Lecture 5.

- Maxwell's Equations.

In Vacuum:

$$\begin{cases} \nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0} \\ \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{B} = \mu_0 \vec{J} + \underbrace{\varepsilon_0 \mu_0 \dfrac{\partial \vec{E}}{\partial t}}_{\text{"displacement curre}} \end{cases}$$

ε_0 — electrical permittivity of free space.

μ_0 — magnetic permeability of free space

$\varepsilon_0 = 8.85 \times 10^{-12}$ F/m.

$\mu_0 = 4\pi \times 10^{-7}$ H/m.

In medium. $\vec{D} = \varepsilon \vec{E}$

(Isotropic homogeneous medium.) $\vec{B} = \mu \vec{H}$

$\vec{E}$ — electric field intensi

$\vec{D}$ — electric flux density

$\vec{B}$ — magnetic flux density

$\vec{H}$ — magnetic field intensi

$\vec{E}$ and $\vec{B}$ represents the strength of the electric and magnetic fields.

$\vec{D}$ and $\vec{H}$ were introduced to make some equations ~~[illegible]~~ easier to use.

- In medium:

$$\boxed{\begin{array}{lll} 4.1a. & \nabla\cdot\vec{D} = \rho & \text{— Gauss' law.} \\ 4.1b. & \nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} & \text{— Faraday's law.} \\ 4.1c. & \nabla\cdot\vec{B} = 0 & \text{— No monopole.} \\ 4.1d & \nabla\times\vec{H} = \vec{J} + \dfrac{\partial\vec{D}}{\partial t}. & \text{— Ampere's law.} \end{array}}$$

Integral form:

$$\nabla\cdot\vec{D} = \rho \qquad \oint_S \vec{D}\cdot d\vec{s} = Q$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \oint_C \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt}.$$

$$\nabla\cdot\vec{B} = 0 \qquad \oint_S \vec{B}\cdot d\vec{s} = 0$$

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}. \qquad \oint_C \vec{H}\cdot d\vec{l} = I + \int_S \frac{\partial\vec{D}}{\partial t}\cdot d\vec{s}$$

- Electrostatics:

$\vec{B}, \vec{H}, \vec{J}, \vec{E}, \vec{D}, \rho$ — time independent.

$$\therefore \quad \nabla\cdot\vec{D} = \rho$$

$$\nabla\times\vec{E} = 0.$$

- magnetostatics.

when $\vec{B}, \vec{H}, \vec{J}, \vec{E}, \vec{D}, \rho$ — time independent.

$$\therefore \quad \nabla\cdot\vec{B} = 0.$$

$$\nabla\times\vec{H} = \vec{J}$$

- Electrostatics

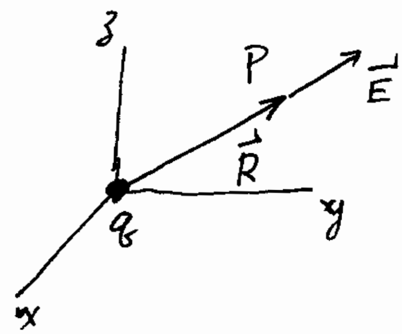
$$\nabla \cdot \vec{D} = \rho \qquad (4.2\text{ a}). \quad \text{Gauss' law}.$$

$$\nabla \times \vec{E} = 0 \qquad (4.2\text{ b}) \quad \vec{E}\text{-field is conservative}.$$

It can be shown that Gauss' Law is equivalent to Coulomb's law:

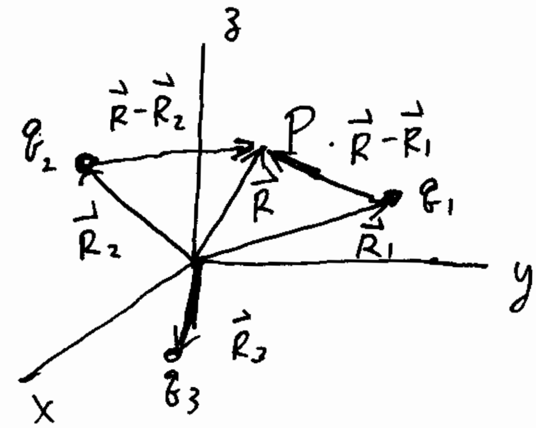
(in vacuum): $\vec{E} = \hat{R} \dfrac{q}{4\pi \varepsilon_0 R^2}$

In medium: $\vec{E} = \hat{R} \dfrac{q}{4\pi \varepsilon R^2}$



- When there are many charges $q_1, q_2, q_3 \cdots\cdots$, located at $\vec{R}_1, \vec{R}_2, \vec{R}_3 \cdots\cdots$,

If we want find the $\vec{E}$ field at location $\vec{R}$.



$$\vec{E}(\vec{R}) = \sum_{i=1}^{N} \vec{E}_i = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i(\vec{R}-\vec{R}_i)}{|\vec{R}-\vec{R}_i|^3}$$

$\vec{E}_i$ — $\vec{E}$ field due to charge q_i located at $\vec{R}_i$

$\vec{R}-\vec{R}_i$ — Vector from $\vec{R}_i$ to $\vec{R}$

$\vec{R}_i$ ↑ Source point

$\vec{R}$ ↑ field point.

- For continuous charge distribution in volume V' with charge density ρ_v

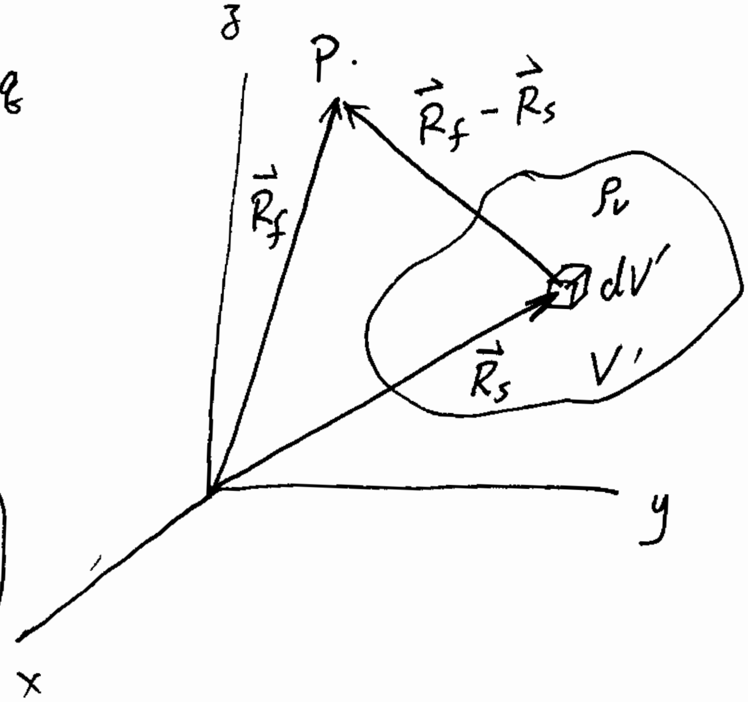
— charge element dq

$$dq = \rho_v \, dV'$$

located at $\vec{R}_s$

— Field point : $\vec{R}_f$

(We want to find $\vec{E}$ at $\vec{R}_f$)



— distance vector from source dq to field point $\vec{R}_f$:

$$(\vec{R}_f - \vec{R}_s) \triangleq \vec{R}'$$

← [Some books use $(\vec{R} - \vec{R}')$ We used it briefly last time]

Total $\vec{E}$-field at $\vec{R}_f$:

$$\vec{E} = \frac{1}{4\pi\varepsilon} \int \frac{\hat{\vec{R}}'}{R'^2} dq = \frac{1}{4\pi\varepsilon} \int_{V'} \frac{\hat{\vec{R}}' \rho_v \, dV'}{R'^2} \qquad (4.21a)$$

for a charged wire: $dq = \rho_\ell \, d\ell$ (4.21c)

for a charged sheet: $dq = \rho_s \, ds$. (4.21b)

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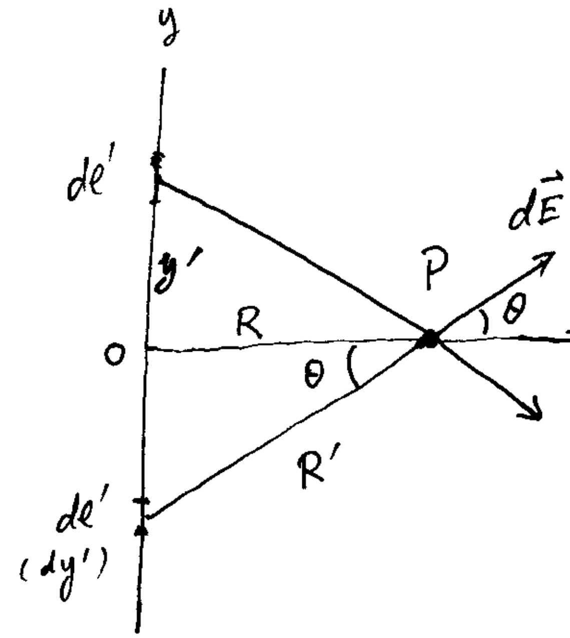
Lecture 6.



e.g. A straight rod of length $2L$ carries a uniform linear charge density ρ_ℓ. Determine the electric field at point P, a distance R from the rod along the perpendicular bisector.

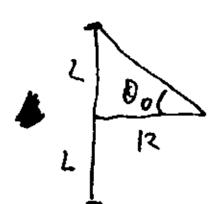
$$dq = dl' = dy'$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \hat{R}'\,\frac{\rho_\ell\, dl'}{R'^2} = \frac{\rho_\ell}{4\pi\varepsilon_0}\int_{-L}^{L}\frac{\hat{R}'\,dy'}{(y'^2+R^2)}$$

by symmetry, $\vec{E} = \hat{x}\,E_x$ (y-component cancels)

$$E_x = \frac{\rho_\ell}{4\pi\varepsilon_0}\int_{-L}^{L}\frac{\cos\theta\, dy'}{y'^2+R^2} = \frac{\rho_\ell}{2\pi\varepsilon_0}\int_{0}^{L}\frac{\cos\theta\, dy'}{y'^2+R^2}$$

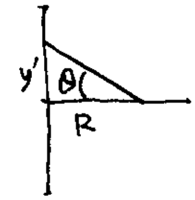
$$\boxed{E_x = \frac{\rho_\ell L}{2\pi\varepsilon_0 R\sqrt{R^2+L^2}}}$$



OR:

$$E_x = \frac{\rho_\ell \sin\theta_0}{2\pi\varepsilon_0 R}$$

OR: $\tan\theta = \dfrac{y'}{R}$ $\qquad$ $\cos\theta = \dfrac{R}{\sqrt{R^2+y'^2}}$



$$y' = R\tan\theta$$

$$dy' = R\cdot\sec^2\theta\cdot d\theta = \frac{R\,d\theta}{\cos^2\theta}$$

$$\int_0^L \frac{\cos\theta\,dy'}{y'^2+R^2} = \int_0^{\theta_0} \frac{R\cos\theta\,d\theta}{\cos^2\theta\,(y'^2+R^2)}$$

$$= \frac{1}{R}\int_0^{\theta_0} \frac{R^2\,d\theta}{(y'^2+R^2)\cos\theta}$$

$$= \frac{1}{R}\int_0^{\theta_0} \frac{\cos^2\theta\,d\theta}{\cos\theta}$$

$$= \frac{1}{R}\int_0^{\theta_0} \cos\theta\,d\theta$$

$$= \left.\frac{\sin\theta}{R}\right|_0^{\theta_0}$$

$$= \frac{\sin\theta_0}{R}. \qquad \sin\theta_0 = \frac{L}{\sqrt{R^2+L^2}}$$

$$= \frac{L}{R\sqrt{R^2+L^2}}$$

when $L \gg R$. (very long rod).

$$\vec{E} = \hat{x}\,\frac{\rho_\ell}{2\pi\varepsilon_0 R}$$

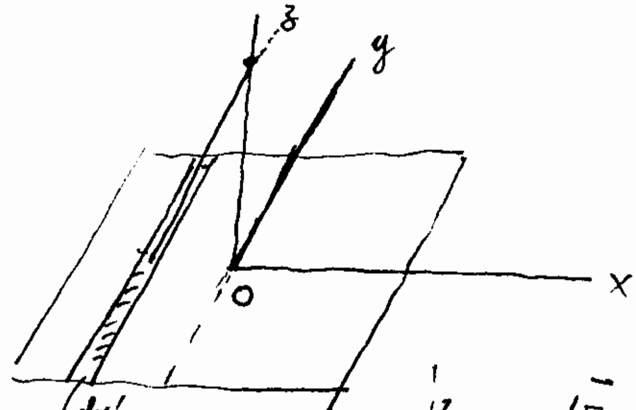
when $R \gg L$.

$$\vec{E} = \hat{x}\,\frac{\rho_\ell L}{2\pi\varepsilon_0 R^2} = \hat{x}\,\frac{Q}{4\pi\varepsilon_0 R^2}$$ (just like a point charge)

- For a large flat sheet of charge with a uniform area density of charge. ρ_s (charge per unit area).

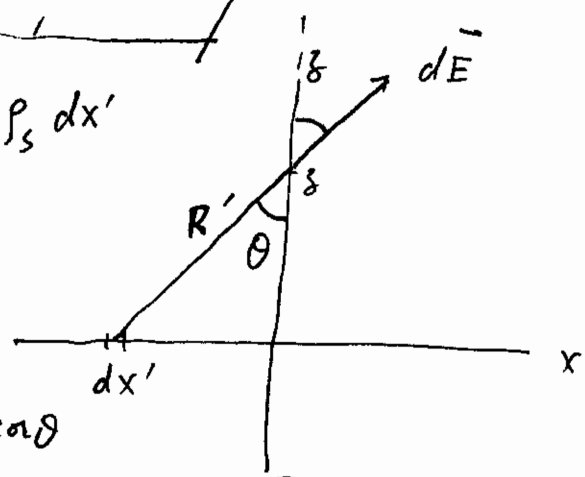
  O — centre of the area.

  Find $\vec{E}$-field at a point on z-axis.



$$\vec{E} = \hat{z}\int_{-L}^{L}\frac{\rho_s \cos\theta \cdot dx'}{2\pi\varepsilon_0 R'}\Bigg|_{L\to\infty}$$

$\rho_\ell = \dfrac{dq}{\text{Length}} = \rho_s\, dx'$



($E_x = 0$, $E_y = 0$ by symmetry).

$$\frac{x'}{z} = \tan\theta, \quad dx' = \frac{z\, d\theta}{\cos^2\theta}, \quad \frac{z}{R'} = \cos\theta$$

$$\vec{E} = \hat{z}\cdot 2\int_0^{L\to\theta_0}\frac{\rho_s \cos\theta\; z\; d\theta}{2\pi\varepsilon_0 R'\cos^2\theta}\Bigg|_{\substack{L\to\infty\\ \theta_0\to\frac{\pi}{2}}} = \hat{z}\,\frac{\rho_s}{\pi\varepsilon_0}\int_0^{\frac{\pi}{2}}\frac{z\; d\theta}{\cos\theta\; R'} = \hat{z}\frac{\rho_s}{\pi\varepsilon_0}\int_0^{\frac{\pi}{2}} d$$

$$= \hat{z}\,\frac{\rho_s}{2\varepsilon_0}$$ (p.76) (4.25)

- Gauss' Law.

$$\nabla \cdot \vec{D} = \rho_v \qquad (4.26)$$

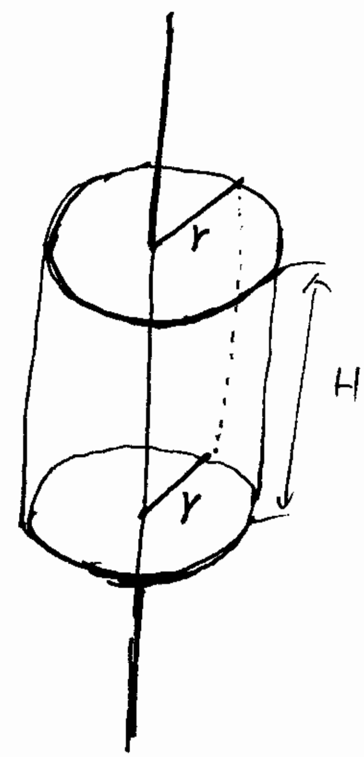
$$\oint_S \vec{D} \cdot \vec{ds} = Q \qquad (4.29)$$

$\vec{D} = \varepsilon \vec{E} = \varepsilon_0 \varepsilon_r \vec{E}$ for istropic and uniform medium

ρ_v — free charge per unit volume.

e.g.: A Long straight wire with a uniform linear charge density ρ_ℓ.

Find $\vec{E}$ field.



Gaussian surface.

by symmetry: $\vec{E} = \hat{r} E_r$.

$\vec{D} \cdot \vec{ds} = 0$ on top and bottom.

E_r = const on lateral surface $\doteq E$

$$Q \doteq \oint_S \vec{D} \cdot \vec{ds} = \oint_S \varepsilon_0 \vec{E} \cdot \vec{ds}$$

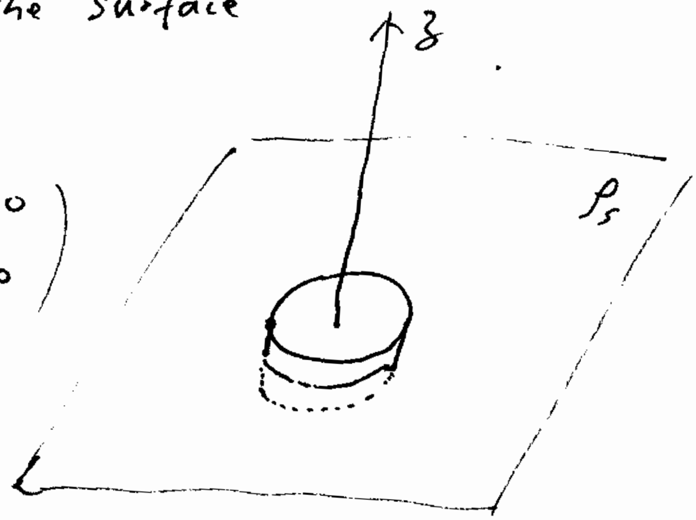
use cylindrical coord.

$$= \varepsilon_0 E_r \cdot 2\pi r \cdot H$$

$$E_r = \frac{Q}{2\pi \varepsilon_0 H} = \frac{\rho_\ell}{2\pi \varepsilon_0}.$$ (same as previous example)

e.g. For a large flat sheet of charge with a uniform area density of charge ρ_s.

Find $\vec{E}$-field neer the surface

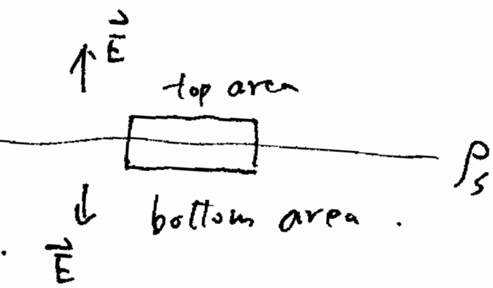


by symmetry.

$E_x = 0$. $E_y = 0$. $\begin{pmatrix} E_r = 0 \\ E_\phi = 0 \end{pmatrix}$

Thin drum-shaped Gaussian surface.

$$\oint_S \vec{D} \cdot d\vec{s} = \varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = Q.$$



Since $E_r = 0$

$\vec{E} \cdot d\vec{s} = 0$ on lateral area.

on top and bottom : $\vec{E} \parallel d\vec{s}$. also, $E = \text{const}$.

$$\therefore Q = \varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = \varepsilon_0 \left(E A_{Top} + E A_{bottom} \right)$$

$$= 2 \varepsilon_0 E A.$$

$$E = \frac{Q}{2 \varepsilon_0 A} = \frac{\rho_s}{2 \varepsilon_0}$$

(same as previous example)