Phys 221

Lecture 6, 7, 8.

Electrostatics — scalar potential V.

Idea : get arround the vector sum.

- For t-independent $\vec{E}$. ($\vec{B}$).

  $\nabla \times \vec{E} = 0$, — everywhere

  $\oint_C \vec{E} \cdot d\vec{l} = 0$ — for any closed C.

  i.e. $\vec{E}$ — conservative.

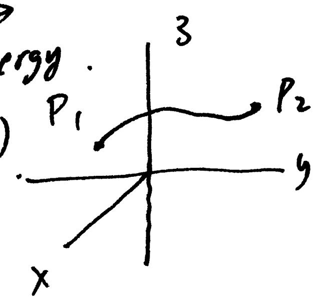
  $\therefore \quad \vec{E} = -\nabla V$ (curl of gradient $= 0$)

  V — electric scalar potential.

- Meaning of V: $\quad \vec{E} = \dfrac{\vec{F}}{q}$

$$\underbrace{\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}}_{\text{work done by } \vec{E}\text{-field (on a unit charge)}} = \int_{P_1}^{P_2} -\nabla V \cdot d\vec{l} = -\int_{P_1}^{P_2} dV = -(V_2 - V_1) = \underbrace{-\Delta V}_{\text{Loss in potential energy (for unit charge)}}.$$

work done by $\vec{E}$-field (on a unit charge) = Loss in potential energy. (for unit charge)



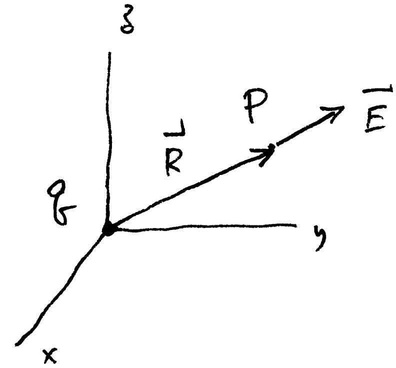
$PE = qV$

Force $= \vec{E} q$.

- For a point charge q at the origin

  — We can define the potential at ∞ as 0 $(V(\infty)=0)$ $\left\{\begin{array}{l}\text{NOT} \\ \text{always} \\ \text{possible !}\end{array}\right.$

Then, the potential at point P is .



$$V(\vec{R}) = V(\vec{R}) - V(\infty)$$

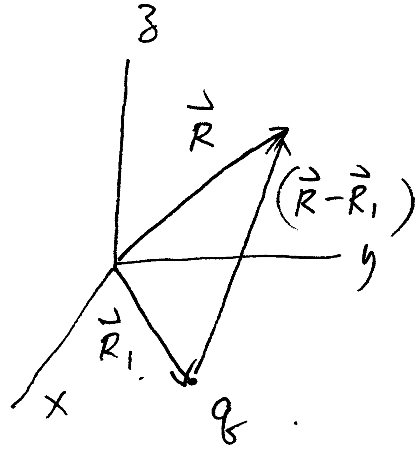
$$= -\int_{\infty}^{P} \vec{E}\cdot d\vec{l}$$

$$= -\int_{\infty}^{R} \frac{q}{4\pi\varepsilon_0 R^2}\cdot dR$$

ε_0 — in vacuum

$$V(\vec{R}) = \frac{q}{4\pi\varepsilon_0 R} \qquad [q \text{ at } (0,0,0)]$$

- If the charge is at $\vec{R}_1$



$$V(\vec{R}) = \frac{q}{4\pi\varepsilon_0 |(\vec{R}-\vec{R}_1)|}.$$

$|\vec{R}-\vec{R}_1|$ — distance from source to field point.

V — scalar !

- For many point charges, $q_1, q_2, \cdots q_N$ at $\vec{R}_1, \vec{R}_2, \cdots \vec{R}_N$.

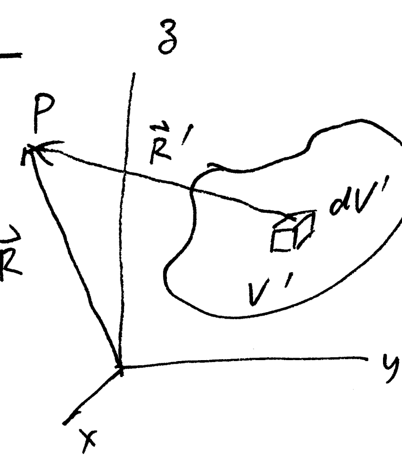
$$V(\vec{R}) = \frac{1}{4\pi\varepsilon_0} \underbrace{\sum_{k=1}^{N}}_{\text{scalar sum ! (algebraic)}} \frac{q_k}{|\vec{R} - \vec{R}_k|}$$

ε — in medium.

↑ electric potential at $\vec{R}$.

- For continuous charge distribution.

$$V(\vec{R}) = \frac{1}{4\pi\varepsilon} \int_{V'} \frac{\rho_v \, dv'}{R'}$$



R' — distance from dv' to the field point P.

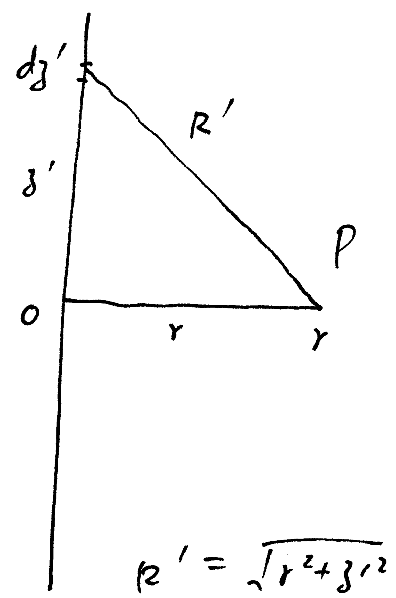
$\rho_v \, dv' = dq$.

for surface charges: $dq = \rho_s \, ds$

for line charges: $dq = \rho_\ell \, d\ell$.

e.g. Infinitely long straight line charge with a uniform line charge density ρ_l in air.

Find the potential V at P.



Try: $V = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_{-\infty}^{+\infty} \dfrac{\rho_l\, dz'}{\sqrt{r^2+z'^2}}$

$= \dfrac{\rho_l}{4\pi\varepsilon_0} \ln\left(z' + \sqrt{z'^2+r^2}\right)\Bigg|_{z'\to-\infty}^{z'\to+\infty}$

$= \displaystyle\lim_{L\to\infty} \dfrac{\rho_l}{4\pi\varepsilon_0} \ln\left(\dfrac{L+\sqrt{L^2+r^2}}{-L+\sqrt{L^2+r^2}}\right)$

$= \displaystyle\lim_{L\to\infty} \dfrac{\rho_l}{4\pi\varepsilon_0} \ln \dfrac{\left(L+\sqrt{L^2+r^2}\right)^2}{r^2} \Longrightarrow \infty$

Problem: when the size of charge is infinite, we can't assume the zero potential point at infinity.

How to do it properly:

(Only the potential difference ΔV is uniquely defined)

$\therefore$ potential difference between r and r_0:

general case.

$$\boxed{V(r) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho_v\, dv'}{R'} + C}$$

$$\Delta V = V(r) - V(r_0)$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty} \frac{\rho_e\, dz'}{\sqrt{r^2+z'^2}} - \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} \frac{\rho_e\, dz'}{\sqrt{r_0^2+z^2}}.$$

$$= \frac{\rho_\ell}{4\pi\varepsilon_0} \ln \frac{z'+\sqrt{z'+r^2}}{z'+\sqrt{z'^2+r_0^2}}\Bigg|_{z'\to-\infty}^{z'\to\infty}$$

$$= \frac{\rho_\ell}{4\pi\varepsilon_0} \ln\left(\frac{L+\sqrt{L^2+r^2}}{L+\sqrt{L^2+r_0^2}} \cdot \frac{-L+\sqrt{L^2+r_0^2}}{-L+\sqrt{L^2+r^2}}\right)_{L\to\infty}$$

$$= \frac{\rho_\ell}{4\pi\varepsilon_0} \ln\left(\frac{1+\sqrt{1+\frac{r^2}{L^2}}}{1+\sqrt{1+\frac{r_0^2}{L^2}}} \cdot \frac{-1+\sqrt{1+(r_0/L)^2}}{-1+\sqrt{1+(r/L)^2}}\right)$$

$$= \frac{\rho_\ell}{4\pi\varepsilon_0} \ln\left(\frac{\left(1+\sqrt{1+r^2/L^2}\right)^2}{\left(1+\sqrt{1+\left(\frac{r_0}{L}\right)^2}\right)^2} \frac{\left(\frac{r_0}{L}\right)^2}{\left(\frac{r}{L}\right)^2}\right)_{L\to\infty}$$

$$V(r)-V(r_0) = \frac{\rho_0}{4\pi\varepsilon_0} \ln \frac{r_0^2}{r^2}. = -\frac{\rho_\ell}{2\pi\varepsilon_0} \ln \frac{r}{r_0}.$$

choose $\boxed{V(r_0)=0: \quad V(r) = -\frac{\rho_\ell}{2\pi\varepsilon_0} \ln \frac{r}{r_0}}$

$\vec{E}$ - field:

$$\vec{E} = -\nabla V.$$

$$= -\hat{r}\frac{\partial V}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial V}{\partial \phi} + \hat{z}\frac{\partial V}{\partial z}.$$

$$= -\hat{r}\frac{\partial V}{\partial r}$$

$$= -\hat{r}\left(\frac{-\rho_\ell}{2\pi\varepsilon_0}\right)\frac{1}{r}$$

$$= \hat{r}\frac{\rho_\ell}{2\pi\varepsilon_0 r}$$ — same as before.

Compare 3 Methods:

① . Coulomb's law — intuitive.
but hard to deal with vector sum.

②. Gauss' law — Easier math.
but requires high symmetry.

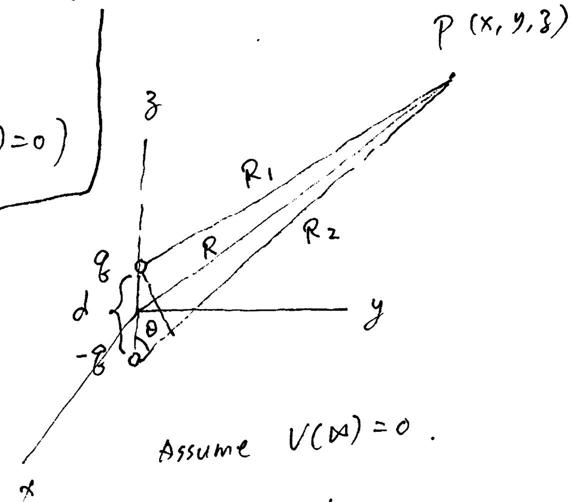
③ potential function — No more vector (sum) integral
Even works for finite length!
but concept and math
could be challenging.

~~A3-4~~ 6-7

2(5/15 marks). An electric dipole consists of two charges q and –q located at (0, 0, d/2) and (0, 0, -d/2) respectively.
(a) Determine the electric potential V at any point P(x, y, z) in free space, given that P is far away from the dipole.
(b) Determine the electric field **E** (both magnitude and direction) at point P.

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R_1} - \frac{q}{R_2}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\,\frac{R_2 - R_1}{R_1 R_2} \qquad (V(\infty)=0)$$



Assume $V(\infty) = 0$.

when $R \gg d$,

$R_1 R_2 \approx R^2$

$R_2 - R_1 \approx d\cos\theta$

$$\cos\theta = \frac{z}{R} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$R = \sqrt{x^2+y^2+z^2}$$

a). when $R \gg d$.

$$V \approx \frac{q}{4\pi\varepsilon_0}\,\frac{d\cos\theta}{R^2}$$

or: $$V = \frac{q\, d\cdot z}{4\pi\varepsilon_0 (x^2+y^2+z^2)^{3/2}}.$$

b). $\vec{E} = -\nabla V$.

$$= -\left[\hat{R}\frac{\partial V}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\phi}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\right]$$

$$= -\hat{R}\frac{\partial V}{\partial R} - \hat{\theta}\frac{1}{R}\frac{\partial V}{\partial \theta}.$$

$$E_R = -\frac{\partial V}{\partial R} = \frac{q}{2\pi\varepsilon_0}\,\frac{d\cos\theta}{R^3}$$

$$E_\theta = -\frac{1}{R}\frac{\partial V}{\partial \theta} = \frac{1}{R}\,\frac{q\cdot d\cdot \sin\theta}{4\pi\varepsilon_0 R^2} = \frac{q d\cdot\sin\theta}{4\pi\varepsilon_0 R^3}$$

$$E_\phi = 0$$

$$\therefore\quad \vec{E} = \hat{R}\,\frac{q\,d\cos\theta}{2\pi\varepsilon_0 R^3} + \hat{\theta}\,\frac{q\,d\sin\theta}{4\pi\varepsilon_0 R^3} \qquad (R \gg d)$$