

PHYS 221 Midterm Examination #2

July 4, 2008

Name Key

Time: 50 minutes

Student No. (27) + (1)*****
1 (1/20 marks). Write down Maxwell's equations in the differential and integral forms:

$$\nabla \cdot \vec{D} = \rho_v$$

$$\oint_S \vec{D} \cdot d\vec{s} = Q$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\oint_C \vec{E} \cdot d\vec{\ell} = - \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

$$\nabla \cdot \vec{B} = 0$$

$$\oint_S \vec{B} \cdot d\vec{s} = 0$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{s}$$

2 (1/20 marks). Write down the boundary conditions for $\vec{E}$, $\vec{D}$, $\vec{H}$ and $\vec{B}$.

$$\hat{n} \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s$$

$$\hat{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0$$

$$\hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0$$

$$\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s$$

(a, b)

3 (6/20 marks). The space between two parallel conducting plates each having an area S is filled with two parallel dielectric layers of thickness d_1 and d_2 . The permittivity is ϵ_1 and ϵ_2 respectively. A d-c voltage V_0 is applied across the plates.

- (a) Determine the electric field intensity between the plates.
 (b) Determine the capacitance.
 (c) If the conductivity of the dielectric layers is σ_1 and σ_2 respectively (i.e., they are not perfect insulators), determine the distribution of free charges.

$$(a) \oint_S \vec{D} \cdot d\vec{s} = Q \quad \vec{D} = -\hat{y} D$$

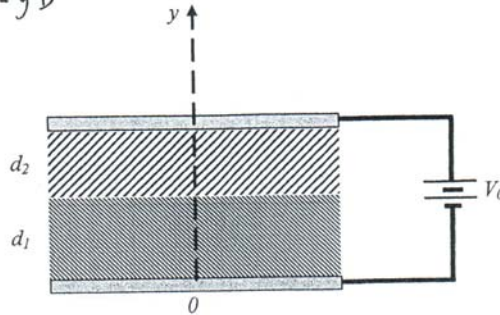
$$DS = P_S \cdot S$$

$$\therefore D = P_S$$

$$\vec{D} = -\hat{y} P_S$$

$$\vec{E}_1 = \frac{\vec{D}}{\epsilon_1} = -\hat{y} \frac{P_S}{\epsilon_1}$$

$$\vec{E}_2 = \frac{\vec{D}}{\epsilon_2} = -\hat{y} \frac{P_S}{\epsilon_2}$$



$$V_0 = -\int \vec{E} \cdot d\vec{l} = \frac{P_S d_1}{\epsilon_1} + \frac{P_S d_2}{\epsilon_2} \Rightarrow P_S = \frac{V_0}{d_1/\epsilon_1 + d_2/\epsilon_2}$$

$$\therefore \vec{E}_1 = -\hat{y} \frac{V_0/\epsilon_1}{d_1/\epsilon_1 + d_2/\epsilon_2} = -\hat{y} \frac{V_0 \epsilon_2}{d_1 \epsilon_2 + d_2 \epsilon_1}$$

$$\vec{E}_2 = -\hat{y} \frac{V_0/\epsilon_2}{d_1/\epsilon_1 + d_2/\epsilon_2} = -\hat{y} \frac{V_0 \epsilon_1}{d_1 \epsilon_2 + d_2 \epsilon_1}$$

$$(b) \quad C = \frac{Q}{V_0} = \frac{P_S S}{\frac{P_S d_1}{\epsilon_1} + \frac{P_S d_2}{\epsilon_2}} = \frac{S}{d_1/\epsilon_1 + d_2/\epsilon_2} = \frac{S \epsilon_1 \epsilon_2}{d_1 \epsilon_2 + d_2 \epsilon_1}$$

$$\left(\begin{aligned} \text{OR: } C &= \frac{C_1 C_2}{C_1 + C_2} \quad C_1 = \frac{S \epsilon_1}{d_1} \quad C_2 = \frac{S \epsilon_2}{d_2} \\ &= \frac{\frac{S \epsilon_1}{d_1} \frac{S \epsilon_2}{d_2}}{\frac{S \epsilon_1}{d_1} + \frac{S \epsilon_2}{d_2}} = \frac{S \epsilon_1 \epsilon_2}{\epsilon_1 d_2 + \epsilon_2 d_1} \end{aligned} \right)$$

- (c) 3(6/20 marks). The space between two parallel conducting plates each having an area S is filled with two parallel dielectric layers of thickness d_1 and d_2 . The permittivity is ϵ_1 and ϵ_2 respectively. A d-c voltage V_0 is applied across the plates.
- Determine the electric field intensity between the plates.
 - Determine the capacitance.
 - If the conductivity of the dielectric layers is σ_1 and σ_2 respectively (i.e., they are not perfect insulators), determine the distribution of free charges.

(c) DC: $J_{1n} = J_{2n}$, $J_{1t} = 0$, $J_{2t} = 0$, $y \uparrow$

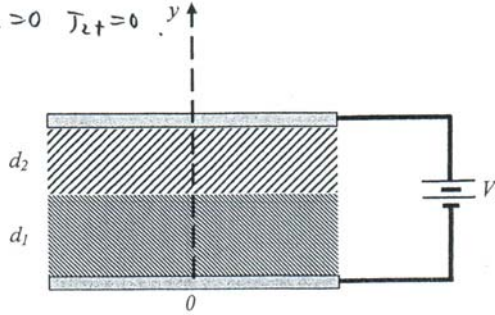
$$\vec{J}_1 = \vec{J}_2 = -\hat{y} J$$

$$\vec{J} = \sigma \vec{E}$$

$$\vec{E}_1 = \frac{\vec{J}_1}{\sigma_1} = -\hat{y} \frac{J}{\sigma_1}$$

$$\vec{E}_2 = \frac{\vec{J}_2}{\sigma_2} = -\hat{y} \frac{J}{\sigma_2}$$

$$\vec{D}_1 = \epsilon_1 \vec{E}_1 = -\hat{y} \frac{J \epsilon_1}{\sigma_1}, \quad \vec{D}_2 = \epsilon_2 \vec{E}_2 = -\hat{y} \frac{J \epsilon_2}{\sigma_2}$$



$$J: E_1 d_1 + E_2 d_2 = V_0 \Rightarrow \frac{J d_1}{\sigma_1} + J \frac{d_2}{\sigma_2} = V_0$$

$$J = \frac{V_0}{d_1/\sigma_1 + d_2/\sigma_2} = \frac{V_0 \sigma_1 \sigma_2}{d_1 \sigma_2 + d_2 \sigma_1}$$

Surface charge density on upper plate:

$$\rho_{su}: \rho_{su} = \vec{D}_2 \cdot (-\hat{y}) = \frac{J \epsilon_2}{\sigma_2} = \frac{V_0 \epsilon_2 \sigma_1}{d_1 \sigma_2 + d_2 \sigma_1}$$

$$\text{lower plate: } \rho_{sl} = \vec{D}_1 \cdot (\hat{y}) = -\frac{J \epsilon_1}{\sigma_1} = \frac{-V_0 \sigma_2 \epsilon_1}{d_1 \sigma_2 + d_2 \sigma_1}$$

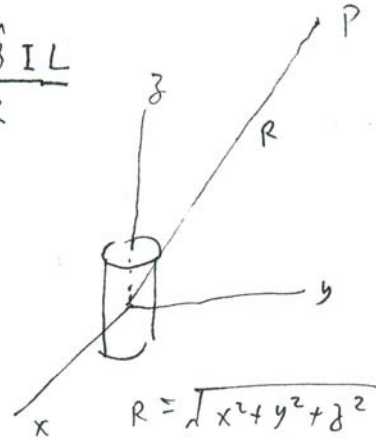
$$\text{Interface between two media: } \rho_{si} = -\hat{y} \cdot (\vec{D}_1 - \vec{D}_2) = J \left(\frac{\epsilon_1}{\sigma_1} - \frac{\epsilon_2}{\sigma_2} \right) \\ = \frac{V_0 (\epsilon_1 \sigma_2 - \epsilon_2 \sigma_1)}{d_1 \sigma_2 + d_2 \sigma_1}$$

4(6/20 marks). A thin current element extending between $z = -L/2$ and $z = L/2$ carries a current I along the z -direction through a circular cross-section of radius a .

(a) Find the magnetic vector potential $\mathbf{A}$ at a point P located very far from the origin (assume that the distance from P to every point in the current element is the same).

(b) Determine the corresponding $\mathbf{H}$.

$$(a) \quad \vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \cdot dV'}{R} = \frac{\mu_0 \hat{z} I L}{4\pi R}$$



$$\begin{aligned} (b) \quad \vec{H} &= \nabla \times \frac{\vec{A}}{\mu_0} \\ &= \nabla \times \frac{I L}{4\pi} \hat{z} \frac{1}{R} \\ &= \frac{I L}{4\pi} \nabla \times \frac{\hat{z}}{R} = \frac{I L}{4\pi} \nabla \times \frac{\hat{z}}{\sqrt{x^2 + y^2 + z^2}} \\ &= \frac{I L}{4\pi} \left\{ \hat{x} \frac{\partial A_z}{\partial y} - \hat{y} \frac{\partial A_z}{\partial x} \right\} \\ &= \frac{I L}{4\pi} \left\{ \hat{x} \frac{-y}{(x^2 + y^2 + z^2)^{3/2}} - \hat{y} \frac{-x}{(x^2 + y^2 + z^2)^{3/2}} \right\} \\ &= \frac{I L}{4\pi R^{3/2}} \left(-\hat{x} y + \hat{y} x \right) \end{aligned}$$

5 (6/20 marks). The circular disk shown in the figure below lies in the x - y plane and rotates with uniform angular velocity ω about the z -axis. The disk is of radius a and is present in a uniform magnetic flux density $\mathbf{B}$ in the z -direction. Obtain an expression for the emf induced at the rim relative to the center of the disk.

$$\begin{aligned}
 V_{\text{emf}} &= \int_0^a \vec{u} \times \vec{B} \cdot d\vec{\ell} \quad (d\vec{\ell} = d\vec{r} = \hat{r} dr) \\
 &= \int_0^a (\vec{u} \times \vec{B}) \cdot \hat{r} dr \\
 &= \int_0^a \omega \cdot r B dr \\
 &= \frac{1}{2} \omega B r^2 \Big|_0^a \\
 &= \frac{1}{2} \omega B a^2.
 \end{aligned}$$

