.



$$\tau = FR = maR = mR^2\alpha$$

$$(\text{since } a = \alpha R)$$

This is for a single point mass; what about an extended object?

As the angular acceleration is the same for the whole object, we can write:

$$\Sigma\tau = \left(\Sigma mR^2\right)\alpha.$$

Newton's Law for rotation

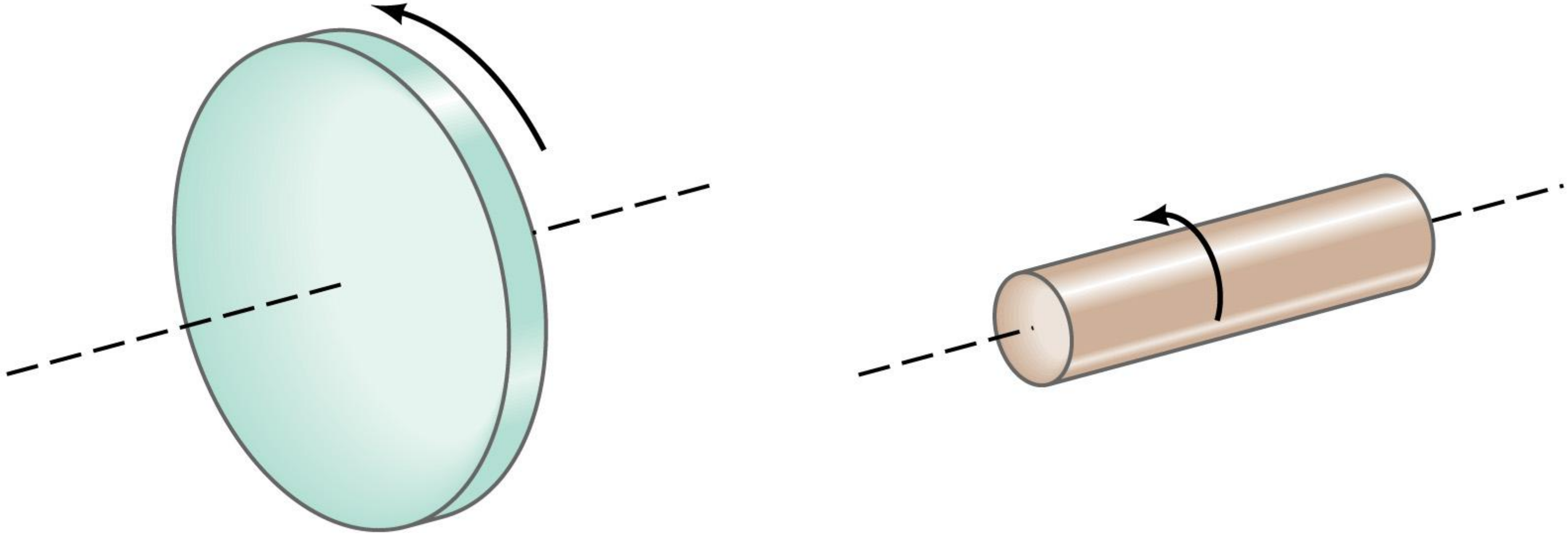
$$\sum \tau = I\alpha \quad \text{where} \quad I = \sum \left(mR^2\right)$$

I – moment of inertia.

Rotational Dynamics; Torque and Rotational Inertia

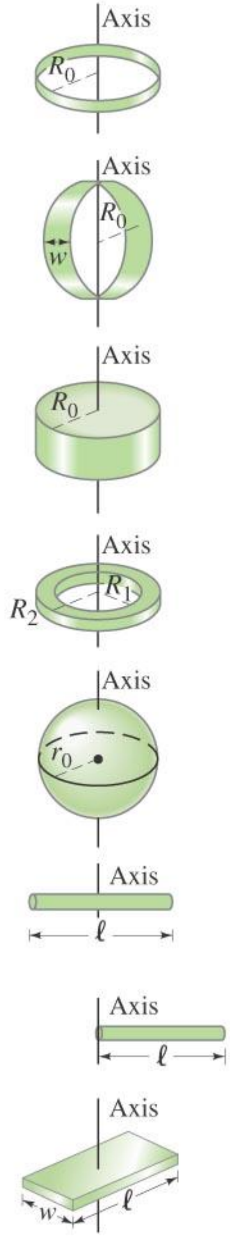
The quantity $I = \Sigma m_i R_i^2$ is called the **rotational inertia** (moment of inertia) of an object.

The **distribution** of mass matters here—these two objects have the same mass, but the one on the left has a greater **rotational inertia**, as so much of its mass is far from the axis of rotation.



Rotational Dynamics; Torque and Rotational Inertia

	Object	Location of axis	Moment of inertia
(a)	**Thin hoop,** radius R_0	Through center	MR_0^2
(b)	**Thin hoop,** radius R_0 width w	Through central diameter	$\frac{1}{2}MR_0^2 + \frac{1}{12}Mw^2$
(c)	**Solid cylinder,** radius R_0	Through center	$\frac{1}{2}MR_0^2$
(d)	**Hollow cylinder,** inner radius R_1 outer radius R_2	Through center	$\frac{1}{2}M(R_1^2 + R_2^2)$
(e)	**Uniform sphere,** radius r_0	Through center	$\frac{2}{5}Mr_0^2$
(f)	**Long uniform rod,** length ℓ	Through center	$\frac{1}{12}M\ell^2$
(g)	**Long uniform rod,** length ℓ	Through end	$\frac{1}{3}M\ell^2$
(h)	**Rectangular thin plate,** length ℓ, width w	Through center	$\frac{1}{12}M(\ell^2 + w^2)$



The rotational inertia of an object depends not only on its mass distribution but also the location of the axis of rotation—compare (f) and (g), for example.

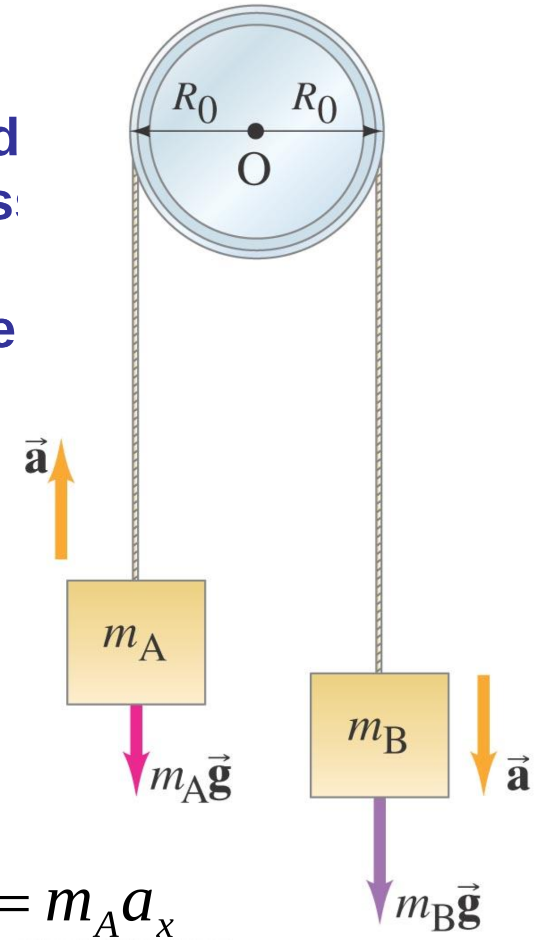
$$I = \sum \left(mR^2 \right)$$

Newton's Law for rotation

$$\sum \tau = I\alpha$$

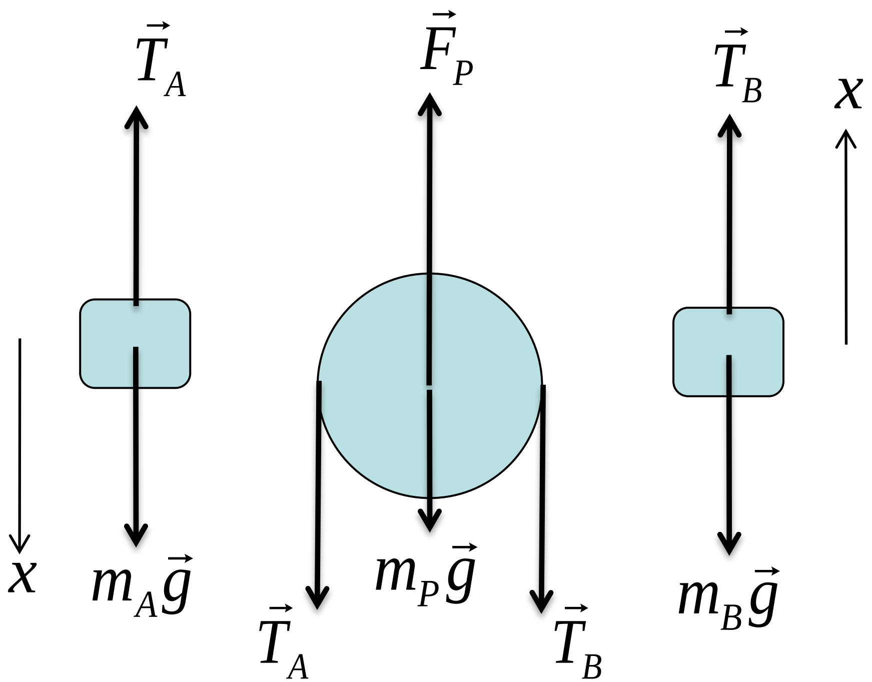
Example: Atwood's machine.

An Atwood machine consists of two masses, m_A and m_B, which are connected by a cord of negligible mass that passes over a pulley. If the pulley has radius R_0 and moment of inertia I about its axle, determine the acceleration of the masses m_A and m_B.



[Solution] FBD for each object

- Don't use a point to represent a rotating object;
- Don't shift forces sideways.



$$m_A g - T_A = m_A a_x$$

$$T_B - m_B g = m_B a_x$$

$$T_A R_0 - T_B R_0 = I\alpha$$

$$a_x = R_0 \alpha$$

$$4 \text{ unknowns}: \ a_x, \alpha, T_A, T_B.$$

$$m_A g - T_A = m_A a_x \qquad (1)$$

$$T_B - m_B g = m_B a_x \qquad (2)$$

$$T_A R_0 - T_B R_0 = I\alpha \qquad (3)$$

$$a_x = R_0 \alpha \qquad (4)$$

subs (4) into (3): $R_0 (T_A - T_B) = \dfrac{I a_x}{R_0} \quad \Rightarrow \quad T_A - T_B = \dfrac{I a_x}{R_0^2} \qquad (5)$

(1) + (2): $(m_A - m_B) g - T_A + T_B = (m_A + m_B) a_x \qquad (6)$

(5) + (6): $(m_A - m_B) g = (m_A + m_B) a_x + \dfrac{I a_x}{R_0^2}$

Solve for a_x: $a_x = \dfrac{(m_A - m_B) g}{(m_A + m_B) + \dfrac{I}{R_0^2}}$

α, T_A, and T_B can be determined using (4) and then (1) and (2).

Rotational Kinetic Energy

The **kinetic energy** of a rotating object is given by

$$K = \Sigma\left(\tfrac{1}{2}mv^2\right).$$

By substituting the rotational quantities, we find that the rotational kinetic energy can be written as:

$$\text{rotational } K = \tfrac{1}{2}I\omega^2.$$

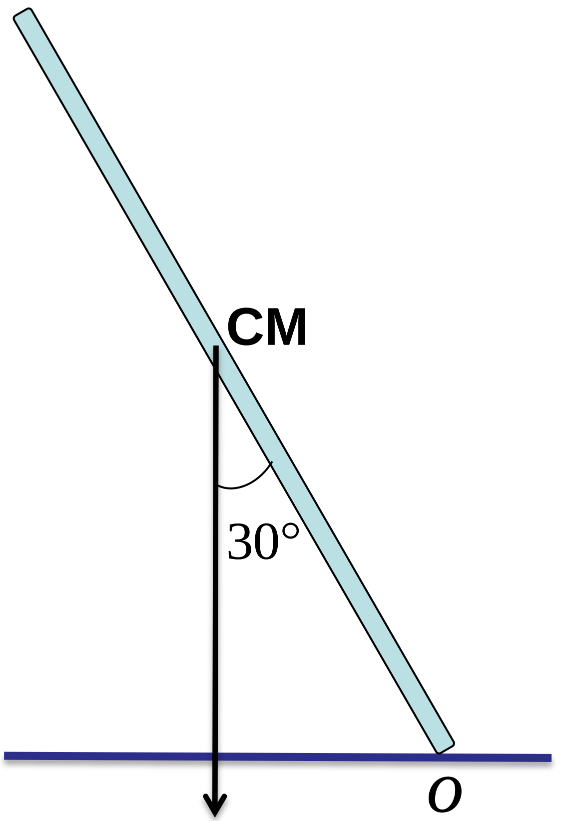
$$K = \sum\left(\frac{1}{2}mv^2\right) = \frac{1}{2}\sum\left(mR^2\omega^2\right) = \frac{1}{2}\omega^2\sum\left(mR^2\right) = \frac{1}{2}\omega^2 I$$

An object in both **translational** and **rotational** motion has both translational and rotational **kinetic energy**:

$$K = \tfrac{1}{2}Mv_{\mathrm{CM}}^2 + \tfrac{1}{2}I_{\mathrm{CM}}\omega^2.$$

Example (Old final exam, #19):

A pencil, 16 cm long ($l=0.16m$), is released from a vertical position with the eraser end resting on a table. The eraser does not slip. Treat the pencil like a uniform rod.
(a) What is the angular acceleration of the pencil when it makes a 30° angle with the vertical?
(b) What is the angular speed of the pencil when it makes a 30° angle with the vertical?



(a) Rotational dynamics (about o)

$$\tau = I\alpha, \quad \left(I = \frac{1}{3}ml^2, \quad \tau = mg\frac{l}{2}\sin 30° \right)$$

$$mg\left(\frac{l}{2}\right)\sin 30° = \frac{1}{3}ml^2\alpha$$

$$\alpha = \frac{3g\sin 30°}{2l} = \frac{3\times 9.8\times 0.5}{2\times 0.16} = 46 \ \ (\text{rad/s}^2)$$

(b) Mechanical energy is conserved because there is no non-conservative work (friction does not do work here).

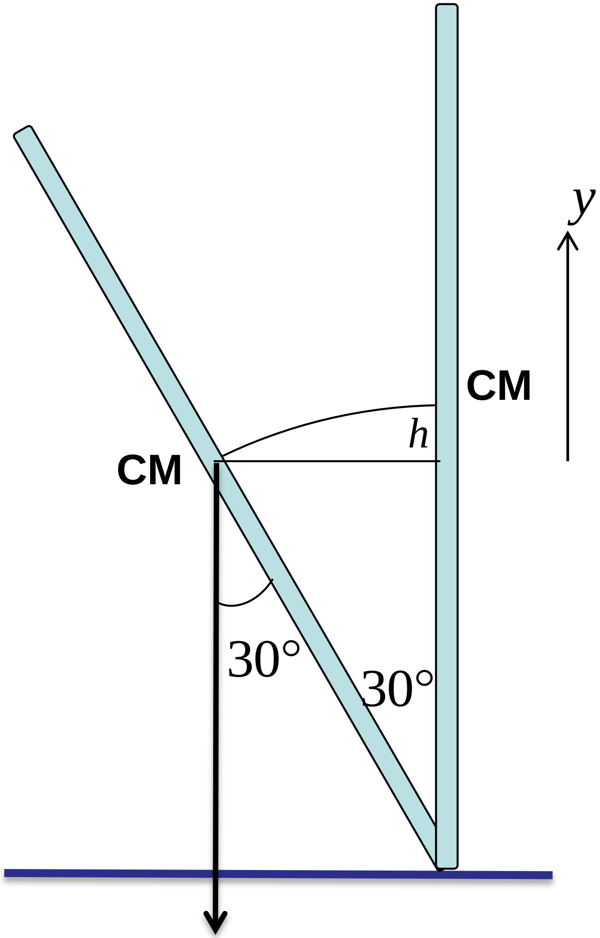
$$E_i = E_f$$

$$h = \frac{l}{2} - \frac{l}{2}\cos 30° = \frac{l}{2}(1-\cos 30°)$$

$$mgh = \frac{1}{2}I\omega^2$$

$$I = \frac{1}{3}ml^2$$

$$\omega = \sqrt{\frac{2mgh}{I}}$$



$$\omega = \sqrt{\frac{2mgl(1-\cos 30°)/2}{ml^2/3}}$$

$$= \sqrt{\frac{3g(1-\cos 30°)}{l}}$$

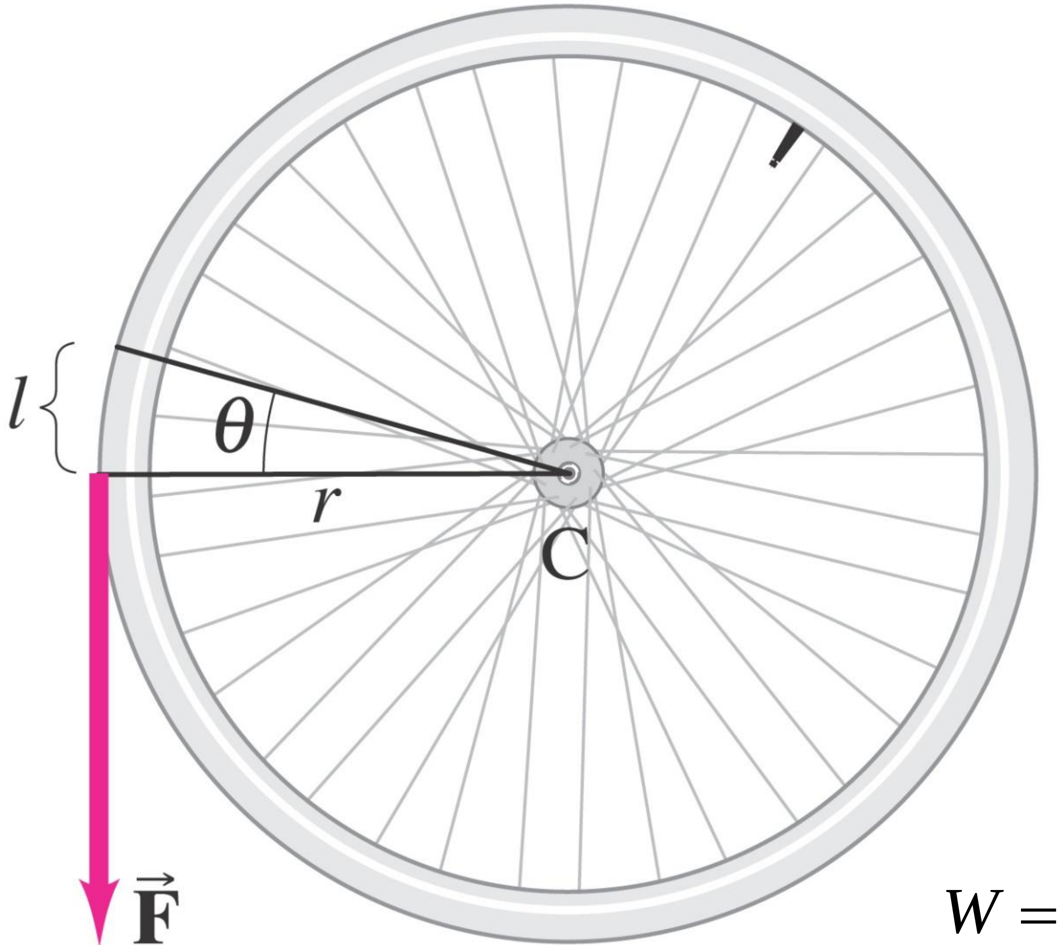
$$= \sqrt{\frac{3(9.8)(1-\cos 30°)}{0.16}}$$

4.96 (rad/s)

Rotational Work

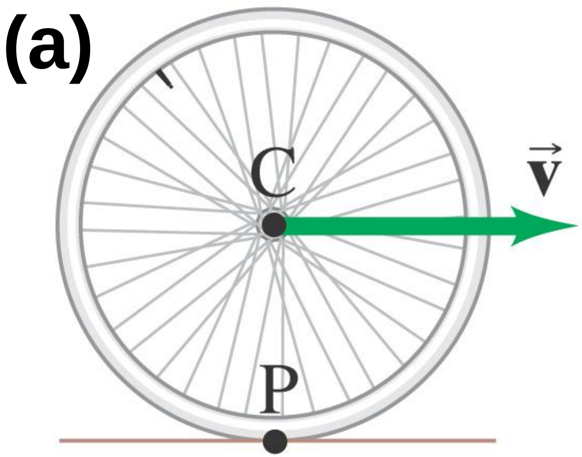
The torque does work as it moves the wheel through an angle *θ*:

$$W = \tau \Delta \theta.$$

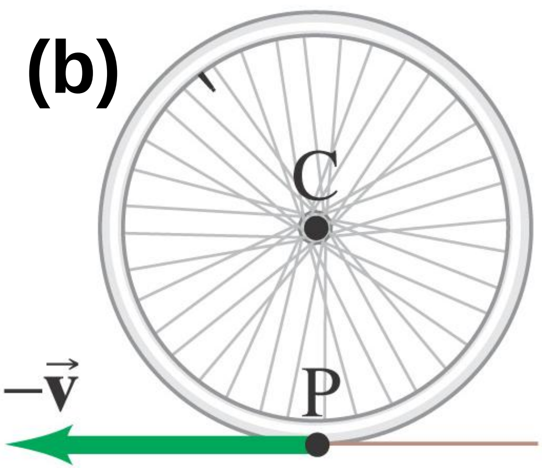


$$W = Fl = F\,R\,\Delta\theta = \tau\,\Delta\theta$$

Rotational Plus Translational Motion; Rolling



In (a), a wheel is **rolling without slipping**. The point P, touching the ground, is instantaneously at **rest**, and the center moves with velocity $\vec{v}$.



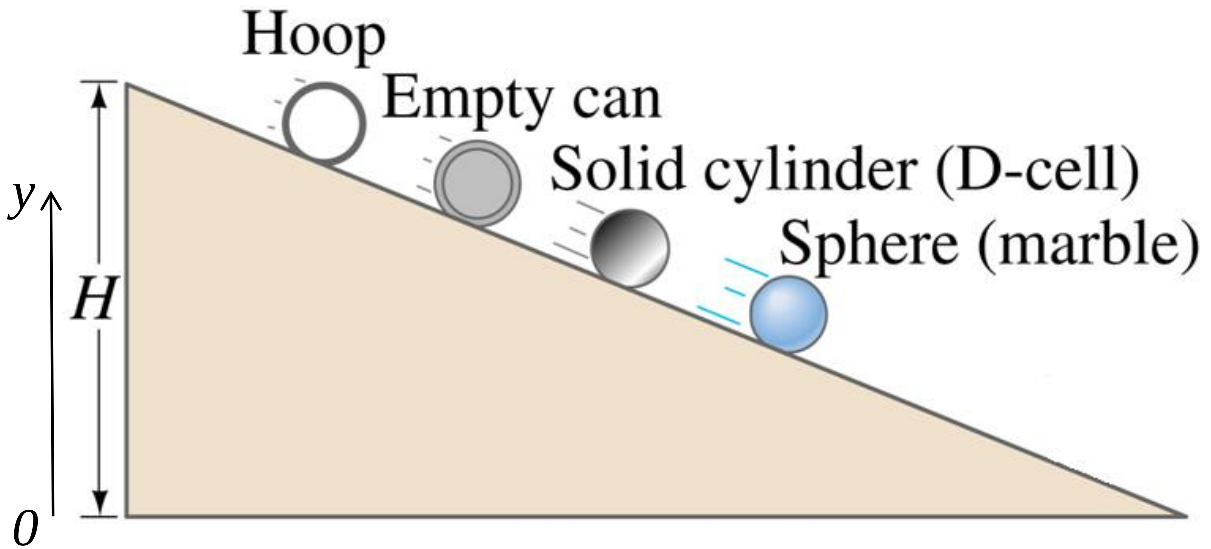
In (b) the same wheel is seen from a **reference frame** where C is at rest. Now point P is moving with velocity $-\vec{v}$.

The **linear speed** of the wheel is related to its **angular speed**:

$$v = R\omega.$$

Demo: Which one reaches the bottom first? Why?

W_{nc}=0, ME is conserved (ignore energy loss due to rolling friction).



For each object,

$$mgH = \frac{1}{2}mv_{CM}^2 + \frac{1}{2}I\omega^2$$

and $v_{CM} = v = r\omega$

$$\therefore \quad mgH = \frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{r^2} = \frac{1}{2}mv^2\left(1+\frac{I}{mr^2}\right)$$

$$v = \sqrt{\frac{2gh}{1+\frac{I}{mr^2}}}$$

$$\frac{I}{mr^2}\begin{cases} 1 - \text{hoop} \\ 1/2 - \text{solid cylinder} \\ 2/5 - \text{solid sphere} \end{cases}$$

The smaller the $\frac{I}{mr^2}$, the faster the v.

$\therefore$ Solid sphere 1st, then solid cylinder, and the hoop is the last.

The hoop:
- mass is distributed away from the axis of rotation (large I).
- Large I means large rotational KE,
- which means small translational KE (since $KE_R + KE_T = mgH = \text{const.}$),
- which means slow v.

Angular Momentum—Objects Rotating About a Fixed Axis

The rotational analog of linear momentum is angular momentum, *L*:

$$L = I\omega.$$

Then the rotational analog of Newton's second law is:

$$\Sigma\tau = \frac{dL}{dt}.$$

This form of Newton's second law is valid even if *I* is not constant.

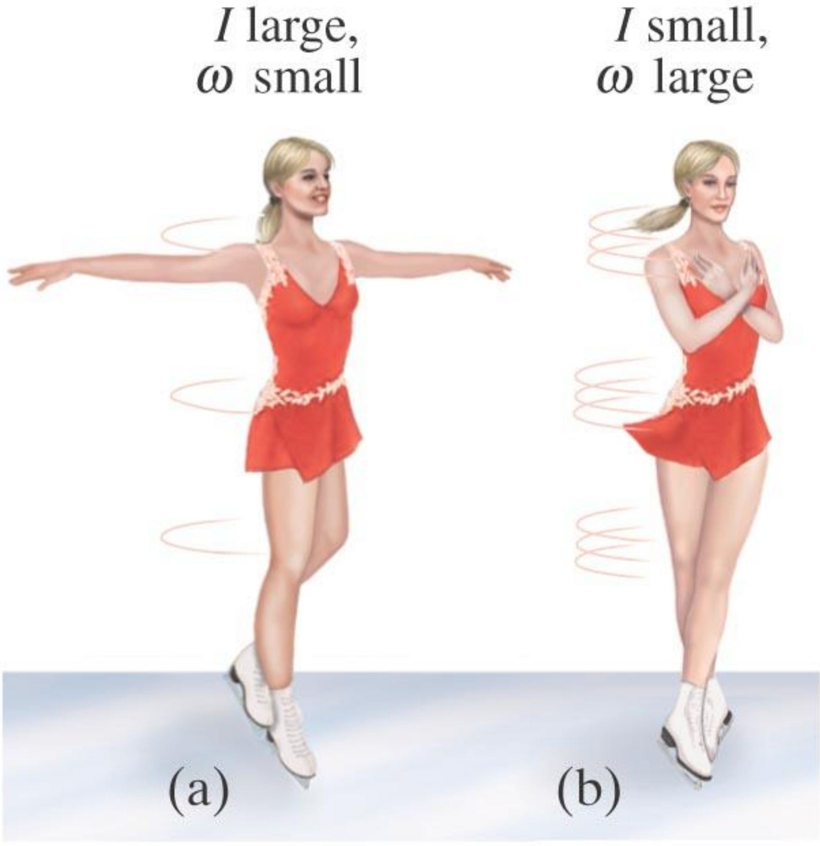
Conservation of Angular Momentum

In the absence of an external torque, angular momentum is conserved:

$$\frac{dL}{dt} = 0 \text{ and } L = I\omega = \text{constant.}$$

The total angular momentum of a system remains constant if the net external torque is zero.

Therefore, a figure skater can increase her angular speed by reducing her moment of inertia.



Demo

i-clicker question 18-1

The condition for angular momentum to be conserved is

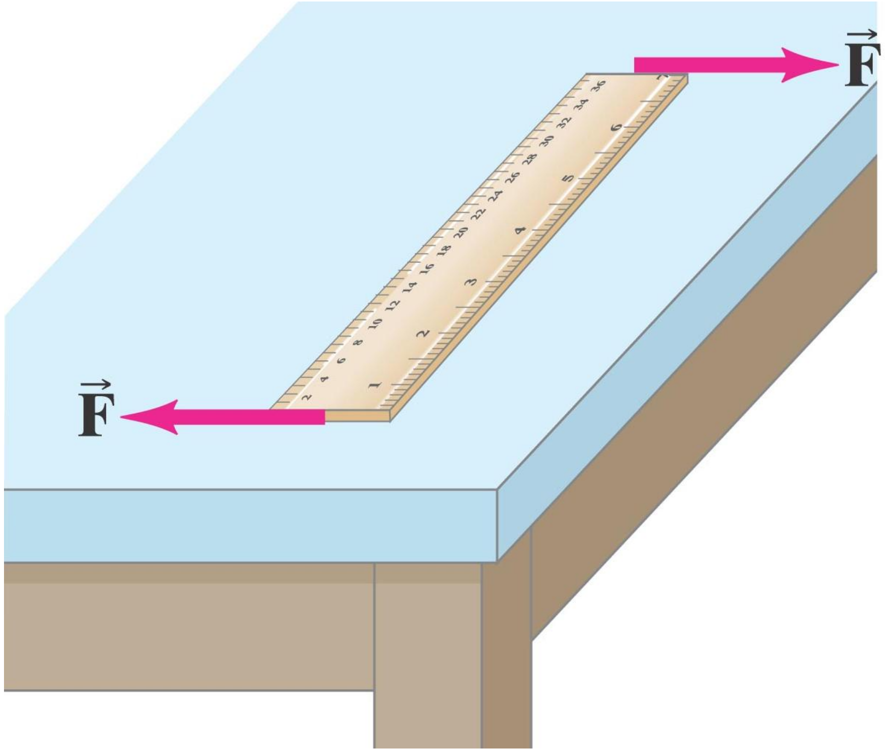
(A) It's a closed system.

(B) The net external force is zero.

(C) No non-conservative work.

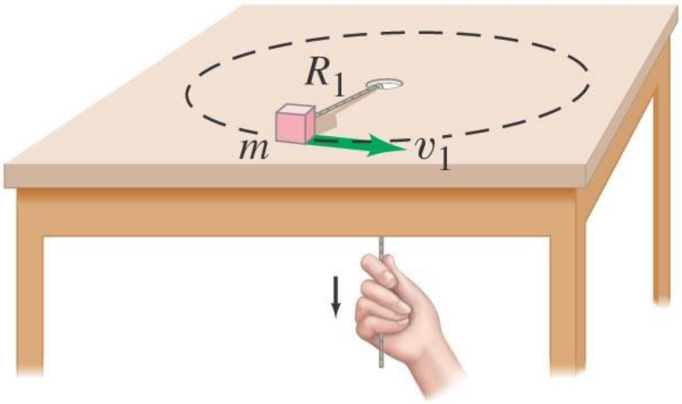
(D) The net external torque is zero.

(E) The angular momentum is always conserved.



Example: Object rotating on a string of changing length.

A small mass *m* attached to the end of a string revolves in a circle on a frictionless tabletop. The other end of the string passes through a hole in the table. Initially, the mass revolves with a speed v_1 = 2.4 m/s in a circle of radius R_1 = 0.80 m. The string is then pulled slowly through the hole so that the radius is reduced to R_2 = 0.48 m. What is the speed, v_2, of the mass now?



[Solution] Net torque = 0. Angular momentum is conserved.

$$I_i\,\omega_i = I_f\,\omega_f$$

$$\left(mR_1^2\right)\frac{v_1}{R_1} = \left(mR_2^2\right)\frac{v_2}{R_2}$$

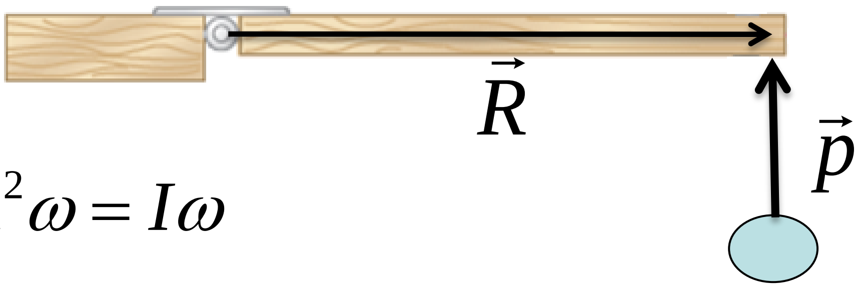
$$R_1 v_1 = R_2 v_2$$

$$v_2 = \frac{R_1 v_1}{R_2} = \frac{0.80 \times 2.4}{0.48} = 4.0\, m/s$$

Given: v_1, R_1, R_2;
Want: v_2.

Angular Momentum of a Moving Particle

You could open a door by throwing a large piece of play-doh at the door.



When $\vec{p} \perp \vec{R}$,

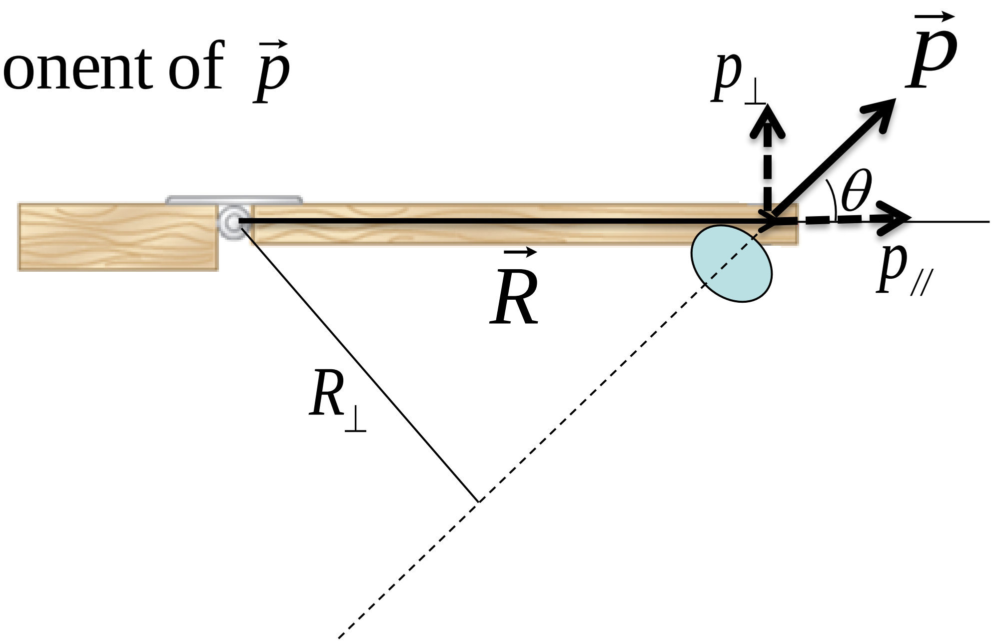
$$L = pR = mvR = m(R\omega)R = mR^2\omega = I\omega$$

When $\vec{p}$ is not perpendicular to $\vec{R}$,

only the perpendicular component of $\vec{p}$

contributes to L:

$$L = p_{\perp}R = pR\sin\theta$$

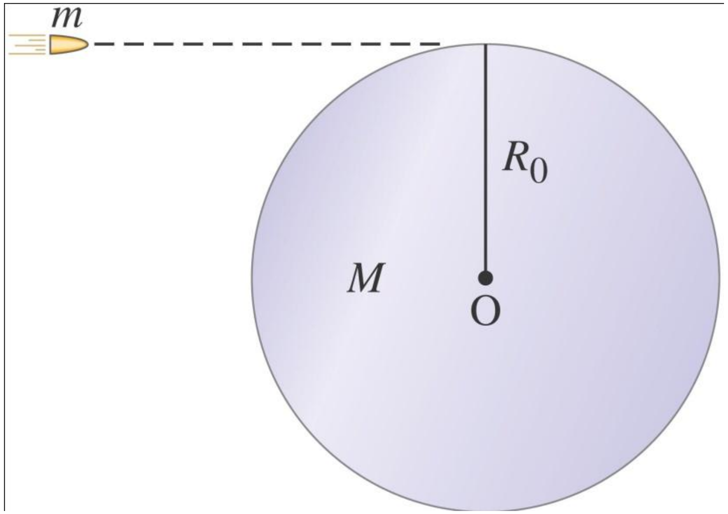


That is equivalent to

$$L = pR_{\perp} = pR\sin\theta$$

$$i.e., \quad L = mvR\sin\theta$$

Example: A bullet of mass m moving with velocity v strikes and becomes embedded at the edge of a cylinder of mass M and radius R_0. The cylinder, initially at rest, begins to rotate about its symmetry axis, which remains fixed in position. Assuming no frictional torque, what is the angular velocity of the cylinder after this collision? Is kinetic energy conserved?



[Solution] Net external torque = 0. Angular momentum is conserved.

$$L_i = L_f$$

$$mvR_0 = \left(mR_0^2 + \frac{1}{2} MR_0^2 \right) \omega$$

$$\omega = \frac{mvR_0}{mR_0^2 + \frac{1}{2} MR_0^2} = \frac{2mv}{(2m + M)R_0}$$

$$\text{KE}: \qquad K_i = \frac{1}{2} mv^2$$

$$K_f = \frac{1}{2} I\omega^2 = \frac{1}{2} \left(mR_0^2 + \frac{1}{2} MR_0^2 \right) \omega^2$$

Kinetic energy is not conserved. As the bullet being embedded into the cylinder, some kinetic energy is converted into heat.
Internal forces can change the kinetic energy of the system!

Vector Nature of Angular Quantities*

The **angular velocity** vector points along the **axis** of rotation, with the direction given by the right-hand rule. If the direction of the rotation axis does not change, the **angular acceleration** vector points along it as well.