PHYS 221 Midterm examination #1

June 18, 2010 — Name: Key

Time: 50 minutes — Student No.

Please show complete solutions and explain your reasoning, stating any principles that you have used.

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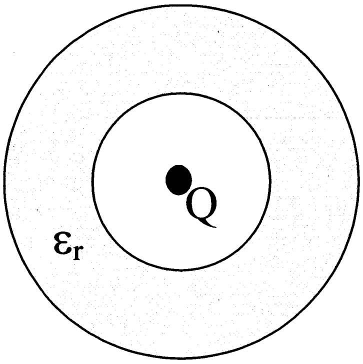
1(5/20 marks). A positive charge Q is at the centre of a spherical dielectric shell of an inner radius R_i and an outer radius R_o. The dielectric constant of the shell is ε_r. Determine the electric field **E** as a function of the radial distance R.



Gauss' Law : $\oint_S \vec{D}\cdot d\vec{s} = Q$

Use spherical Gaussian surface with a radius R.

$\vec{D} = \hat{R} D_R$ — by symmetry.

Then : $D_R \cdot 4\pi R^2 = Q$.

$D_R = \dfrac{Q}{4\pi R^2}$, $\vec{D} = \hat{R}\dfrac{Q}{4\pi R^2}$

for $R \leq R_i$: $\vec{E} = \hat{R}\dfrac{Q}{4\pi\varepsilon_0 R^2}$

for $R_i \leq R \leq R_o$: $\vec{E} = \hat{R}\dfrac{Q}{4\pi\varepsilon_0\varepsilon_r R^2}$

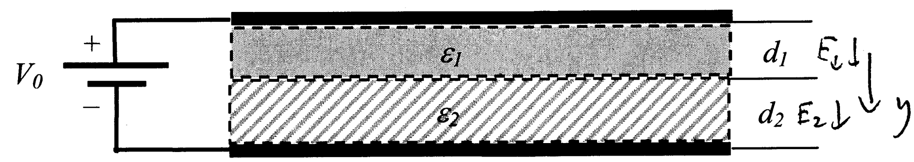
for $R \geq R_o$: $\vec{E} = \hat{R}\dfrac{Q}{4\pi\varepsilon_0 R^2}$

2(5/20 marks). A DC voltage V_0 is applied across a parallel-plate capacitor. The space between the conducting plates is filled with two different lossless dielectrics of thicknesses d_1 and d_2, permittivity ε_1 and ε_2, respectively. The area of each conducting plate is S.

(a) Determine the electric field intensity **E** in the dielectrics;

(b) Determine the capacitance.

~~(c) Find the surface free charge density on the plates.~~



(a) $V_0 = E_1 d_1 + E_2 d_2.$

$D_1 = D_2. \qquad \varepsilon_1 E_1 = \varepsilon_2 E_2. \qquad E_2 = \frac{\varepsilon_1}{\varepsilon_2} E_1.$

$$V_0 = E_1 \left(d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2 \right)$$

$$\vec{E}_1 = \hat{y} \frac{V_0 \varepsilon_2}{d_1 \varepsilon_2 + \varepsilon_1 d_2}. \qquad \vec{E}_2 = \hat{y} \frac{V_0 \varepsilon_1}{d_1 \varepsilon_2 + \varepsilon_1 d_2}.$$

(b). $W = d_1 S \cdot \frac{1}{2} \cdot \varepsilon_1 E_1^2 + d_2 S \frac{1}{2} \cdot \varepsilon_2 E_2^2.$

$$= \frac{1}{2} S \left(\frac{d_1 \varepsilon_1 \varepsilon_2^2 V_0^2}{(d_1 \varepsilon_2 + \varepsilon_1 d_2)^2} + \frac{d_2 \varepsilon_2 \varepsilon_1^2 V_0^2}{(d_1 \varepsilon_2 + \varepsilon_1 d_2)^2} \right)$$

$$= \frac{1}{2} S V_0^2 \left(\frac{d_1 \varepsilon_1 \varepsilon_2^2 + d_2 \varepsilon_2 \varepsilon_1^2}{(d_1 \varepsilon_2 + \varepsilon_1 d_2)^2} \right)$$

$$= \frac{1}{2} S V_0^2 \frac{\varepsilon_1 \varepsilon_2}{d_1 \varepsilon_2 + \varepsilon_1 d_2} = \frac{1}{2} C V_0^2$$

$$C = \frac{S \varepsilon_1 \varepsilon_2}{d_1 \varepsilon_2 + d_2 \varepsilon_1}$$

3(5/20 marks). A very long solid cylinder with a radius R is carrying a steady current I, which is uniformly distributed over the cross section. Figure A below depicts its cross section.
(a) Determine the magnetic field **B** at point P(2R/3, 0).
(b) If the rod has a cylindrical hole along the rod, with a radius R/2, as shown in figure B below, determine the magnetic field **B** at point P(2R/3, 0).

current density: $J = \frac{I}{\pi R^2}$

By symmetry.

$\vec{B} = \hat{\phi} B$

$\frac{2R}{3} - \frac{R}{2} = \frac{R}{6}$

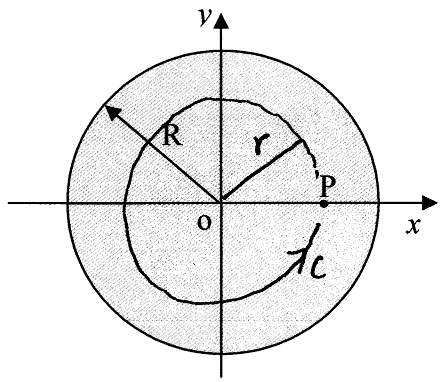


Figure A

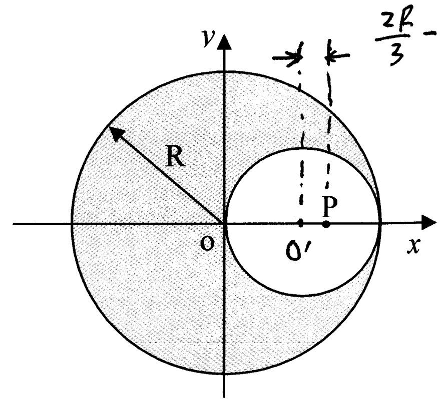


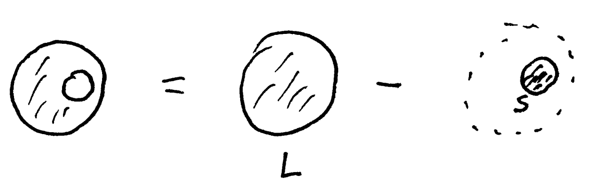
Figure B

(a). Ampere's law. $\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \frac{I}{\pi R^2} \cdot \pi r^2 = \frac{\mu_0 I r^2}{R^2}$

$$B \cdot 2\pi r = \frac{\mu_0 I r^2}{R^2}.$$

$$B = \frac{\mu_0 I r}{2\pi R^2}, \quad \vec{B} = \hat{\phi} \frac{\mu_0 I r}{2\pi R^2}.$$

at $r = \frac{2R}{3}$: $\vec{B} = \hat{\phi} \frac{\mu_0 I}{2\pi R^2} \cdot \frac{2R}{3} = \frac{\mu_0 I}{3\pi R} \hat{\phi}$

(b). 

, i.e, $\vec{B} = \vec{B}_L - \vec{B}_S$

$$\vec{B} = \hat{\phi} \left(\frac{\mu_0 I}{2\pi R^2}\right)\left(\frac{2R}{3} - \frac{R}{6}\right) = \hat{\phi} \frac{\mu_0 I}{2\pi R^2} \cdot \frac{R}{2} = \hat{\phi} \frac{\mu_0 I}{4\pi R}.$$

(Note: at point P, $\hat{\phi} \parallel \hat{y}$. $\therefore \vec{B} = \hat{y} \frac{\mu_0 I}{4\pi R}$)

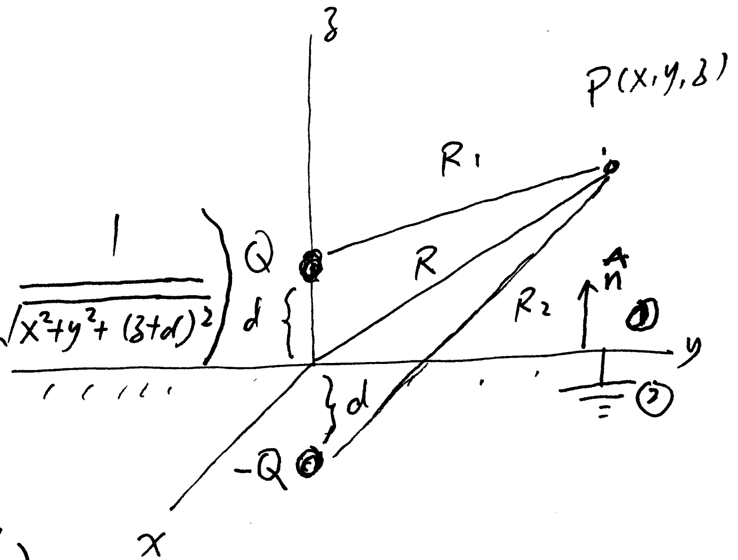
4 (5/20 marks) A charge Q is at a distance d above a grounded conducting plane which is defined as the x-y plane. The space above the conducting plane is in vacuum.

(a) Determine the electric potential V at an arbitrary point P(x, y, z) above the conducting plane, i.e., z>0;

(b) Determine the electric field **E** at point P(x, y, z);.

(c) Determine the surface charge density on the surface of the conducting plane.

Use the image method



(a). $$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R_1} - \frac{Q}{R_2}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right)$$

(b). $$\vec{E} = -\nabla V$$

$$= -\left(\hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{\hat{x}x + \hat{y}y + \hat{z}(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{\hat{x}x + \hat{y}y + \hat{z}(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right)$$

(c). $$\rho_s = D_{1n} - D_{2n} = -\varepsilon_0 E_z - 0$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{z-d}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{z+d}{[x^2+y^2+(z+d)^2]^{3/2}}\right]_{z=0}$$

$$= \frac{-Qd}{2\pi\varepsilon_0}\cdot\frac{1}{(x^2+y^2+z^2)^{3/2}}.$$