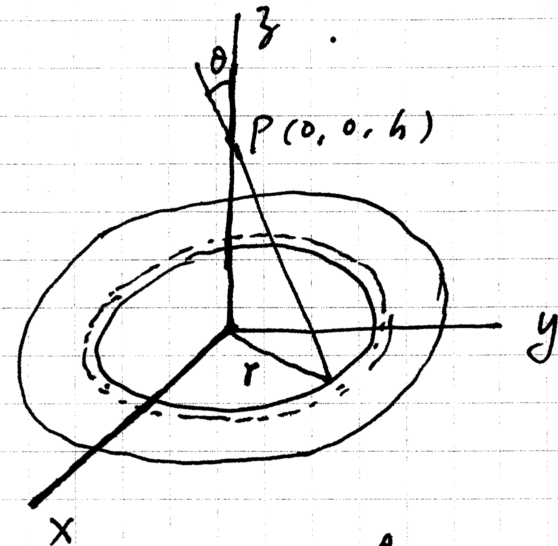
Phys 221

Assignment #3. Solutions.

4.15. Find $\vec{E}$ at $P(0,0,h)$ due to a circular disk charge in the x-y plane with a surface charge density $\rho_s = \rho_{s0} r^2$

$$\vec{E} = \hat{z} E_z \quad \text{(by symmetry)}.$$



$$E_z = \frac{1}{4\pi\varepsilon_0}\int_0^a \frac{\rho_s\, 2\pi r\, dr \cdot \cos\theta}{r^2+h^2}$$

$$\cos\theta = \frac{h}{\sqrt{h^2+r^2}}$$

$$= \frac{1}{2\varepsilon_0}\int_0^a \frac{\rho_{s0} r^3\, h\, dr}{(r^2+h^2)^{3/2}}$$

$$= \frac{\rho_{s0} h}{2\varepsilon_0}\int_0^a \frac{r^3\, dr}{(r^2+h^2)^{3/2}}.$$

let $u = r^2+h^2$. $du = 2r\,dr$.

when $r=0$, $u=h^2$.

when $r=a$, $u=a^2+h^2$

$$= \frac{\rho_{s0} h}{4\varepsilon_0}\int_{h^2}^{a^2+h^2} \frac{(u-h^2)\, du}{u^{3/2}}$$

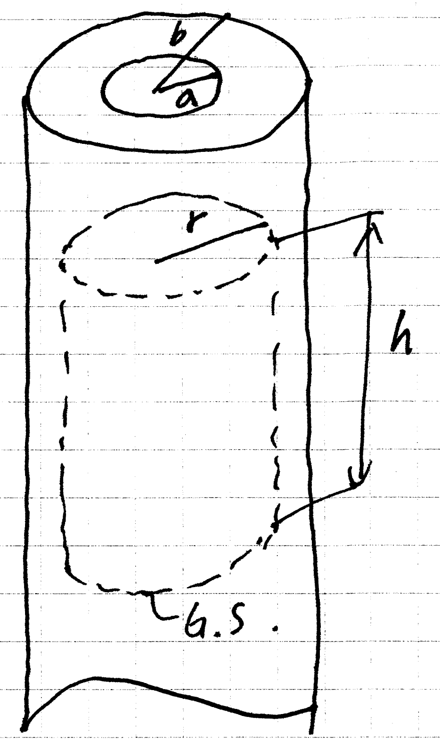
$$= \frac{\rho_{s0} h}{4\varepsilon_0}\left[2u^{\frac{1}{2}} + 2h^2 u^{-\frac{1}{2}}\right]_{h^2}^{a^2+h^2}$$

$$= \frac{\rho_{s0} h}{2\varepsilon_0}\left[\sqrt{a^2+h^2} - h + \frac{h^2}{\sqrt{a^2+h^2}} - h\right]$$

$$\therefore \vec{E} = \hat{z}\,\frac{\rho_{s0} h}{2\varepsilon_0}\left(\sqrt{a^2+h^2} + \frac{h^2}{\sqrt{a^2+h^2}} - 2h\right).$$

4.25 Gauss's law.

$$\oint \vec{D} \cdot d\vec{s} = Q$$



By symmetry. $\vec{D} = \hat{r} D_r$.

On side of G.S: $D_r = \text{const}$.

On Top and bottom of G.S. : $\vec{D} \cdot d\vec{s} = 0$.

$$\therefore \int_{\text{lateral}} \vec{D} \cdot d\vec{s} = Q$$

$$D_r \cdot 2\pi r \cdot h = Q.$$

$$D_r = \frac{Q}{2\pi r h}$$

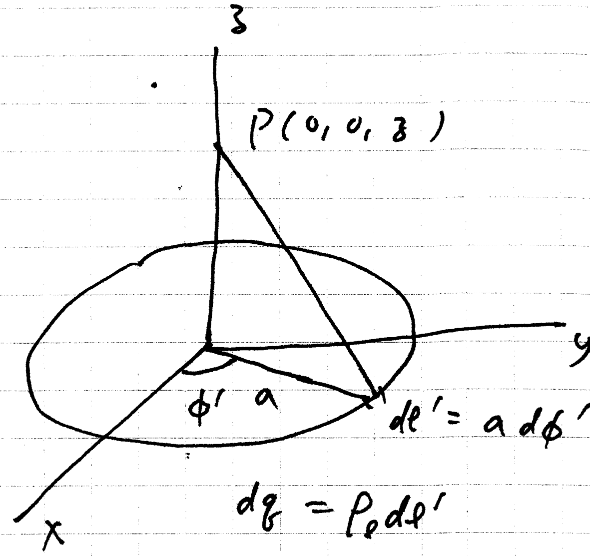
$$Q = \pi (r^2 - a^2)\, h \cdot \rho_{v0}.$$

$r < a$: $Q = 0$. $\vec{D} = 0$.

$a < r < b$: $Q = \pi (r^2 - a^2)\, h \cdot \rho_{v0}$, $\vec{D} = \hat{r} \dfrac{\rho_{v0}(r^2 - a^2)}{2r} = \hat{r} \dfrac{\rho_{v0}(r^2 - 1)}{2r}$

$r > b$: $Q = \pi (b^2 - a^2)\, h\, \rho_{v0}$, $\vec{D} = \hat{r} \dfrac{\rho_{v0}(b^2 - a^2)}{2r} = \hat{r} \dfrac{4\rho_{v0}}{r}$

4.29. Assume: $V=0$ at $r \to \infty$



$$dq = \rho_l \, dl' = \rho_l \cdot a \, d\phi'$$

a) $$V = \frac{1}{4\pi\varepsilon_0} \int_0^{2\pi} \frac{\rho_l \, a \cdot d\phi'}{\sqrt{a^2+z^2}}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{\rho_l \, a}{\sqrt{a^2+z^2}} \int_0^{2\pi} d\phi'$$

$$= \frac{\rho_l \cdot a}{2\varepsilon_0 \sqrt{a^2+z^2}}$$

b). by symmetry: $\vec{E} = \hat{z} E_z = -\hat{z} \frac{\partial V}{\partial z}$.

$$\vec{E}(0,0,z) = -\hat{z} \frac{\partial V(0,0,z)}{\partial z}$$

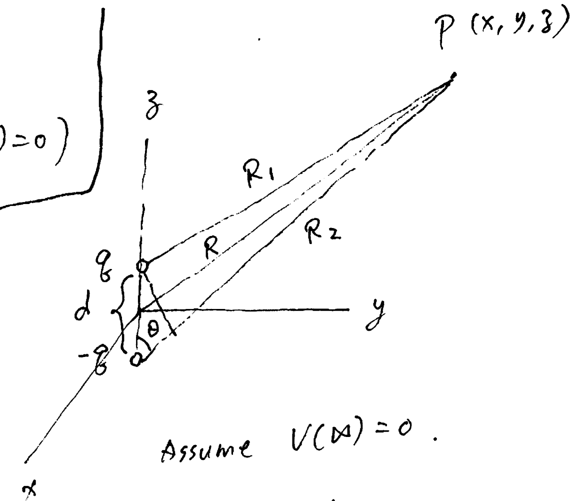
$$= -\frac{\rho_l a}{2\varepsilon_0} \cdot \left(-\frac{1}{2}\right) \frac{2z}{(a^2+z^2)^{3/2}} \cdot \hat{z}$$

$$= \hat{z} \frac{\rho_l \, a \cdot z}{2\varepsilon_0 (a^2+z^2)^{3/2}}$$

2(5/15 marks). An electric dipole consists of two charges q and –q located at (0, 0, d/2) and (0, 0, -d/2) respectively.
(a) Determine the electric potential *V* at any point P(x, y, z) in free space, given that P is far away from the dipole.
(b) Determine the electric field **E** (both magnitude and direction) at point P.

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R_1} - \frac{q}{R_2}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\,\frac{R_2 - R_1}{R_1 R_2} \qquad (V(\infty)=0)$$



Assume $V(\infty) = 0$.

when $R \gg d$,

$R_1 R_2 \approx R^2$

$R_2 - R_1 \approx d\cos\theta$

$\cos\theta = \dfrac{z}{R} = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

$R = \sqrt{x^2+y^2+z^2}$

a). when $R \gg d$.

$$V \approx \frac{q}{4\pi\varepsilon_0}\,\frac{d\cos\theta}{R^2}$$

OR:

$$V = \frac{q\, d \cdot z}{4\pi\varepsilon_0 (x^2+y^2+z^2)^{3/2}}.$$

b). $\vec{E} = -\nabla V$.

$$= -\left[\hat{R}\frac{\partial V}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\phi}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\right]$$

$$= -\hat{R}\frac{\partial V}{\partial R} - \hat{\theta}\frac{1}{R}\frac{\partial V}{\partial \theta}.$$

$$E_R = -\frac{\partial V}{\partial R} = \frac{q}{2\pi\varepsilon_0}\,\frac{d\cos\theta}{R^3}$$

$$E_\theta = -\frac{1}{R}\frac{\partial V}{\partial \theta} = \frac{1}{R}\,\frac{q \cdot d \cdot \sin\theta}{4\pi\varepsilon_0 R^2} = \frac{qd\cdot\sin\theta}{4\pi\varepsilon_0 R^3}$$

$$E_\phi = 0$$

$$\therefore\ \vec{E} = \hat{R}\,\frac{q\,d\cos\theta}{2\pi\varepsilon_0 R^3} + \hat{\theta}\,\frac{q\,d\sin\theta}{4\pi\varepsilon_0 R^3} \qquad (R \gg d)$$