Physics 221.

Lecture 3.

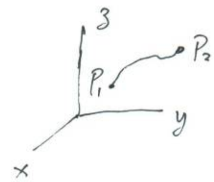
- Differential of V: $\quad V = V(x, y, z)$

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} \cdot dy + \frac{\partial V}{\partial z} dz$$

$$= \nabla V \cdot d\vec{l}$$

Difference of V between locations P_1 and P_2.

$$\int_{P_1}^{P_2} dV = \int_{P_1}^{P_2} \nabla V \cdot d\vec{l}$$



$$V_2 - V_1 = \int_{P_1}^{P_2} (-\vec{E}) \cdot d\vec{l}$$

i.e.

$$\boxed{V_2 - V_1 = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}} \qquad (4.39)$$

$\vec{E}$ — electric field. $\quad \vec{E} = \vec{F}/q$.

= electric force per unit charge.

V — electric potential.

= potential energy per unit charge.

(4-39) can be rewritten as

$$\int_{P_1}^{P_2} \vec{E} \cdot \vec{d\ell} = -(V_2 - V_1) = -\Delta V$$

Meaning:

work done by $\vec{E}$-field = loss of potential energy

- Divergence of a vector field.

Given a vector field $\vec{E}$, its divergence is:

$$\nabla \cdot \vec{E} \triangleq div\,\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \qquad (3.28)$$

The result is a scalar function.

- Meaning of divergence.

e.g. Gauss' Law.

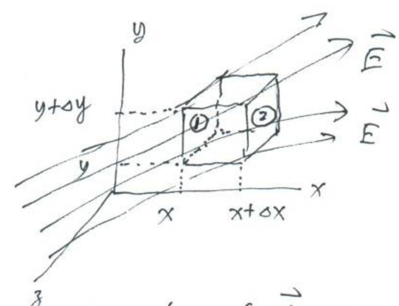
$$\oint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0}$$

Q – Total charge inside the enclosed surface S.

Consider a small cube whose surface area is S.

Total flux of $\vec{E}$:

$$\oint_S \vec{E} \cdot d\vec{s}.$$



direction of $d\vec{s}$ — outward.

Flux through face ①:

$$F_1 = \int_{①} \vec{E} \cdot d\vec{s}$$

$$= \int_{①} (\hat{x} E_x + \hat{y} E_y + \hat{z} E_z) \cdot (-\hat{x})\, dy\, dz \approx -E_x(1) \cdot \Delta y \Delta z.$$

Flux through face ②.

$$F_2 = \int_{(2)} \vec{E} \cdot d\vec{s}$$

$$= \int_{(2)} (\hat{x} E_x + \hat{y} E_y + \hat{z} E_z) \cdot (\hat{x})\, dy\, dz = E_x(2) \cdot \Delta y \Delta z$$

$$\text{but} \quad E_x(2) = E_x(1) + \frac{\partial E_x}{\partial x} \Delta x$$

$$F_2 = \left(E_x(1) + \frac{\partial E_x}{\partial x} \Delta x \right) \Delta y \Delta z \ .$$

Sum of fluxes out of face ① and ②:

$$F_1 + F_2 = \frac{\partial E}{\partial x} \Delta x \Delta y \Delta z \ .$$

∴ Total flux out of the surface area of the cube:

$$\oint_S \vec{E} \cdot d\vec{s} = \Big(\underbrace{\frac{\partial E}{\partial x}}_{\text{left-right}} + \underbrace{\frac{\partial E}{\partial y}}_{\text{Top-bottom}} + \underbrace{\frac{\partial E}{\partial z}}_{\text{front and back}} \Big) \cdot \underbrace{\Delta x \Delta y \cdot \Delta z}_{\Delta V \text{ volume element}} \ .$$

i.e.

$$\oint_S \vec{E} \cdot d\vec{s} = (\nabla \cdot \vec{E}) \cdot \Delta V.$$

$$\therefore \quad \boxed{\nabla \cdot \vec{E} = \lim_{\Delta V \to 0} \frac{\oint \vec{E} \cdot d\vec{s}}{\Delta V}} \qquad \left(= \frac{Q}{\varepsilon_0 \Delta V} \, . \quad Q - \text{charge in } \Delta V \text{ i.e. source of } \vec{E} \text{ lines.} \right)$$

(3.27).

$+Q$: source of $\vec{E}$-lines.

$-Q$: sink of $\vec{E}$-lines.

$\frac{Q}{\Delta V} = \rho$ — density of charge.
(strength of source (+) or sink (−)).

$\therefore$ — $\nabla \cdot \vec{E}$ is a scalar.

— It's the net outward flux of $\vec{E}$ per unit volume.

— when $\nabla \cdot \vec{E} > 0$, it's a source

$\nabla \cdot \vec{E} < 0$, it's a sink.

(If $\nabla \cdot \vec{E} = 0$ everywhere,
Then. $\vec{E}$ is divergenceless, ("sourceless")
OR. Solenoidal.)

— $\nabla \cdot \vec{E}$ represents the strength of source of $\vec{E}$ at a point. (Note. $\Delta V \Rightarrow 0$)

Note: $\nabla \cdot \vec{B} = 0$ in magetostatics. (5.1a).

why? $\because$ There is no "magnetic charge"!

• When the volume in (3.27) is not small:

$$\boxed{\int_V \nabla\cdot\vec{E}\cdot dV = \oint_S \vec{E}\cdot d\vec{S}} \qquad 3-30$$

called the divergence theorem.

(V is the volume enclosed by S).

• Calculations of divergence in different coordinate systems:

plug in the formulas !

e.g. 3-3. p.55.