Phys 221

Lecture 5.

Ref: 3-2, 3-3, 3-4;
4-1, 4-2, 4-3, 4-4, 4-5;
6-1.

- A few very important results of vector calculus.

  - Gauss' theorem (Divergence thm) $\int_V \nabla\cdot\vec{E}\, dV = \oint_S \vec{E}\cdot d\vec{S}$

  - Stoke's theorem $\int_S (\nabla\times\vec{B})\cdot d\vec{S} = \oint_C \vec{B}\cdot d\vec{l}$

  - Curl of gradient: $\nabla\times\nabla V = 0$

  - Divergence of curl: $\nabla\cdot(\nabla\times\vec{A}) = 0$

- Laplacian operator ∇^2

$$\nabla^2 V \equiv \nabla\cdot(\nabla V)$$

In cartesian coordinates: $\nabla^2 V = \frac{\partial^2}{\partial x^2} V + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

∇^2 can act on a vector: $\nabla^2\vec{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\vec{E}$

$$= \hat{x}\,\nabla^2 E_x + \hat{y}\,(\nabla^2 E_y) + \hat{z}\,\nabla^2 E_z.$$

please see textbook for other coordinates.

- Maxwell's equations. (time-varying).

  (foundation of electromagnetics theory).

In vacuum:

$$\left\{\begin{aligned} &\oint_S \vec{E}\cdot d\vec{s} = \frac{Q}{\varepsilon_0} \\ &\oint_C \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt} \\ &\oint_S \vec{B}\cdot d\vec{s} = 0 \\ &\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I + \mu_0\varepsilon_0 \int \frac{\partial \vec{E}}{\partial t}\cdot d\vec{s} \end{aligned}\right.$$

$$\left\{\begin{aligned} &\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} && \text{Gauss' law} \\ &\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} && \text{Faraday's law.} \\ &\nabla\cdot\vec{B} = 0 && \text{No monopole.} \\ &\nabla\times\vec{B} = \mu_0\vec{J} + \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} && \end{aligned}\right.$$

Ampere's law.

ε_0 — electrical permitivity of free space $= 8.85\times10^{-12}$ F/m

μ_0 — magnetic permeability of free space $= 4\pi\times10^{-7}$ H/m.

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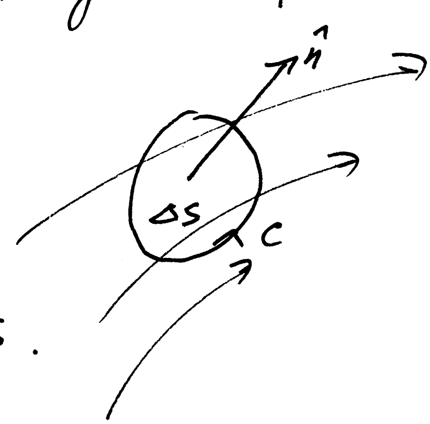
e.g. Faraday's law: For a time-dependent magnetic field $\vec{B}(x,y,z,t)$, consider an arbitrary small loop C (fixed)

$$\mathcal{E} = -\frac{d\Phi}{dt}.$$

$\mathcal{E}$ ↑ electromotive force in C.

$\frac{d\Phi}{dt}$ ↑ time rate of change in $\vec{B}$-flux through ΔS.



Φ — $\vec{B}$ - flux

$$\Phi = \int_{\Delta S} \vec{B} \cdot \vec{ds}$$

$$\mathcal{E} = \oint_C \vec{E} \cdot \vec{dl}$$

(unlike electrostatics, with t-dependence, $\oint_C \vec{E} \cdot \vec{dl} \neq 0$ i.e. $\vec{E}$ is no longer conservative)

$$\therefore \quad \oint_C \vec{E} \cdot \vec{dl} = -\frac{d}{dt} \int_{\Delta S} \vec{B} \cdot \vec{ds}$$

$$\int_{\Delta S} (\nabla \times \vec{E}) \cdot \vec{ds} = -\int_{\Delta S} \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}$$ — true for any ΔS.

$$\therefore \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$ — differential form of Faraday's law.

meaning : t-varying $\vec{B}$ gives rise to a non-zero circulation of $\vec{E}$.

$(\nabla \times \vec{E} \neq 0)$

- Maxwell proposed : A t-dependent $\vec{E}$ should contribute to $\nabla \times \vec{B}$ as well. i.e, $\frac{\partial \vec{E}}{\partial t} \longrightarrow \nabla \times \vec{B}$.

$\therefore$ Ampere's law needs to be modified :

$$\nabla \times \vec{B} = \mu_0 \vec{J} \Longrightarrow \nabla \times \vec{B} = \mu_0 \Big(\vec{J} + \underbrace{\varepsilon_0 \frac{\partial \vec{E}}{\partial t}}_{\text{"displacement current"}} \Big)$$

- In medium. (Homogeneous and isotropic)

$\varepsilon_0 \longrightarrow \varepsilon = \varepsilon_r \varepsilon_0$

$\mu_0 \longrightarrow \mu = \mu_r \cdot \mu_0$.

$$\left.\begin{aligned} \vec{D} &= \varepsilon \vec{E} \\ \vec{B} &= \mu \vec{H} \end{aligned}\right\}$$

$\vec{E}$ — electric field intensity

$\vec{D}$ — electric flux density

$\vec{B}$ — magnetic flux density

$\vec{H}$ — magnetic field intensity.

- Maxwell's equations:

Gauss' law: $\nabla \cdot \vec{D} = \rho$ $\qquad \oint_S \vec{D} \cdot d\vec{s} = Q$.

Faraday's law: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\qquad \oint_C \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$.

No monopole: $\nabla \cdot \vec{B} = 0$ $\qquad \oint_S \vec{B} \cdot d\vec{s} = 0$

Ampere's law: $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$. $\qquad \oint_C \vec{H} \cdot d\vec{l} = I + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s}$.

- Note: $\vec{E}$ and $\vec{B}$ describe the strength of the electric and magnetic fields.

$\vec{D}$ and $\vec{H}$ sometimes make calculations easier (in media).

why? They are related to free sources.

(No need to worry about induced charges or currents)

(bound charges / bound currents)

- Electrostatics. (No t-dependence)

$$\nabla \cdot \vec{D} = \rho \qquad (4.2a) \qquad \text{Gauss' law.}$$

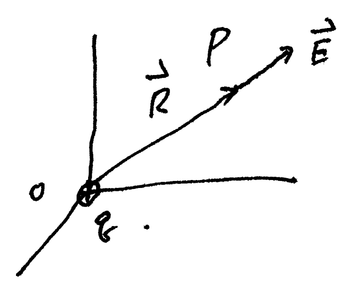
$$\nabla \times \vec{E} = 0 \qquad (4.2b) \qquad \vec{E}\text{-field is conservative.}$$

i.e. $\vec{E} = -\nabla V$.

Also, it can be shown that Gauss' law is equivalent to coulomb's law:

$$\vec{E} = \frac{\hat{R}\, q}{4\pi \varepsilon_0 R^2}$$

$\vec{R}$ — from source to field.



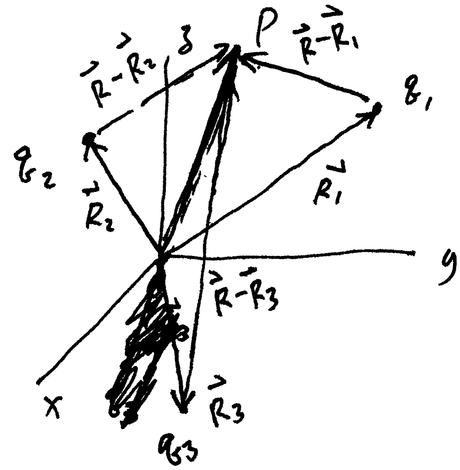
- 3 Methods to determine the $\vec{E}$-field of given charges.

  1. coulomb's law.
  2. Gauss' law
  3. potential.

- When there are many charges, $q_1, q_2, q_3, \ldots$ located at $\vec{R}_1, \vec{R}_2, \vec{R}_3, \ldots$ If we want to find the $\vec{E}$-field at location $\vec{R}$

$$\vec{E}(\vec{R}) = \sum_i \vec{E}_i$$

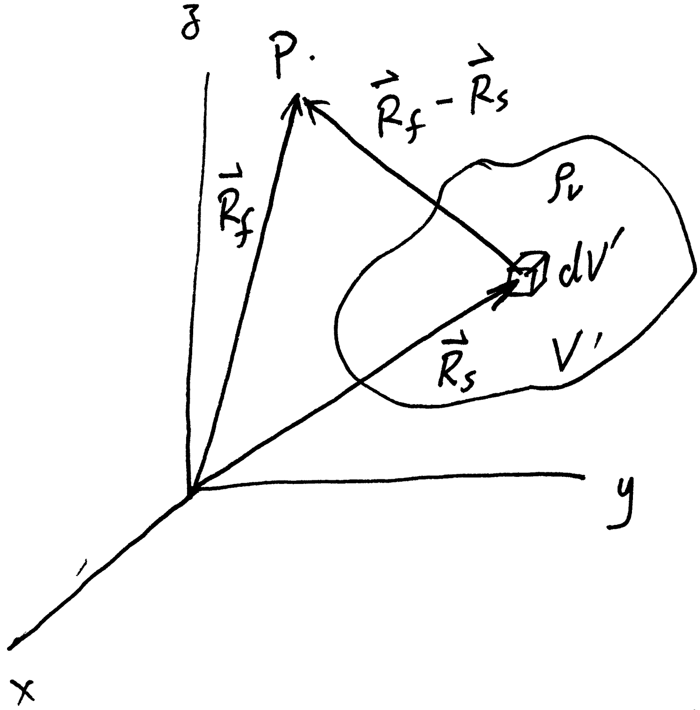
$$= \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i(\vec{R} - \vec{R}_i)}{(|\vec{R} - \vec{R}_i|)^3}$$



$\vec{R} - \vec{R}_i$ — vector from $\vec{R}_i$ to $\vec{R}$

source point — field point.

- For continuous charge distribution in volume V' with charge density ρ_v



— charge element dq

$$dq = \rho_v \, dV'$$

located at $\vec{R}_s$

— Field point : $\vec{R}_f$

(We want to find $\vec{E}$ at $\vec{R}_f$)

— distance vector from source dq to field point $\vec{R}_f$:

$$(\vec{R}_f - \vec{R}_s) \triangleq \vec{R}'$$

~~Some books use $(\vec{R} - \vec{R}')$ We used it briefly last time~~

Total $\vec{E}$-field at $\vec{R}_f$:

$$\vec{E} = \frac{1}{4\pi\varepsilon} \int \frac{\hat{\vec{R}}'}{R'^2} \, dq = \frac{1}{4\pi\varepsilon} \int_{V'} \frac{\hat{\vec{R}}' \rho_v \, dV'}{R'^2} \qquad (4.21a)$$

for a charged wire: $dq = \rho_\ell \, d\ell'$ (4.21c)

for a charged sheet: $dq = \rho_s \, ds'$. (4.21b)