

2-1.
Phys 221 Lecture 2.

Vector Calculus.

Ref: 3-1, 2; 4-1, 2, 3, 4, 5.

Assignment #1.

Due next Friday 9:20 am.

p. 44. # 2.5

p. 63. # 3.1, 3.12, 3.17.

- Gradient of a scalar field $V(x, y, z)$.

$$\text{grad } V \triangleq \nabla V \triangleq \hat{x} \frac{\partial V}{\partial x} + \hat{y} \frac{\partial V}{\partial y} + \hat{z} \frac{\partial V}{\partial z}$$

— it's a vector

— represents the magnitude and direction of the maximum space rate of change of $V(x, y, z)$.

∇ — "del" — is a vector operator

$$\nabla \triangleq \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}.$$

e.g. Electrostatics

V — electric potential

$\vec{E}$ — electric field.

$$\vec{E} = -\nabla V, \text{ i.e., } E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}.$$

⌞ "—" sign — $\vec{E}$ direction is from higher V to lower V .

• Differential of $V(x, y, z)$:

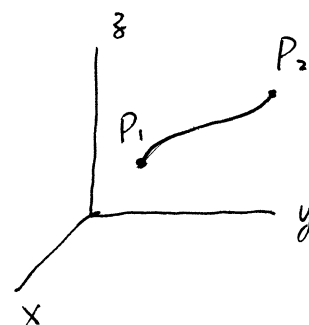
$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

$$= \nabla V \cdot d\vec{l}$$

Difference of V between locations P_1 and P_2

(Imagine moving a charge q
from P_1 to P_2)

$$V_2 - V_1 = \int_{P_1}^{P_2} dV = \int_{P_1}^{P_2} \nabla V \cdot d\vec{l}$$



$$\therefore V_2 - V_1 = - \int_{P_1}^{P_2} \vec{E} \cdot d\vec{l} \quad (\text{since } \vec{E} = -\nabla V)$$

(4-39)

Electric field $\vec{E}$ — $\vec{E} = \vec{F}/q$, electric force per unit charge.

Electric potential V — potential energy per unit charge.

(4-39) can be rewritten as.

$$\underbrace{\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}} = -(V_2 - V_1) = -\Delta V$$

Work done by $\vec{E}$ -field = loss of potential energy.

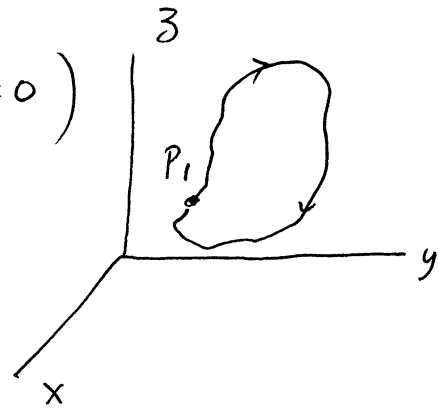
(on a unit charge)

Along a closed path (starts from P_i and ends at P_i)

$$\oint \vec{E} \cdot d\vec{\ell} = 0 \quad (\because V_{P_i} - V_{P_i} = 0)$$

i.e. circulation of $\vec{E} = 0$.

meaning. $\vec{E}$ -field is conservative.



- work does not depend on path.
only depends on initial and final positions.

$$\left(\text{e.g.: } \int_1^2 \vec{E} \cdot d\vec{\ell} = -(V_2 - V_1) \right)$$

- OR:
- work along any closed path = 0.

We just have shown: Any ~~the~~ vector field (such as $\vec{E}$)

that can be defined as a gradient of a scalar function is a conservative field.

$$\vec{E} = -\nabla V.$$

• Gradient operator in cylindrical and spherical coordinates.

cylindrical: $\nabla = \hat{r} \frac{\partial}{\partial r} + \hat{\phi} \frac{1}{r} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z}$.

spherical: $\nabla = \hat{R} \frac{\partial}{\partial R} + \hat{\theta} \frac{1}{R} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{R \sin \theta} \frac{\partial}{\partial \phi}$.

e.g: Given the electric potential of a point charge q located at the origin:

$$V = \frac{q}{4\pi\epsilon_0} \frac{1}{R} \quad \left(\begin{array}{l} \text{We choose spherical coord.} \\ \text{Then, } V \text{ is independent of } \theta, \phi \\ \text{so is } \vec{E}. \end{array} \right)$$

Electric field:

$$\vec{E} = -\nabla V = -\hat{R} \frac{\partial}{\partial R} \left(\frac{q}{4\pi\epsilon_0} \frac{1}{R} \right)$$

$$= \hat{R} \frac{q}{4\pi\epsilon_0 R^2} \quad \left(\begin{array}{l} \text{That's Coulomb's} \\ \text{Law} \end{array} \right)$$

• Divergence of a vector field.

Given a vector field $\vec{E}(x, y, z)$ its divergence is

$$\begin{aligned}\nabla \cdot \vec{E} &\triangleq \operatorname{div} \vec{E} = \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot (\hat{x} E_x + \hat{y} E_y + \hat{z} E_z) \\ &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \quad (3-28)\end{aligned}$$

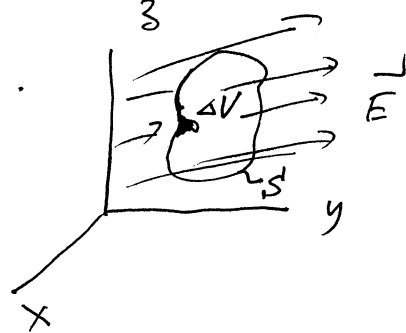
OR. (equivalent definition)

$$\nabla \cdot \vec{E} = \lim_{\Delta V \rightarrow 0} \frac{\oint_S \vec{E} \cdot d\vec{S}}{\Delta V}$$

ΔV — small volume enclosed by S .

$\vec{E} \cdot d\vec{S}$ — flux of $\vec{E}$ through ds .

$\oint_S \vec{E} \cdot d\vec{S}$ — ^{Total} outward flux ~~from~~.



Meaning:

①. $\nabla \cdot \vec{E}$ is a scalar.

② It's the total outward flux of $\vec{E}$ per unit volume.

③ If $\nabla \cdot \vec{E} > 0$, it's a source.

If $\nabla \cdot \vec{E} < 0$, it's a sink.

④ The magnitude of $\nabla \cdot \vec{E}$ represents the strength of the source (or sink).

e.g. For an electric field $\vec{E}$.

Gauss' Law: $\oint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\epsilon_0}$

then.

$$\nabla \cdot \vec{E} = \lim_{\Delta V \rightarrow 0} \frac{\oint_S \vec{E} \cdot d\vec{s}}{\Delta V} = \lim_{\Delta V \rightarrow 0} \frac{\frac{Q}{\epsilon_0}}{\Delta V} = \frac{\rho}{\epsilon_0}$$

ρ — charge density (charge per unit volume)

$+q$ — source of $\vec{E}$

$-q$ — sink of $\vec{E}$

e.g.: For a magnetic field $\vec{B}$.

$$\nabla \cdot \vec{B} = 0 \quad \text{(4.1c Maxwell's equation)}$$

Why? Because there is no "magnetic charges".
(monopoles)

• When $\nabla \cdot \vec{B} = 0$ everywhere,

We say that $\vec{B}$ is divergenceless (sourceless)

or solenoidal

↳ the field lines go back to themselves (never stop, never start at "charges").

- Gauss' Theorem (The divergence theorem).

when the volume V is not small.

$$\int_V \nabla \cdot \vec{E} \, dV = \oint_S \vec{E} \cdot d\vec{S}$$

V — volume enclosed by surface S .

- calculation of divergence in different coordinate systems:

just plug in the formulas.

(need some practice)

e.g.: 3-3. on page 55.

(a). $\vec{E} = \hat{x} 3x^2 + \hat{y} 2y + \hat{z} x^2 z$, find $\nabla \cdot \vec{E}$ at $(2, -2, 0)$
 (x, y, z) .

(b). $\vec{E} = \hat{R} \left(\frac{a^3 \cos \theta}{R^2} \right) - \hat{\theta} \left(\frac{a^3 \sin \theta}{R^2} \right)$, find $\nabla \cdot \vec{E}$ at $\left(\frac{a}{2}, 0, \pi \right)$
 (R, θ, ϕ)

Next: 4-1, 2, 3, 4, 5