

Phys 221

Lecture 25.

Ref. 7-5.

Assignment #10: 7.31, 43, 47, 51, 55. Due: Fri. July 17.

- Phasor equations in charge-free, lossy medium:

$$\nabla \cdot \tilde{\vec{E}} = 0$$

$$\nabla \times \tilde{\vec{E}} = -j\omega\mu\tilde{\vec{H}}$$

$$\nabla \cdot \tilde{\vec{H}} = 0$$

$$\nabla \times \tilde{\vec{H}} = j\omega\epsilon_c\tilde{\vec{E}}$$

$$\left(\rho_v = 0, \sigma \neq 0, \tilde{\vec{J}} = \sigma \tilde{\vec{E}} \neq 0 \right)$$

$$\epsilon_c = \epsilon' - j\epsilon'' \quad \left\{ \begin{array}{l} \epsilon' = \epsilon = \epsilon_0 \epsilon_r \\ \epsilon'' = \frac{\sigma}{\omega} \end{array} \right.$$

$$\left(\text{For lossless, } \sigma = 0, \tilde{\vec{J}} = 0, \text{ we have } \epsilon_c = \epsilon. \right. \\ \left. \text{i.e. } \epsilon'' = 0 \right)$$

- Wave equations: for lossy medium.

$$\nabla^2 \tilde{\vec{E}} - \gamma^2 \tilde{\vec{E}} = 0 \quad (7.70)$$

$$\nabla^2 \tilde{\vec{H}} - \gamma^2 \tilde{\vec{H}} = 0 \quad (7.71)$$

$$\gamma^2 = -\omega^2 \mu \epsilon_c$$

• Plane wave in lossy medium.

- Relationship between $\vec{E}$ and $\vec{H}$.

$$\vec{H} = \frac{1}{\eta_c} \hat{k} \times \vec{E}$$

$\left(\begin{array}{l} \vec{k} - \text{wave vector :} \\ \text{direction - propagation direction} \\ \text{magnitude - } k = \frac{2\pi}{\lambda} \end{array} \right)$

λ - wavelength in the medium

η_c - intrinsic impedance

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon'}} (1 - j \frac{\epsilon''}{\epsilon'})^{-\frac{1}{2}} \quad (7.125)$$

- γ, α, β .

$$\gamma = \alpha + j\beta$$

$\left\{ \begin{array}{l} \gamma - \text{propagation constant} \\ \beta - \text{phase constant} \\ \alpha - \text{attenuation constant} \end{array} \right.$

$$\gamma^2 = -\omega^2 \mu \epsilon_c \quad \epsilon_c = \epsilon' - j\epsilon''$$

Q: How do we determine α and β from $\epsilon', \epsilon'', \mu$?

$$\begin{aligned}\gamma^2 &= -\omega^2 \mu (\epsilon' - j\epsilon'') \\ &= -\omega^2 \mu \epsilon' + j\omega^2 \mu \epsilon''\end{aligned}$$

$$\begin{aligned}\text{Also: } \gamma^2 &= (\alpha + j\beta)^2 \\ &= \alpha^2 - \beta^2 + j2\alpha\beta.\end{aligned}$$

$$\therefore \begin{cases} \alpha^2 - \beta^2 = -\omega^2 \mu \epsilon' \\ 2\alpha\beta = \omega^2 \mu \epsilon'' \end{cases}$$

Solve for α, β :

$$\left. \begin{aligned}\alpha &= \omega \left\{ \frac{\mu \epsilon'}{2} \left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'} \right)^2} - 1 \right] \right\}^{1/2} \\ \beta &= \omega \left\{ \frac{\mu \epsilon'}{2} \left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'} \right)^2} + 1 \right] \right\}^{1/2}\end{aligned} \right\} \begin{array}{l} (7.121) \\ p. 225 \end{array}$$

- plane wave solution (polarized along $\hat{x}$)

$$\vec{E} = \hat{x} E_{x0} e^{-\alpha z} = \hat{x} E_{x0} e^{-\alpha z} \cdot e^{-j\beta z}$$

$$\vec{H} = \hat{y} \frac{E_{x0}}{\eta_c} e^{-\alpha z} e^{-j\beta z}$$

Note: Since η_c is complex in lossy medium,

$\vec{H}$ and $\vec{E}$ are not in phase.

i.e. There is a phase shift between $\vec{H}$ and $\vec{E}$
in a lossy medium.

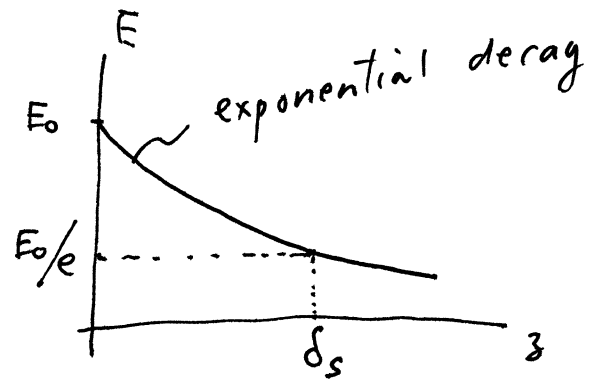
($\vec{H}$ and $\vec{E}$ are in phase in lossless media)

- Attenuation factor.

$$e^{-\alpha z}$$

Skin depth (penetration depth).

$$\delta_s = \frac{1}{\alpha}$$



$$\begin{aligned} \text{at } z = \delta_s = \frac{1}{\alpha}, \quad E &= E_0 e^{-\frac{\alpha}{\alpha}} = E_0 e^{-1} = \frac{E_0}{e} \\ &= 37\% E_0. \end{aligned}$$

• Table 7-2 (p. 228): α , β , η , λ . $\lambda = v_p/f$

1. General (any medium)

2. lossless $\sigma = 0$.

3. low-loss $\epsilon''/\epsilon' \ll 1$. (10^{-2} or smaller)

4. good conductor: $\frac{\epsilon''}{\epsilon'} \gg 1$. (10^2 or greater)

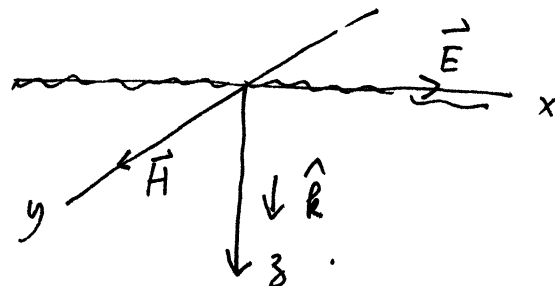
Note: $\epsilon'' = \sigma/\omega$ — depends on frequency!

• e.g. plane wave in seawater.

$$\epsilon_r = 80,$$

$$\mu_r = 1$$

$$\sigma = 4 \text{ S/m}$$



$$\vec{H}(0, t) = \hat{y} 100 \cos(2\pi \times 10^3 t + 15^\circ) \text{ (mA/m)}$$

(a). Find $\vec{E}(z, t)$ and $\vec{H}(z, t)$

(b) Find $z = d$ so that $\frac{\tilde{E}_o(d)}{\tilde{E}_o(0)} = 1\%$ (amplitude is 1% of its value at $z = 0$)

(a). Since, $\hat{k} = \hat{z}$, $\hat{H} = \hat{y}$.

$$\hat{E} = \hat{x}.$$

$$a) \quad \hat{\underline{E}}(z) = \hat{x} E_{x0} e^{-\alpha z - j\beta z}$$

$$\hat{\underline{H}}(z) = \hat{y} \frac{E_{x0}}{\eta_c} e^{-\alpha z} e^{-j\beta z}$$

Need to determine α, β, η_c : Since $\frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega \epsilon} = \frac{4}{2\pi \times 10^3 \times \epsilon_0 \epsilon_r}$

$$= 9 \times 10^9 \gg 1$$

$\therefore$ Sea water is good conductor
(at $f = 1 \text{ kHz}$)

(but, for $f = 10^{15}$ (visible light),
 $\frac{\epsilon''}{\epsilon'} = 9 \times 10^{-10} \ll 1$.
 low-loss!
 Transparent!)

$$\therefore \alpha = \sqrt{\pi f \mu \sigma} = 0.126$$

$$\beta = \alpha = 0.126 \quad (\alpha = \beta \text{ for good conductors})$$

$$\eta_c = (1+j) \frac{\alpha}{\sigma} = \sqrt{2} e^{j\frac{\pi}{4}} \frac{0.126}{4} = 0.044 e^{j\frac{\pi}{4}}$$

Note: ① E_{x0} is a complex constant. $E_{x0} = |E_{x0}| e^{j\phi_0}$.

② $e^{j\frac{\pi}{4}}$ in η_c means the phase shift between $\hat{\underline{H}}$

and $\hat{\underline{E}}$ is 45° .

$$\begin{aligned}\therefore \vec{E}(z, t) &= \text{Re} \left[\hat{x} E_{x_0} e^{-\alpha z} e^{-j\beta z} \cdot e^{j\omega t} \right] \\ &= \hat{x} |E_{x_0}| e^{-0.126z} \cos(2\pi \times 10^3 t - 0.126z + \phi_0)\end{aligned}$$

$$\begin{aligned}\vec{H}(z, t) &= \text{Re} \left[\hat{y} \frac{|E_{x_0}|}{0.044 e^{j\pi/6}} e^{-\alpha z} e^{-j\beta z} e^{j\omega t} \right] \\ &= \hat{y} \frac{|E_{x_0}|}{0.044} e^{-0.126z} \cos(2\pi \times 10^3 t - 0.126z + \phi_0 - \frac{\pi}{6})\end{aligned}$$

$$\begin{aligned}\text{at } z=0 : \quad \vec{H}(0, t) &= \hat{y} \frac{|E_{x_0}|}{0.044} \cdot \cos(2\pi \times 10^3 t + \phi_0 - 45^\circ) \\ &= \hat{y} \cdot 100 \cos(2\pi \times 10^3 t + 15^\circ)\end{aligned}$$

$$\therefore \frac{E_{x_0}}{0.044} = 100, \quad \phi_0 - 45^\circ = 15^\circ$$

$$E_{x_0} = 4.4 \text{ (V/m)} \quad \phi_0 = 60^\circ$$

$$\begin{aligned}\therefore \vec{E}(z, t) &= \hat{x} 4.4 e^{-0.126z} \cos(2\pi \times 10^3 t - 0.126z + 60^\circ) \\ \vec{H}(z, t) &= \hat{y} 100 e^{-0.126z} \cos(2\pi \times 10^3 t - 0.126z + 15^\circ)\end{aligned}$$