Phys 221.

Assignment # 2. Solutions.

Part 1.

3.1.

(a) $$T = \frac{z}{x^2+z^2}.$$

$$\nabla T = \hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z}.$$

$$= \hat{x}\frac{-4x}{(x^2+z^2)^2} + \hat{z}\frac{-4z}{(x^2+z^2)^2}$$

$$= \frac{-4(x+z)}{(x^2+z^2)^2}(\hat{x}+\hat{z})$$

(b). $$V = xy^2z^3$$

$$\nabla V = \hat{x}\, y^2z^3 + \hat{y}\, 2xyz^3 + \hat{z}\, 3xy^2z^2$$

(c) $$U = \frac{z\cos\phi}{1+r^2}.$$

$$\nabla V = \hat{r}\frac{\partial U}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial U}{\partial \phi} + \hat{z}\frac{\partial U}{\partial z}.$$

$$= \hat{r}\frac{-2rz\cos\phi}{(1+r^2)^2} + \hat{\phi}\frac{1}{r}\cdot\frac{(-z)\sin\phi}{1+r^2} + \hat{z}\frac{\cos\phi}{1+r^2}.$$

$$= -\hat{r}\frac{2rz\cos\phi}{(1+r^2)^2} - \hat{\phi}\frac{z\cos\phi}{r(1+r^2)} + \hat{z}\frac{\cos\phi}{1+r^2}$$

(d) $W = e^{-R}\sin\theta$

$$\nabla W = \hat{R}\frac{\partial W}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial W}{\partial \theta} + \hat{\phi}\frac{1}{R\sin\theta}\frac{\partial W}{\partial \phi}$$

$$= \hat{R}\left(-e^{-R}\sin\theta\right) + \hat{\theta}\left(\frac{1}{R}e^{-R}\cos\theta\right)$$

$$= e^{-R}\left(-\hat{R}\sin\theta + \hat{\theta}\frac{\cos\theta}{R}\right)$$

(e): $S = x^2 e^{-z} + y^2$

$$\nabla S = \hat{x}\frac{\partial S}{\partial x} + \hat{y}\frac{\partial S}{\partial y} + \hat{z}\frac{\partial S}{\partial z}.$$

$$= \hat{x}\left(2xe^{-z}\right) + \hat{y}\,(2y) + \hat{z}\left(-x^2 e^{-z}\right)$$

$$= \hat{x}\,2xe^{-z} + \hat{y}\,2y - \hat{z}\,x^2 e^{-z}.$$

(f). $N = r^2\cos\phi$

$$\nabla N = \hat{r}\frac{\partial N}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial N}{\partial \phi} + \hat{z}\frac{\partial N}{\partial z}$$

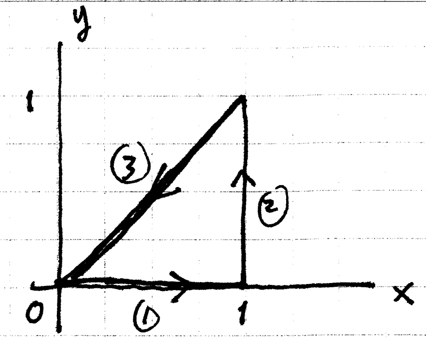
$$= \hat{r}\,2r\cos\phi - \hat{\phi}\,r\sin\phi.$$

(g). $M = R\cos\theta\sin\phi$.

$$\nabla M = \hat{R}\cos\theta\sin\phi + \hat{\theta}\left(-\sin\theta\sin\phi\right) + \hat{\phi}\frac{\cos\theta}{\sin\theta}\cdot\cos\phi$$

$$= \hat{R}\cos\theta\sin\phi - \hat{\theta}\sin\theta\sin\phi + \hat{\phi}\frac{\cos\theta\cdot\cos\phi}{\sin\theta}$$

3.12. $\vec{E} = \hat{x}\,xy - \hat{y}\,(x^2 + 2y^2)$



(a) $\oint_C \vec{E}\cdot d\vec{l}$

$= \int_{①} \vec{E}\cdot d\vec{l} + \int_{②} \vec{E}\cdot d\vec{l} + \int_{③} \vec{E}\cdot d\vec{l}$

③ : $y = x$.

$= 0 + \int_0^1 \{-[(1^2) + 2y^2]\}dy + \int_{③} xy\,dx - (x^2+2y^2)\,dy$.

$= -\int_0^1 (1+2y^2)\,dy + \int_1^0 x^2 dx - 3x^2 dx$

$= -\left[y - \frac{2}{3}y^3\right]_0^1 + \int_1^0 -2x^2\,dx$

$= -\left(1 - \frac{2}{3}\right) - \left[\frac{2}{3}x^3\right]_1^0$

$= -1 + \frac{2}{3} + \frac{2}{3}$

$= -1.$

(b). $\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -x^2-2y^2 & 0 \end{vmatrix}$

$= \hat{x}(0) + \hat{y}(0) + \hat{z}(-2x - x)$

$= -\hat{z}\,3x$

$\int (\nabla\times\vec{E})\cdot d\vec{s} = -\iint 3x\,dx\,dy = -\int_0^1 dx \int_0^x 3x\,dy = -\int_0^1 3x(x-0)\,dx$

$= -\int_0^1 3x^2\,dx = -x^3\Big|_0^1 = -1.$

3.17.

(a). $\vec{A} = \hat{x}\, 2xy - \hat{y}\, y^2$

$$\nabla \cdot \vec{A} = \frac{\partial (2xy)}{\partial x} + \frac{\partial (-y^2)}{\partial y} = 2y - 2y = 0 \quad \text{— solenoidal}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & -y^2 & 0 \end{vmatrix} = \hat{x}(0) - \hat{y}(0) + \hat{z}\, 2x \neq 0.$$

Not conservative.

(b). $\vec{B} = \hat{x}\, x^2 - \hat{y}\, y^2 + \hat{z}\, 2z$

$$\nabla \cdot \vec{B} = 2x - 2y + 2 \neq 0 \quad \text{Not solenoidal}$$

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial x & \partial y & \partial z \\ x^2 & -y^2 & 2z \end{vmatrix} = \hat{x}(0) - \hat{y}(0) + \hat{z}(0) = 0.$$

conservative.

(c). $\vec{C} = \hat{r}\, \frac{\sin\phi}{r^2} + \hat{\phi}\, \frac{\cos\phi}{r^2}$

$$\nabla \cdot \vec{C} = \frac{1}{r} \frac{\partial}{\partial r}\left(\frac{r \sin\phi}{r^2}\right) + \frac{1}{r} \frac{\partial}{\partial \phi}\left(\frac{\cos\phi}{r^2}\right) + 0$$

$$= \frac{1}{r}\left(-\frac{\sin\phi}{r^2}\right) + \frac{1}{r} \cdot \frac{-\sin\phi}{r^2}$$

$$= -\frac{2\sin\phi}{r^3} \neq 0 \quad \text{Not solenoidal}$$

$$\nabla \times \vec{C} = \hat{z}\, \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r \frac{\cos\phi}{r^2}\right) - \frac{\partial}{\partial \phi}\left(\frac{\sin\phi}{r^2}\right)\right]$$

$$= \hat{z}\left(\frac{-\cos\phi}{r^3} - \frac{\cos\phi}{r^3}\right) = -\hat{z}\, \frac{2\cos\phi}{r^3} \neq 0. \quad \text{Not conservative}$$

(d). $\vec{D} = \hat{R}\frac{1}{R}$

$$\nabla\cdot\vec{D} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{1}{R}\right) = \frac{1}{R^2} \neq 0.$$ Not solenoidal.

$$\nabla\times\vec{D} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{R} & \hat{\theta}R & \hat{\phi}R\sin\theta \\ \partial_R & \partial_\theta & \partial_\phi \\ \frac{1}{R} & 0 & 0 \end{vmatrix} = 0$$ conservative.

(e) $\vec{E} = \hat{r}\left(3 - \frac{r}{1+r}\right) + \hat{z}z$

$$\nabla\cdot\vec{E} = \frac{1}{r}\cdot\frac{\partial}{\partial r}\left(3r - \frac{r^2}{1+r}\right) + \frac{\partial z}{\partial z}$$

$$= \frac{1}{r}\left[3 - \frac{2r(1+r) - r^2}{(1+r)^2}\right] + 1$$

$$= \frac{1}{r}\left[3 - \frac{2r+2r^2-r^2}{(1+r)^2}\right] + 1$$

$$= \frac{1}{r}\left[3 - \frac{r(2+r)}{(1+r)^2}\right] + 1 \neq 0$$ NOT solenoidal

$$\nabla\times\vec{E} = \hat{r}(0) + \hat{\phi}(0) + \hat{z}(0) = 0$$ conservative.

(f). $\vec{F} = \frac{\hat{x}y + \hat{y}x}{x^2+y^2}$

$$\nabla\cdot\vec{F} = \frac{-2xy}{(x^2+y^2)^2} + \frac{-2xy}{(x^2+y^2)^2} = -\frac{4xy}{(x^2+y^2)^2} \neq 0$$

NOT solenoidal

$$\nabla\times\vec{F} = \hat{x}(0) + \hat{y}(0) + \hat{z}\left(\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} - \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right)$$

$$= \hat{z}\frac{2(y^2-x^2)}{(x^2+y^2)^2} \neq 0$$ Not conservative.

(g). $\vec{G} = \hat{x}(x^2+z^2) + \hat{y}(y^2+x^2) + \hat{z}(y^2+z^2)$

$\nabla \cdot \vec{G} = 2x + 2y + 2z \neq 0$ Not solenoidal

$\nabla \times \vec{G} = \hat{x}(2y-0) + \hat{y}(2z-0) + \hat{z}(2x-0)$

$= \hat{x}2y + \hat{y}2z + \hat{z}2x \neq 0$ NOT conservative.

(h). $\vec{H} = \hat{R}(Re^{-R})$

$\nabla \cdot \vec{H} = \dfrac{1}{R^2}\dfrac{\partial}{\partial R}\left(R^2 \cdot R\, R^{-R}\right)$

$= \dfrac{1}{R^2}\left[3R^2 e^{-R} - R^3 e^{-R}\right]$

$= (3-R)e^{-R}$

$\neq 0$ (NOT solenoidal)

$\nabla \times \vec{H} = 0$. conservative.

(Note: $\vec{H} = \nabla\left[-(1+R)e^{-R}\right]$

$\therefore \quad \nabla \times \vec{H} = \nabla \times \nabla\left[-(1+R)e^{-R}\right] = 0$)

Part 2. use coulomb's law.

By symmetry, $\mathbf{E}_y = 0$.

i.e. $\vec{E} = \hat{x} E_x + \hat{z} E_z$.

$$E_x = \frac{-1}{4\pi\varepsilon_0} \int \frac{dq \cdot \sin\theta \cos\phi'}{R'^2}$$

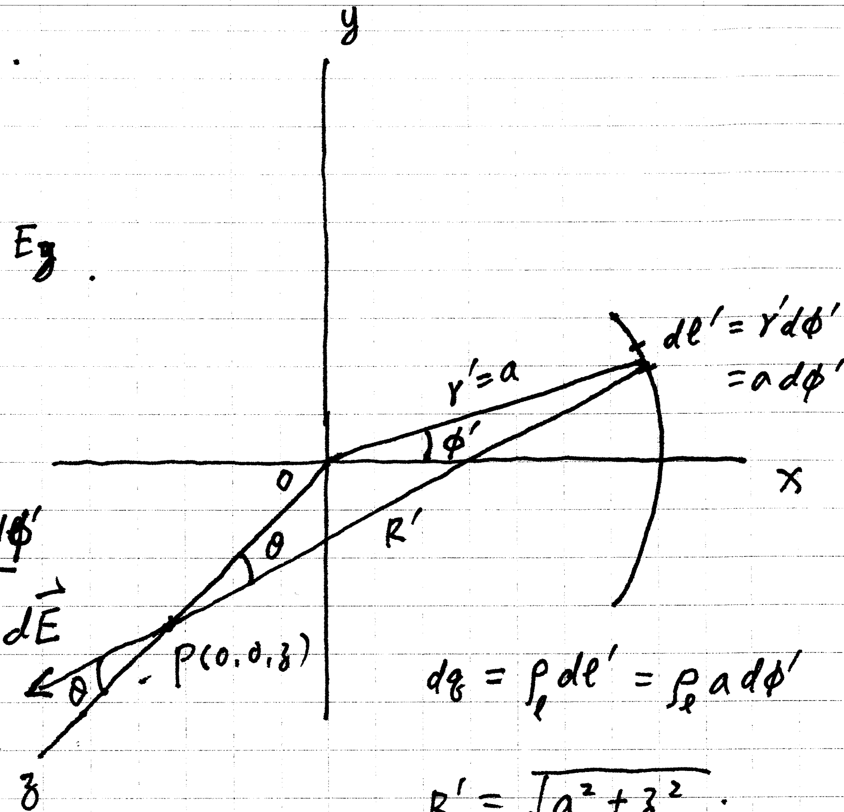
$$= \frac{-1}{4\pi\varepsilon_0} \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \frac{\sin\theta \cdot \cos\phi' \cdot \rho_\ell a\, d\phi'}{R'^2}$$

$$= \frac{-\rho_\ell a}{4\pi\varepsilon_0} \frac{\sin\theta}{R'^2} \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \cos\phi' d\phi'$$

$$= \frac{-2\rho_\ell a \sin\theta \sin\frac{\pi}{8}}{4\pi\varepsilon_0 R'^2}$$

$$= \frac{-\rho_\ell a \cdot a \cdot \sin\frac{\pi}{8}}{2\pi\varepsilon_0 \sqrt{a^2+z^2}\,(a^2+z^2)}$$

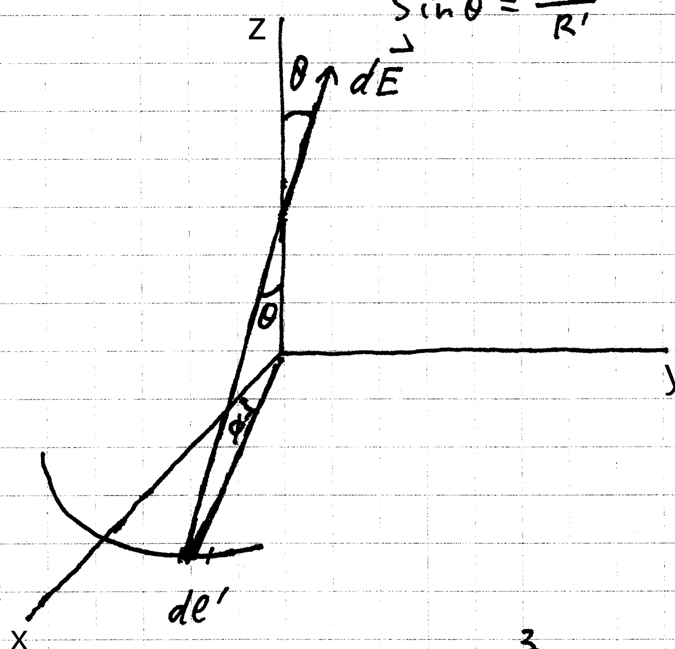
$$= \frac{-\rho_\ell a^2 \sin\frac{\pi}{8}}{2\pi\varepsilon_0 (a^2+z^2)^{3/2}}.$$



$dq = \rho_\ell d\ell' = \rho_\ell a\, d\phi'$

$R' = \sqrt{a^2+z^2}$.

$\sin\theta = \frac{a}{R'} = \frac{a}{\sqrt{a^2+z^2}}$.



$$E_z = \frac{1}{4\pi\varepsilon_0} \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \frac{\rho_\ell a\, d\phi' \cos\theta}{R'^2}$$

$\cos\theta = \frac{z}{R'} = \frac{z}{\sqrt{a^2+z^2}}$

$$= \frac{\rho_\ell a z \frac{\pi}{4}}{4\pi\varepsilon_0 (a^2+z^2)^{3/2}}$$

$$= \frac{\rho_\ell a z}{16\varepsilon_0 (a^2+z^2)^{3/2}}$$