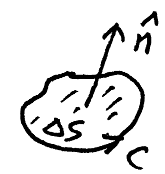
Phys 221.

Lecture 3. Ref: 3-3, 4-4, 5-4, 5-5.

- Curl of a vector field:
(rotation).

$$\nabla \times \vec{B} \triangleq \lim_{\Delta S \to 0} \frac{1}{\Delta S} \left[\hat{n} \oint_C \vec{B} \cdot d\vec{l} \right]_{max} \qquad (3.5)$$

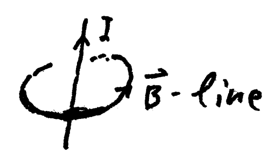
ΔS — area enclosed by contour C



$\hat{n}$ — unit vector normal to ΔS
(RHR)

$\oint_C \vec{B} \cdot d\vec{l}$ — circulation of $\vec{B}$

why max? — The circulation depends on orientation of C.
(Think about $\vec{B}$ field due to a current).



- Meaning of $\nabla \times \vec{B}$
  - It's a vector
  - max circulation of $\vec{B}$ per unit area.
  - Points to the normal direction (RHR)
  - represents the strength of a vortex source that causes the circulation of $\vec{B}$.
    - e.g. long current I.
    - e.g. Magnetic field.
  - When $\nabla \times \vec{B} = 0$, $\vec{B}$ is called an irrotational field. or conservative field.
    (e.g. $\vec{E}$ in electrostatics.

- Calculation of $\nabla \times \vec{B}$ (see formulas in backcover of Textbook).

e.g. In cartesian coordinates.

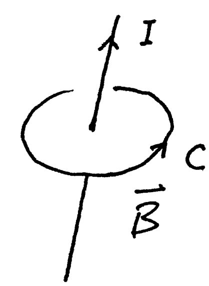
$$\nabla \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

- cylindrical and spherical coordinates : you go through them.

- e.g. Magnetostatics.

Ampere's law $\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I$

current I is the source of $\vec{B}$



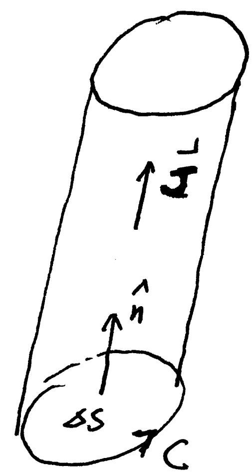
For a current distribution with current density $\vec{J}$

$\vec{J}$: direction : $\hat{n}$

magnitude : $|\vec{J}| = \frac{I}{\Delta S}$

$$I = \int_{\Delta S} \vec{J} \cdot d\vec{s} \qquad (4.12)$$

$$\vec{J} = \hat{n} \frac{I}{\Delta S} \Big|_{\Delta S \to 0}$$



Then.
$$\nabla \times \vec{B} = \lim_{\Delta S \to 0} \frac{\hat{n} \oint_c \vec{B} \cdot d\vec{l}}{\Delta S}$$

$$= \lim_{\Delta S \to 0} \frac{\hat{n}\, \mu_0 I}{\Delta S}$$

$$\therefore \quad \nabla \times \vec{B} = \mu_0 \vec{J}$$

current density $\vec{J}$ represents the strength of vortex source which causes the circulation of $\vec{B}$.

- stoke's theorem

$$\int_S (\nabla \times \vec{B}) \cdot d\vec{s} = \oint_c \vec{B} \cdot d\vec{l}$$

physically, it's consistent with the definition of $\nabla \times \vec{B}$ (3.5), when ΔS is not small, ΔS becomes S.

Mathematically, stoke's theorem converts a line integral to a surface integral, and vice versu.

- Two identities involving curl

① . $\nabla \times \nabla V = 0$ for any scalar function V.

i.e. the curl of a gradient is always zero.

e.g. electrostatics : since $\vec{E} = -\nabla V$

$\nabla \times \vec{E} = 0$.

i.e. $\oint \vec{E} \cdot d\vec{l} = 0$.

$\therefore$ $\vec{E}$ is a conservative field.

Note : This is true when everything is t-independent.

but, when t-dependence is introduced, $\nabla \times \vec{E} \neq 0$.

i.e. $\vec{E}$ is no longer conservative.

e.g: Faraday's law. — later.

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② . $\nabla \cdot (\nabla \times \vec{A}) = 0$. for any vector function $\vec{A}$.

i.e. The divergence of a curl is always zero.

e.g. magnetostatics : Because $\nabla \cdot \vec{B} = 0$ (No monopole)

$\vec{B}$ can be written as

the curl of a vector:

$$\vec{B} = \nabla \times \vec{A} \qquad (5\text{-}53)$$

$\vec{A}$ — vector potential of the magnetic field $\vec{B}$.

- Laplacian operator ∇^2

$$\nabla^2 V \triangleq \nabla \cdot (\nabla V)$$

In Cartesian coord : $\nabla^2 V = \frac{\partial^2}{\partial x^2} V + \frac{\partial^2}{\partial y^2} V + \frac{\partial^2}{\partial z^2} V$

∇^2 can act on a vector function too .

$$\nabla^2 \vec{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \vec{E}$$

$$= \hat{x}\, \nabla^2 E_x + \hat{y}\, \nabla^2 E_y + \hat{z}\, \nabla^2 E_z \, .$$

In cylindrical and spherical systems – see textbook.

∇^2 is very important in wave equations.