Ref: 7-2, 7-3, 7-4.

Assign #9: 7.13, 7.23, 7.25, 7.27, 7.29.

Phys 221

Lecture 24, 23.

- Time-harmonic fields

$$\vec{E}(x,y,z,t) = \mathrm{Re}\big[\underbrace{\tilde{\vec{E}}(x,y,z)}_{\text{varies (Need to work on)}}\ \underbrace{e^{j\omega t}}_{\text{always the same.}}\big]$$

$$\vec{H}(x,y,z,t) = \mathrm{Re}\left[\tilde{\vec{H}}(x,y,z)\, e^{j\omega t}\right]$$

- Charge-free, lossless Maxwell's equations:

$(\rho_v = 0,\ \vec{J} = 0)$
$\sigma = 0$

$$\nabla \cdot \tilde{\vec{E}} = 0$$

$$\nabla \times \tilde{\vec{E}} = -j\omega\mu \tilde{\vec{H}}$$

$$\nabla \cdot \tilde{\vec{H}} = 0$$

$$\nabla \times \tilde{\vec{H}} = j\omega\varepsilon \tilde{\vec{E}}$$

- Charge-free, lossy medium ($\sigma \neq 0$, $\rho_v = 0$. but $\underline{\vec{J} = \sigma \vec{E}} \neq 0$)

$$\nabla \cdot \tilde{\vec{E}} = 0$$

$$\nabla \times \tilde{\vec{E}} = -j\omega\mu \tilde{\vec{H}}$$

$$\nabla \cdot \tilde{\vec{H}} = 0$$

$$\nabla \times \tilde{\vec{H}} = j\omega\varepsilon_c \tilde{\vec{E}}, \qquad \varepsilon_c = \varepsilon' - j\varepsilon''$$

$$\varepsilon' = \varepsilon, \quad \varepsilon'' = \frac{\sigma}{\omega}.$$

Midterm #2: 4-5 — 7-4.

- Wave equations for charge-free medium.

$$\nabla \times \nabla \times \tilde{\vec{E}} = \nabla \times (-j\omega\mu \tilde{\vec{H}})$$

$$\nabla(\nabla \cdot \tilde{\vec{E}}) - \nabla^2 \tilde{\vec{E}} = -j\omega\mu \; \nabla \times \tilde{\vec{H}}$$

$$\underset{0}{\downarrow} \qquad - \nabla^2 \tilde{\vec{E}} = \omega^2 \mu \varepsilon_c \tilde{\vec{E}}$$

$$\therefore \quad \nabla^2 \tilde{\vec{E}} + \omega^2 \mu \varepsilon_c \tilde{\vec{E}} = 0 . \qquad (7.68)$$

(p.210)

define : propagation constant γ :

$$\gamma^2 \triangleq -\omega^2 \mu \varepsilon_c .$$

Then, wave equation for $\tilde{\vec{E}}$:

$$\boxed{\nabla^2 \tilde{\vec{E}} - \gamma^2 \tilde{\vec{E}} = 0} \qquad (7.70)$$

Similarly, wave equation for $\tilde{\vec{H}}$:

$$\nabla^2 \tilde{\vec{H}} - \gamma^2 \tilde{\vec{H}} = 0 . \qquad (7.71)$$

Note :

How to solve (7.70) ?

① depends on boundary. (shape, properties.)

② by guessing : e.g : $\left\{ \begin{array}{l} (e^{-j\gamma x})' = -j\gamma e^{-j\gamma x} \\ (e^{-j\gamma x})'' = -\gamma^2 e^{-j\gamma x} \end{array} \right.$

- Lossless media: $\sigma = 0, \quad \varepsilon'' = 0,$

$$\varepsilon_c = \varepsilon' = \varepsilon .$$

$$\gamma^2 = -\omega^2 \mu \varepsilon .$$

define: wavenumber k: $\quad k = \omega\sqrt{\mu\varepsilon}$. (7.73)

$$(k^2 = -\gamma^2)$$

Then, wave equations:

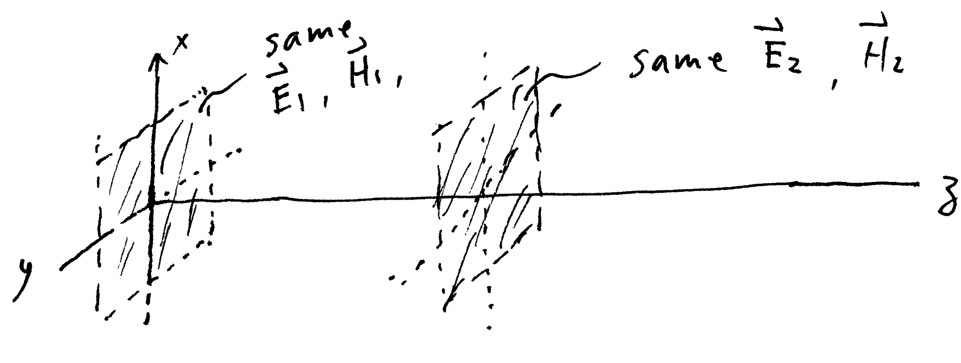
$$\nabla^2 \tilde{\vec{E}} + k^2 \tilde{\vec{E}} = 0 \qquad (7.74)$$

$$\nabla^2 \tilde{\vec{H}} + k^2 \tilde{\vec{H}} = 0$$

Called Helmholtz's equations.
(well studied).

- Plane waves in lossless media:
$\vec{E}$ — same at every point on a plane. ($\vec{H}$ too).

e.g. $\vec{E}$ and $\vec{H}$ — (uniform on x-y plane). ind. of (x, y).



Then. $\tilde{\vec{E}}(x, y, z) = \tilde{\vec{E}}(z)$

$\tilde{\vec{H}}(x, y, z) = \tilde{\vec{H}}(z)$.

Wave equations:

$$\frac{d^2\tilde{\vec{E}}}{dz^2} + k^2\tilde{\vec{E}} = 0$$

$$\left(\frac{d^2\tilde{\vec{H}}}{dz^2} + k^2\tilde{\vec{H}} = 0\right)$$

Solutions: $\underbrace{\tilde{\vec{E}}_0^- e^{jkz}}_{\underbrace{\tilde{\vec{E}}^-(z)}}$ OR. $\underbrace{\tilde{\vec{E}}_0^+ e^{-jkz}}_{\underbrace{\tilde{\vec{E}}^+(z)}}$. ($\tilde{\vec{E}}_0^{+,-}$ constant. (No $-z$))

propagates in $-z$ direction.

propagates in $+z$ directi-

$\tilde{\vec{E}}_0^+ = \vec{E}_0^+ e^{j\phi_0}$

$\tilde{\vec{E}}^+(z) = \tilde{\vec{E}}_0^+ e^{-jkz}$.

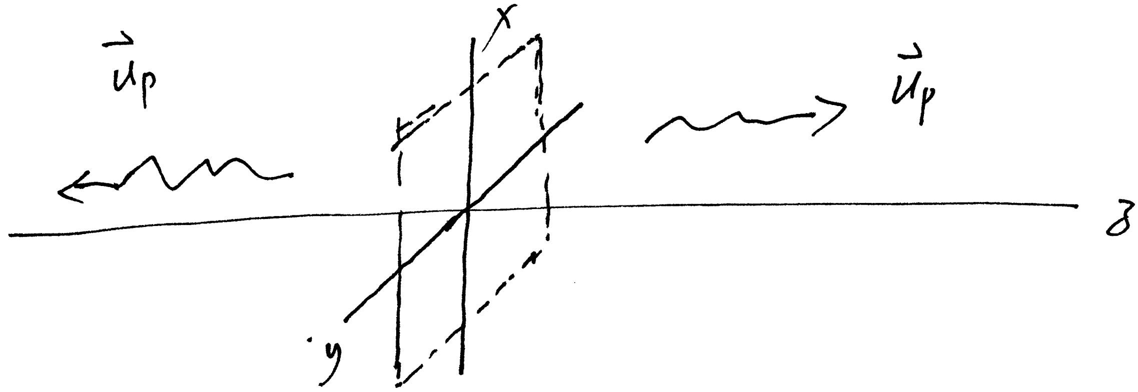
$\vec{E}^+(z,t) = \mathrm{Re}\left[\tilde{\vec{E}}_0^+ e^{-jkz} \cdot e^{j\omega t}\right] = \vec{E}_0^+ \cos(\omega t - kz + \phi_0^+)$

$\vec{E}^-(z,t) = \vec{E}_0^- \cos(\omega t + kz + \phi_0^-)$

~~Simplified notation~~: $\cancel{\vec{E}(z,t) = \vec{E}_0 \cos(\omega t - kz + \phi_0)}$

General solution: $\vec{E}(z,t) = \vec{E}_0^+ \cos(\omega t - kz + \phi_0^+) + \vec{E}_0^- \cos(\omega t + kz + \phi_0^-)$

i.e, it's a wave propagating in $+z$ or $-z$ directions.



• Directions of $\vec{E}$? directions of $\vec{H}$? (in x-y plane).

(It turns out $(\vec{E} \times \vec{H}) \parallel \vec{u}_p$).

How do we know?

$$\nabla \times \tilde{\vec{H}} = j\omega\varepsilon \tilde{\vec{E}}$$

$$(\nabla \times \tilde{\vec{H}})_z = j\omega\varepsilon E_z .$$

$$\underbrace{\frac{\partial \hat{H}_y}{\partial x}}_{0} - \underbrace{\frac{\partial \tilde{H}_x}{\partial y}}_{0} = j\omega\varepsilon \tilde{E}_z .$$

(No x-y dependence).

$$\therefore \tilde{E}_z = 0 . \Rightarrow \vec{E} \perp \hat{z}, \text{ i.e. } \vec{E} \perp \vec{u}_p .$$

Similarly. $\tilde{H}_z = 0$. $\Rightarrow \vec{H} \perp \hat{z}$, i.e. $\vec{H} \perp \vec{u}_p$.

(Since. $\nabla \times \tilde{\vec{E}} = -j\omega\mu\tilde{\vec{H}}$)

e.g. When $\vec{E}$ is along x-axis, we say that the E & M wave is polarized along x-direction.

if it travels along $+z$ direction:

$$\tilde{\vec{E}}(z) = \hat{x} E_0 e^{-jkz}$$

Then, what about $\vec{H}$?

$$\nabla \times \tilde{\vec{E}} = -j\omega\mu \tilde{\vec{H}}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_0 e^{-jkz} & 0 & 0 \end{vmatrix} = -j\omega\mu \left(\hat{x} \tilde{H}_x + \hat{y} \tilde{H}_y + \hat{z} \tilde{H}_z \right)$$

$$\hat{x}\, 0 - \hat{y} \left[0 - \frac{\partial}{\partial z}(E_0 e^{-jkz}) \right] + \hat{z} \left[0 - \frac{\partial}{\partial y}(E_0 e^{-jkz}) \right] = -j\omega\mu \tilde{\vec{H}}$$

$$\hat{y} \left(-jk E_0 e^{-jkz} \right) = -j\omega\mu \left(\hat{x} \tilde{H}_x + \hat{y} \tilde{H}_y + \hat{z} \tilde{H}_z \right)$$

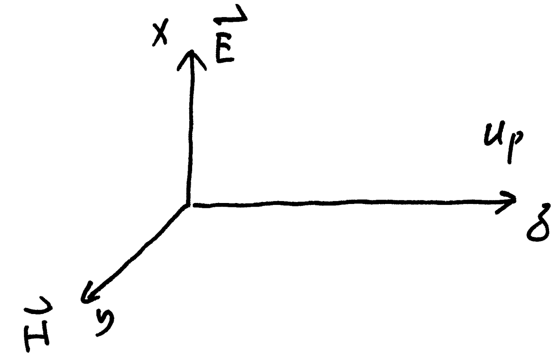
$$\therefore \quad \tilde{H}_x = 0, \quad \tilde{H}_z = 0.$$

$$-jk E_0 e^{-jkz} = -j\omega\mu \tilde{H}_y$$

$$\therefore \quad \tilde{H}_y = \frac{k}{\omega\mu} E_0 e^{-jkz}$$

i.e. $$\tilde{\vec{H}} = \hat{y} \frac{k}{\omega\mu} E_0 e^{-jkz}.$$

$\therefore \ (\vec{E} \times \vec{H}) \parallel \vec{u}_p$.



Transverse wave.
(for $\vec{E}$ and $\vec{H}$)

- $\dfrac{E}{H} = \dfrac{\omega\mu}{k} \equiv \eta$ — intrinsic impedance

since $k = \omega\sqrt{\mu\varepsilon}$,

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}.$$

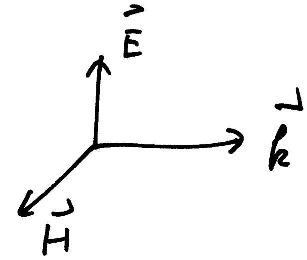
In vacuum: $\eta_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = 377\ \Omega$.

- Wave vector $\vec{k}$: $\quad k = |\vec{k}| = \omega\sqrt{\mu\varepsilon} = \dfrac{2\pi}{\lambda}$

$\hat{k} \parallel \vec{u}_p$ (direction of propagation)

Then: $\tilde{\vec{H}} = \dfrac{1}{\eta}\,\hat{k} \times \tilde{\vec{E}}$

$\vec{E} = -\eta\,\hat{k} \times \tilde{\vec{H}}$



• Polarization.

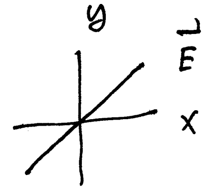
In general, for a plane wave propagating in z direction, $\vec{E}$ can have x- and y- components.

$$\tilde{\vec{E}}(z) = (\hat{x}\, a_x + \hat{y}\, a_y\, e^{j\delta})\, e^{-jkz}$$

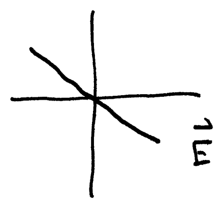
There may be a phase difference δ between the two components.

[Note: If there is only one component — linearly polarized]

① If $\delta = 0$, in phase

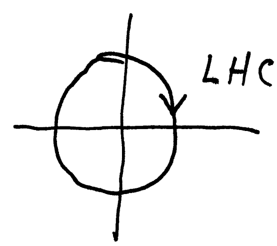


If $\delta = \pi$, out of phase



both are linearly polarized.

② If $\delta = \pm\frac{\pi}{2}$, and $a_x = a_y$.



Then, circularly polarized.

+ : Left handed

− : RHC.

(see page 219. figure 7-15 for the Rules).

③ All others: elliptically polarized.

see figure 7-19 on page 224 for a summary.