Lecture 4.

- Curl of a vector field $\vec{B}$

Definition:

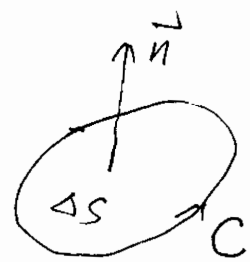
$$\boxed{\nabla \times \vec{B} \triangleq \lim_{\Delta S \to 0} \frac{1}{\Delta S}\left[\hat{n} \oint_C \vec{B}\cdot d\vec{l}\,\right]_{max}} \qquad (3.35)$$

ΔS — area enclosed by the contour C.

$\hat{n}$ — unit vector normal to ΔS.

$\oint_C \vec{B}\cdot d\vec{l}$ — circulation of $\vec{B}$

Why max? — because the circulation depends on the orientation of C.



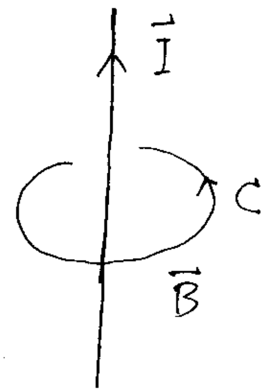
(yields max circulation)

Meaning of $\nabla \times \vec{B}$:

- It's a vector.
- Max circulation of $\vec{B}$ per unit area.
- points to the normal direction (right-hand-rule).
- Represents the strength of a vortex source that causes the circulation of $\vec{B}$.
- When $\nabla \times \vec{B} = 0$, $\vec{B}$ is called an irrotational field. OR conservative field.

• Example. Magnetostatics

Ampere's Law. $\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I$.

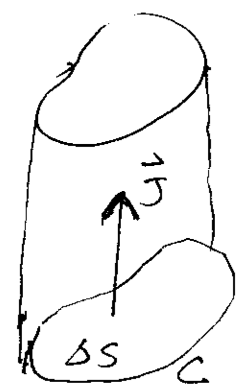


Current I is the source of $\vec{B}$.

For a current distribution with current density $\vec{J}$:

$$I = \int_{\Delta S} \vec{J}\cdot d\vec{S} \qquad (4.12)$$

↑ current flowing through area ΔS.



Then:

$$\nabla \times \vec{B} = \left. \frac{\hat{n} \oint_C \vec{B}\cdot d\vec{l}}{\Delta S} \right|_{\Delta S \to 0}$$

$$= \left. \frac{\hat{n}\,\mu_0 I}{\Delta S} \right|_{\Delta S \to 0}$$

$$= \mu_0 \vec{J}$$

($\vec{J}$: direction : $\vec{J} // \hat{n}$; magnitude : $|\vec{J}| = \frac{I}{\Delta S}$)

i.e. $\boxed{\nabla \times \vec{B} = \mu_0 \vec{J}}$

current density represents the strength of vortex source which is the cause of the circulation of $\vec{B}$.

- Vector identities involving the curl

① . $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$

② $\nabla \cdot (\nabla \times \vec{A}) = 0$ (for any vector $\vec{A}$)

e.g. Magnetostatics : Since $\nabla \cdot \vec{B} = 0$ (no monopo-

$\therefore$ $\vec{B}$ can be written as the curl of a vector $\vec{A}$

$$\vec{B} = \nabla \times \vec{A} . \quad (5.53)$$

($\vec{A}$ — vector potential)

③ . $\nabla \times (\nabla V) = 0$. (for any scalar V)

e.g. electrostatics: Since $\vec{E} = -\nabla V$.

$$\nabla \times \vec{E} = 0 .$$

$\therefore$ $\vec{E}$ field is conservative .

i.e. $\oint \vec{E} \cdot d\vec{l} = 0$.

- Stoke's Theorem.

$$\boxed{\oint_S (\nabla \times \vec{B}) \cdot d\vec{s} = \oint_C \vec{B} \cdot d\vec{l}} \qquad (3.39).$$

It comes from (3.35), by changing ΔS to S.

It converts a line integral of $\vec{B}$ to a surface integral of $\nabla \times \vec{B}$.

Note. Divergence Theorem is also called Gauss' Theorem.

$$\int_V \nabla \cdot \vec{E}\, dV = \oint_S \vec{E} \cdot d\vec{s}$$

It converts a surface integral of $\vec{E}$ to the volume integral of $\nabla \cdot \vec{E}$

- Laplacian operator ∇^2

$$\nabla^2 V \triangleq \nabla \cdot (\nabla V) = \underbrace{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}_{\text{in Cartesian system}}.$$

In cartesian coordinates:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

It can also act on a vector. $\vec{E} = \hat{x} E_x + \hat{y} E_y + \hat{z} E_z$.

$$\nabla^2 \vec{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2} \right) \vec{E}$$

OR.

$$\nabla^2 \vec{E} = \hat{x}\, \nabla^2 E_x + \hat{y}\, \nabla^2 E_y + \hat{z}\, \nabla^2 E_z.$$

- Useful formulas in the back cover of the textbook.