PHYS 221 Midterm examination #1

May 29, 2009 Name Key

Time: 50 minutes Student No.

Please show complete solutions and explain your reasoning, stating any principles that you have used.

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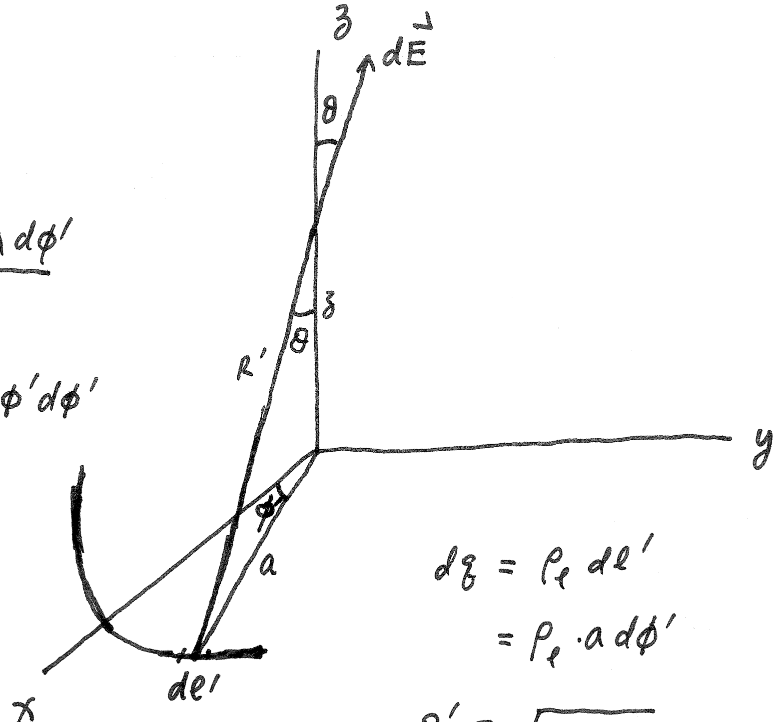
1(5/15 marks). Electric charge is uniformly distributed along an arc located in the x-y plane and defined by $r = a$ and $-\frac{\pi}{4} \le \phi \le \frac{\pi}{4}$. The linear charge density is ρ_l (charge per unit length). Determine the electric filed $\mathbf{E}$ at (0, 0, z).

By symmetry, $E_y = 0$

i.e., $\vec{E} = \hat{x} E_x + \hat{z} E_z$



$$E_x = \frac{1}{4\pi\varepsilon_0} \int \frac{-\sin\theta \cos\phi' \cdot \rho_l a\, d\phi'}{a^2+z^2}$$

$$= \frac{-\rho_l a^2}{4\pi\varepsilon_0} \frac{1}{(a^2+z^2)^{3/2}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos\phi'\, d\phi'$$

$$= \frac{-\rho_l a^2 \sin\frac{\pi}{4}}{2\pi\varepsilon_0 (a^2+z^2)^{3/2}}$$

$$= \frac{-\sqrt{2}\, \rho_l a^2}{4\pi\varepsilon_0 (a^2+z^2)^{3/2}}$$

$$dq = \rho_l\, dl' = \rho_l \cdot a\, d\phi'$$

$$R' = \sqrt{a^2+z^2}$$

$$\sin\theta = \frac{a}{R'} = \frac{a}{\sqrt{a^2+z^2}}$$

$$\cos\theta = \frac{z}{R'} = \frac{z}{\sqrt{a^2+z^2}}$$

$$E_z = \frac{1}{4\pi\varepsilon_0} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos\theta \cdot \rho_l \cdot a\, d\phi'}{a^2+z^2}$$

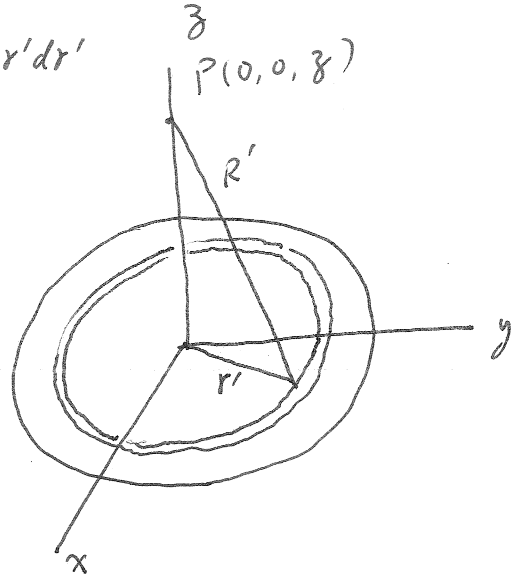
$$= \frac{\rho_l a z}{8\varepsilon_0 (a^2+z^2)^{3/2}}$$

$$\therefore \vec{E} = -\hat{x} \frac{\sqrt{2}\, \rho_l a^2}{4\pi\varepsilon_0 (a^2+z^2)^{3/2}} + \hat{z} \frac{\rho_l a z}{8\varepsilon_0 (a^2+z^2)^{3/2}}$$

2(5/15 marks). A very thin circular disk of radius a has uniform surface charge density ρ_s (charge per unit area). The disk lies in the x-y plane and is centred at the origin.
(a) Determine the electric potential V at point P(0, 0, z) in free space. (for $z>0$).
(b) Determine the electric field **E** at point P(0, 0, z). (for $z>0$).

a). For finite charge distributions, $V(\infty)=0$.

charge element dq : a ring : $dq = \rho_s \cdot 2\pi r' dr'$



$$V(0,0,z) = \frac{1}{4\pi\varepsilon_0}\int_Q \frac{dq}{R'}$$

$$= \frac{1}{4\pi\varepsilon_0}\int_0^a \frac{\rho_s 2\pi r' dr'}{\sqrt{r'^2+z^2}}$$

$$= \frac{\rho_s}{2\varepsilon_0}\left(\sqrt{a^2+z^2} - z\right)$$

(b). By symmetry, at point $P(0,0,z)$.

$$\vec{E} = \hat{z} E_z .$$

$$E_z = -\frac{\partial V}{\partial z}.$$

$$E_z(0,0,z) = -\frac{\rho_s}{2\varepsilon_0}\left(\frac{2z}{2\sqrt{a^2+z^2}} - 1\right)$$

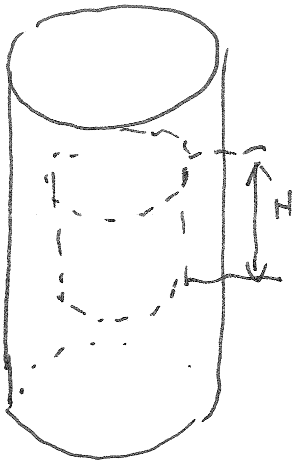
$$\therefore \quad \vec{E}(0,0,z) = \hat{z}\frac{\rho_s}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

Note : $$\int_0^a \frac{r'dr'}{\sqrt{r'^2+z^2}} = \int_{z^2}^{a^2+z^2}\frac{du}{2u^{1/2}}$$

Let $u = r'^2 + z^2$.
$du = 2r'dr$

$$= u^{1/2}\Big|_{z^2}^{a^2+z^2} = \sqrt{a^2+z^2} - z$$

3(5/15). A very long solid cylinder with a radius R is uniformly charged. The charge density (charge per unit volume) is ρ. Figure A below depicts its cross section.
(a) Determine the electric field E at point P. The distance from the center of the rod to point P is 2R/3.
(b) If the rod has a cylindrical hole along the rod, with a radius R/2, as shown in figure B below, determine the electric field at point P(2R/3, 0). Note that the centre of the rod, the centre of the hole and point P are all on the x-axis.
(c) If the cylindrical hole inside the rod has a radius w and the centre of the hole is located at (a, b), find the electric field at an arbitrary point (x, y) inside the hole.



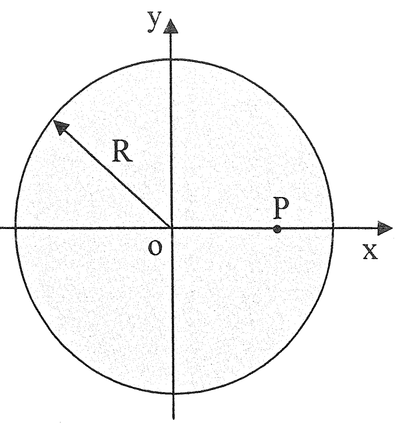


Figure A

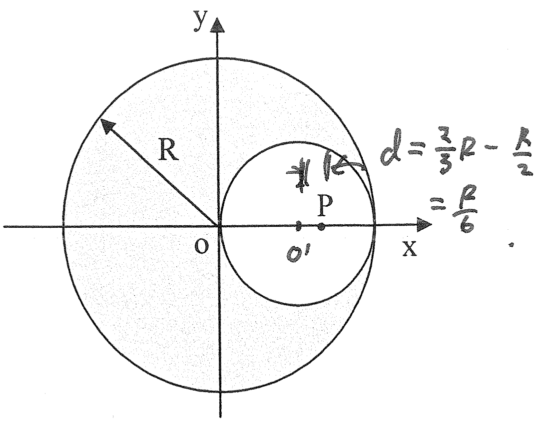


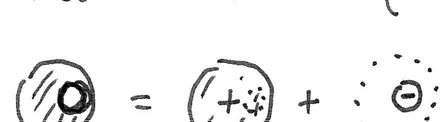
Figure B

(a) Use Gauss' law.

$$\frac{Q}{\varepsilon_0} = \oint_S \vec{E}\cdot d\vec{s} = \int_{\text{lateral}} \vec{E}\cdot d\vec{s} = E_r \cdot 2\pi r H = \frac{\rho \pi r^2 H}{\varepsilon_0}.$$

$$\therefore \vec{E} = \hat{r} E_r = \hat{r}\,\frac{\rho r}{2\varepsilon_0} = \frac{\rho \vec{r}}{2\varepsilon_0} \quad \left(E_z = 0,\ E_\phi = 0 \text{ from symmetry} \right)$$

$$= \hat{r}\,\frac{\rho R}{3\varepsilon_0} \quad \left(r = \frac{2R}{3} \right)$$

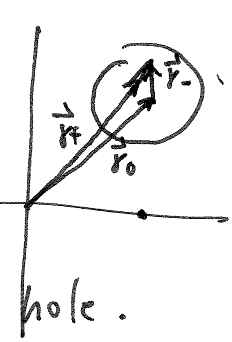
(b). $\vec{E} = \vec{E}_+ + \vec{E}_-$. ()

at P, $\hat{r} = \hat{x}$.

$$\vec{E} = \hat{x}\left(\frac{\rho R}{3\varepsilon_0} - \frac{\rho}{2\varepsilon_0}\cdot\frac{R}{6} \right) = \hat{x}\,\frac{\rho R}{\varepsilon_0}\left(\frac{1}{3} - \frac{1}{12} \right) = \hat{x}\,\frac{\rho R}{4\varepsilon_0}$$

(c). $\vec{E} = \vec{E}_+ + \vec{E}_- = \left(\frac{\rho \vec{r}_+}{2\varepsilon_0} - \frac{\rho \vec{r}_-}{2\varepsilon_0} \right)$

$$= \frac{\rho}{2\varepsilon_0}\left(\vec{r}_+ - \vec{r}_- \right) = \frac{\rho}{2\varepsilon_0}\vec{r}_0 = \frac{\rho}{2\varepsilon_0}\left(\hat{x} a + \hat{y} b \right)$$



$\vec{r}_0$ — center of hole.