Phys 221

Lecture 4. Ref: 4-1, 2, 3, 4, 5.

- Maxwell's Equations. (foundation of electromagnetics theory)

In vacuum. (allow t-dependence).

$$\oint_S \vec{E}\cdot d\vec{s} = \frac{Q}{\varepsilon_0} \qquad \text{Gauss' Law} \qquad \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt} \qquad \text{Faraday's law} \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}.$$

$$\oint_S \vec{B}\cdot d\vec{s} = 0 \qquad \text{No monopole} \qquad \nabla\cdot\vec{B} = 0$$

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I + \varepsilon_0\mu_0\int\frac{\partial\vec{E}}{\partial t}\cdot d\vec{s} \qquad \text{Ampere' law} \qquad \nabla\times\vec{B} = \mu_0\vec{J} + \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

Integral Form — differential Form.

- Note

Faraday's law: electro-magnetic induction.

① Time-varying $\vec{B}$ contributes to the circulation of $\vec{E}$.

($\Phi = \int \vec{B}\cdot d\vec{s}$ — magnetic flux

② . $\vec{E}$-field is not conservative because of $\frac{\partial\vec{B}}{\partial t}$.

Maxwell's idea: Since $\frac{\partial\vec{B}}{\partial t}$ contributes to $\nabla\times\vec{E}$,

$\frac{\partial\vec{E}}{\partial t}$ should contribute to $\nabla\times\vec{B}$.

(Turns out to be correct)

- When there is no time-dependence,

$$\frac{\partial \vec{B}}{\partial t} = 0, \qquad \frac{\partial \vec{E}}{\partial t} = 0.$$

We have:

Electrostatics:

$$\oint_S \vec{E}\cdot d\vec{s} = \frac{Q}{\varepsilon_0} \qquad \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\oint_C \vec{E}\cdot d\vec{l} = 0 \qquad \nabla\times\vec{E} = 0. \quad (\vec{E} = -\nabla V)$$

magnetostatics.

$$\oint_S \vec{B}\cdot d\vec{s} = 0 \qquad \nabla\cdot\vec{B} = 0 \quad (\vec{B} = \nabla\times\vec{A})$$

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I \qquad \nabla\times\vec{B} = \mu_0\vec{J} \quad ($$

ε_0 — electrical permittivity of free space $= 8.85\times10^{-12}$ F/m

μ_0 — magnetic permeability of free space $= 4\pi\times10^{-7}$ H/m

- In medium (Homogeneous and isotropic)

$$\varepsilon_0 \longrightarrow \varepsilon_0\varepsilon_r = \varepsilon.$$

$$\mu_0 \longrightarrow \mu_0\mu_r = \mu$$

$\vec{D} = \varepsilon\vec{E}$ $\qquad$ $\vec{D}$ — electric flux density.

$\vec{B} = \mu\vec{H}.$ $\qquad$ $\vec{B}$ — magnetic flux density

$\qquad\qquad$ $\vec{H}$ — magnetic field density.

- Electrostatics (No t-dependence, in medium ε)

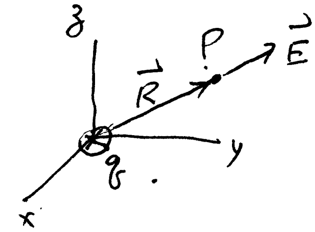
$\nabla \cdot \vec{D} = \rho$ (4.2a) Gauss' Law. $\oint_S \vec{D} \cdot d\vec{s} = Q$

$\nabla \times \vec{E} = 0$ (4.2b) $\vec{E}$ - field is conservative

Then. $\vec{E} = -\nabla V$.

Also, it can be shown that Gauss' law is equivalent to Coulomb's law.

$$\vec{E} = \frac{\hat{R}\, q}{4\pi \varepsilon_0 R^2}$$



- 3 Methods to determine the $\vec{E}$ - field of given charges

① Coulomb's law

② Gauss' law

③ Potential and then $\vec{E} = -\nabla V$.

	Pros	Cons
Coulomb's law :	clear physical meaning.	Hard to add/subtract if there are many charges.
Gauss' law :	Easier to calculate	require high degree of symmetry.
Potential .	~~Vector~~ V is a scalar. easy to add.	Need to calculated V and $\vec{E}$ $\vec{E} = -\nabla V$.

- Coulomb's law.

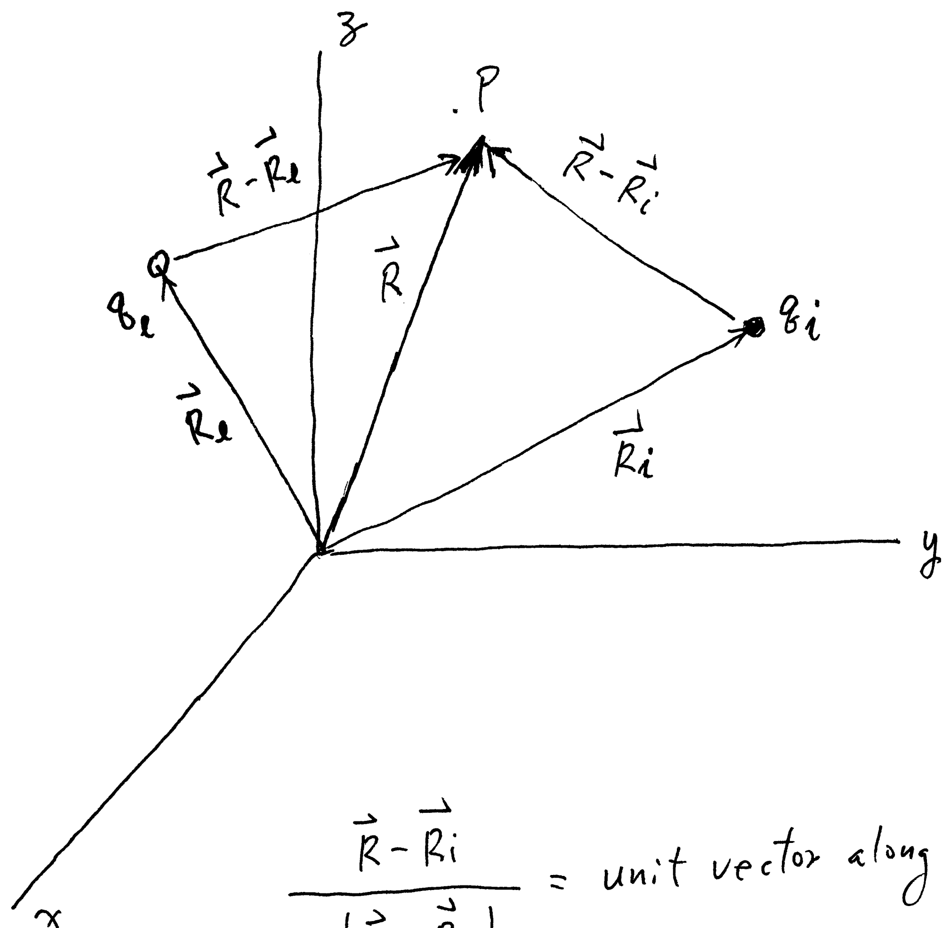
For charges $q_1, q_2, q_3 \cdots\cdots$ located at $\vec{R}_1, \vec{R}_2, \vec{R}_3 \cdots\cdots$

The $\vec{E}$-field at $\vec{R}$:

$$\vec{E}(\vec{R}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i(\vec{R}-\vec{R}_i)}{|\vec{R}-\vec{R}_i|^3}$$

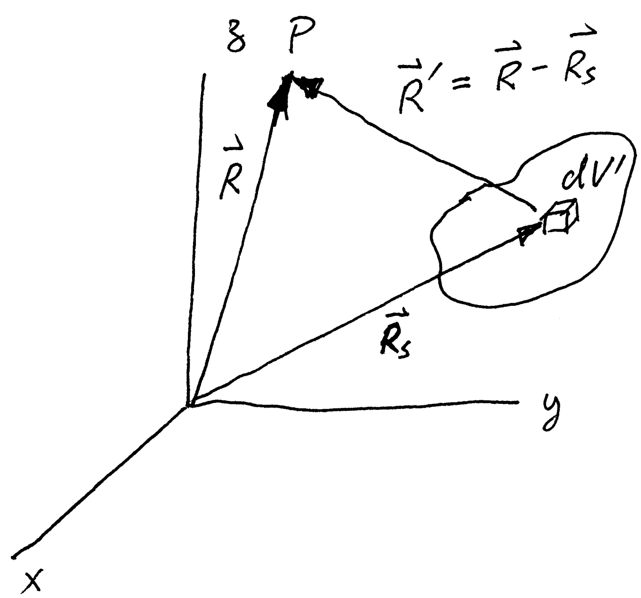
$\vec{R}-\vec{R}_i$ — vector from $\vec{R}_i$ to $\vec{R}$

source field.



$$\frac{\vec{R}-\vec{R}_i}{|\vec{R}-\vec{R}_i|} = \text{unit vector along } \vec{R}-\vec{R}_i.$$

For continuous charge distribution in Volume V' with charge density ρ_v



— charge element dq

$$dq = \rho_v \, dV'$$

located at $\vec{R}_s$

— field Point P at $\vec{R}$

(We want to find $\vec{E}$ at $\vec{R}$)

— distance vector from source dq to field point P:

$$\vec{R}' = \vec{R} - \vec{R}_s$$

Total $\vec{E}$-field at $\vec{R}$:

$$\vec{E} = \frac{1}{4\pi\varepsilon} \int \frac{\hat{\vec{R}}'}{R'^2} \, dq = \frac{1}{4\pi\varepsilon} \int \frac{\hat{\vec{R}}' \rho_v \, dV'}{R'^2} = \frac{1}{4\pi\varepsilon_0} \int \frac{\vec{R}' \rho_v \, dV'}{R'^3}$$

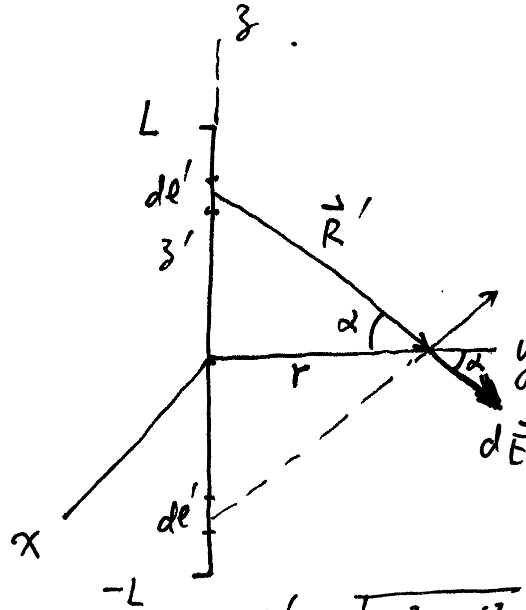
(4.21a)

For a charged wire : $dq = \rho_\ell \, d\ell'$

For a charged sheet : $dq = \rho_s \, dS'$

Note: For integral (or sum), sometimes it's convenient to deal with components.

e.g. A straight wire of length $2L$ carries a uniform linear charge density ρ_ℓ. Determine the electric field $\vec{E}$ at Point P, a distance r from the wire along the perpendicular bisector.



$R' = \sqrt{r^2 + z'^2}$.

$\cos\alpha = \dfrac{r}{R'}$

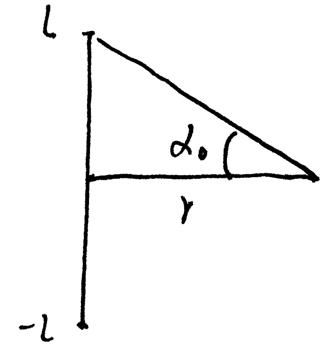
by symmetry, $\vec{E} = \hat{r} E_r$. (cylindrical coord.)

Coulomb's law:

$$E_r = \frac{1}{4\pi\varepsilon_0}\int \frac{dq \cdot \cos\alpha}{R'^2} = \frac{\rho_\ell}{4\pi\varepsilon_0}\int_{-L}^{L} \frac{r\,dz'}{R'^3}$$

$$= \frac{2\rho_\ell}{4\pi\varepsilon_0}\int_0^L \frac{r\,dz'}{(r^2+z'^2)^{3/2}}$$

$$= \frac{\rho_\ell}{2\pi\varepsilon_0}\left[\frac{z'}{r\sqrt{r^2+z'^2}}\right]_0^L$$



$$= \frac{\rho_\ell L}{2\pi\varepsilon_0 r\sqrt{r^2+L^2}}$$

$$= \frac{\rho_\ell \sin\alpha_0}{2\pi\varepsilon_0 r}.$$

$$\frac{L}{\sqrt{r^2+L^2}} = \sin\alpha_0$$

$$\left(= \frac{Q}{4\pi\varepsilon_0 r\sqrt{r^2+L^2}} \right)$$

(r', ϕ', z') — source

(r, ϕ, z) — field.

When $L \gg r$ (very long wire)

$$\vec{E} = \hat{r}\,\frac{\rho_\ell}{2\pi\varepsilon_0 r} \qquad \alpha_0 \to \frac{\pi}{2}, \quad \sin\alpha_0 = 1$$

when $r \gg L$ (very far away)

$$\vec{E} = \hat{r}\,\frac{\rho_\ell L}{2\pi\varepsilon_0 r^2} = \hat{r}\,\frac{Q}{4\pi\varepsilon_0 r^2}$$ (just like a point charge).

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e.g. For a large flat sheet of charge with a uniform area charge density ρ_s (charge per unit area).

Find $\vec{E}$-field at a point on z-axis.

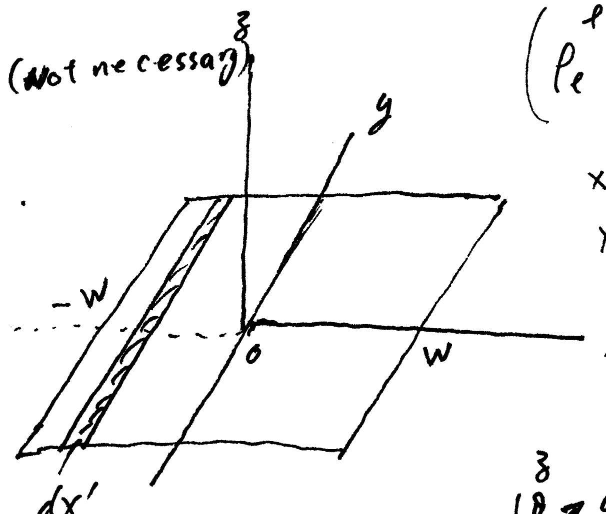
$O \sim$ centre of the area (not necessary)

charge per unit length in y ($\rho_\ell = \rho_s\,dx'$).

x', y', z' — source

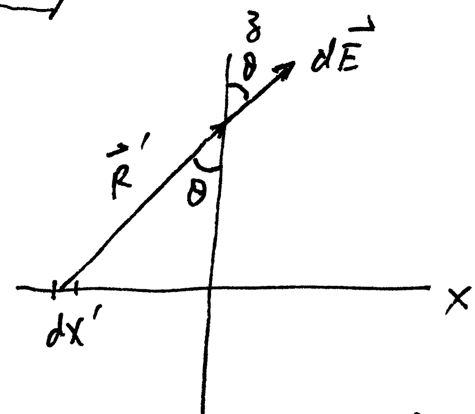
x, y, z — field.

by symmetry, $\vec{E} = \hat{z}\,E_z$.



$$E_z = \int_{-W}^{W} \frac{\rho_s\,dx'\cos\theta}{2\pi\varepsilon_0 R'}$$

$$= \frac{\rho_s}{2\varepsilon_0} \quad (\text{as } W \to \infty).$$



Note: $\frac{x'}{z} = \tan\theta$, $dx' = \frac{z\,d\theta}{\cos^2\theta}$, $\frac{z}{R'} = \cos\theta$

$$E_z = 2\int_0^{\theta_0 \to \frac{\pi}{2}} \frac{\rho_s\,z\cdot\cos\theta\,d\theta}{2\pi\varepsilon_0 R'\cdot\cos^2\theta} = \frac{\rho_s}{\pi\varepsilon_0}\int_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{\rho_s}{2\varepsilon_0}$$

(R' is the r in the wire case)

The field point doesn't have to be at centre, as long as it's closer to surface.