Phys 221

Lecture 9. Ref: 4.5.6.7.8.9.

- Electric potential $V(\vec{R})$ can also be determined by solving its differential equations.

In electrostatics: $\vec{E} = -\nabla V$

since. $\nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon}$

$$\nabla \cdot (\nabla V) = -\frac{\rho_v}{\varepsilon}$$

$$\therefore \quad \boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}} \qquad (4.60)$$

(Poisson's equation)

when there is no charge, $\rho_v = 0$:

$$\boxed{\nabla^2 V = 0} \qquad (4.61)$$

(Laplace's equation).

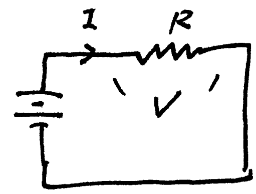
Note: in Cartesian coordinates:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}.$$

(4.60) and (4.61) are well studied and often can be solved along with boundary conditions.

- Ohm's law:

In a circuit: $R = \frac{V}{I}$, $V = RI$



OR: $I = \frac{1}{R} V$ $\qquad R$ – resistance

i.e. $I = GV$. – G – conductance

unit: Siemens (S)

In a field: $\boxed{\vec{J} = \sigma \vec{E}}$ (4.67)

~~conductivity~~.

(4.67) is known as the point form of Ohm's law,

a very useful law in addition to Maxwell's equations.

$\vec{J}$ – current density

σ – conductivity, $\sigma = \frac{1}{\rho}$. ρ – resistivity.

$\vec{E}$ – electric field.

- Perfect conductor: $\sigma = \infty$, $\rho = 0$.

- Perfect insulator: $\sigma = 0$, $\vec{J} = 0$.

- Note, in electrostatics, all charges are at equilibrium (i.e, NOT moving). Then, inside a conductor,

$\vec{E} = 0$. – in conductor.

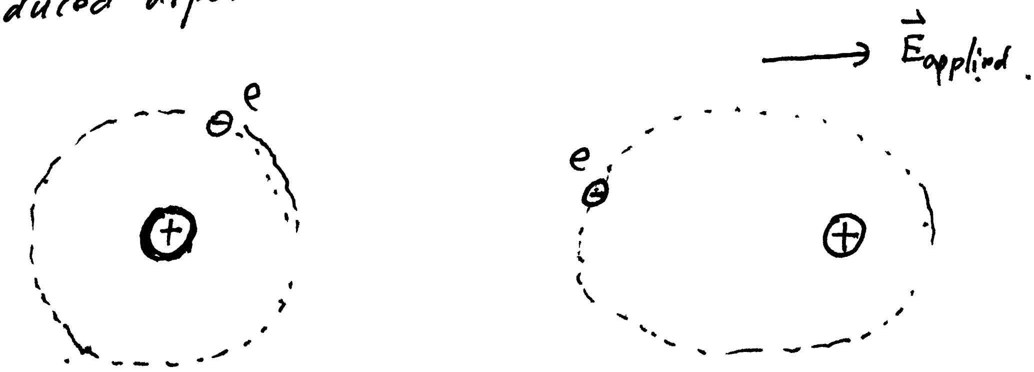
$\therefore$ A conductor is an equipotential body. $\int_1^2 \vec{E} \cdot d\vec{l} = 0$.

$V_1 = V_2$.

- Dielectrics. — Not just insulators.

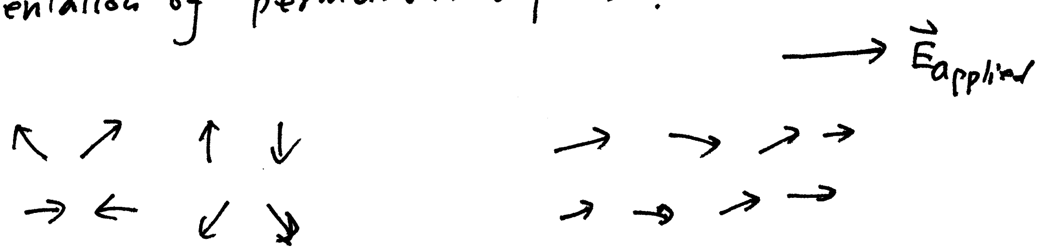
Effects of an applied $\vec{E}$-field on dielectric materials.

— induced dipoles.



$\vec{E}_{applied} = 0$ $\qquad\qquad$ $\vec{E}_{applied} \neq 0$

— Orientation of permanent dipoles.



$\vec{E}_{applied} = 0$ $\qquad\qquad$ $\vec{E}_{applied} \neq 0$

- Polarization field.

$$\vec{P} = \frac{\sum \vec{p}}{V}$$ Total dipole per unit volume.

Note: Lots ~~of cancellations in $\sum \vec{p}$ in dielectrics~~.

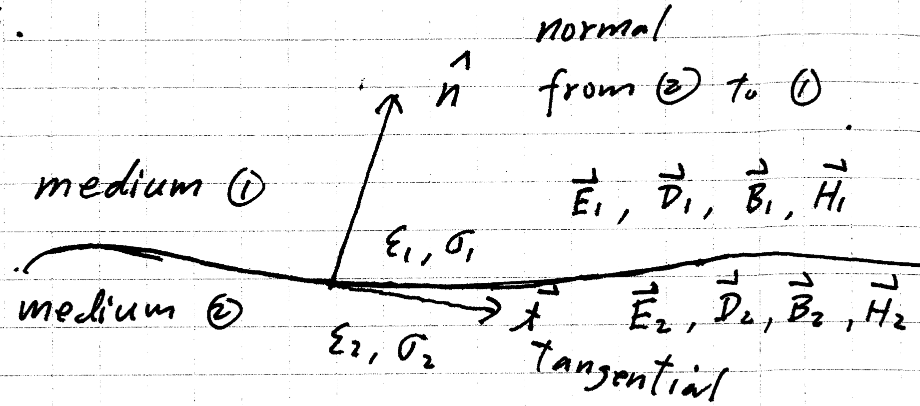
- For linear, isotropic and homogeneous medium:

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}, \quad \vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 (1+\chi_e)\vec{E} = \varepsilon_0 \varepsilon_r \vec{E}$$

χ_e — electric susceptibility.

- Boundary conditions.

Typical situation



Difficulty: physical quantities $(\vec{E}, \vec{D}, \vec{B}, \vec{H}, \varepsilon, \sigma)$ are discontinuous at the boundary.

i.e, derivatives are undefined. $(\nabla, \nabla\cdot, \nabla\times$, etc.)

Solution: Use the integral form of Maxwell's equations.
(All we need is that the physical quantities are finite, i.e, integrable).

Results: <u>Table 4-3</u> on page 96.
↑ will <u>NOT</u> be provided in exams!

most important ones:

(4.90) $\quad E_{1t} = E_{2t}$ (Tangential component of $\vec{E}$ is continuous)

(4.94) $\quad D_{1n} - D_{2n} = \rho_s$ (ρ_s — free charge surface density)

i.e. $\quad \hat{n}\cdot(\vec{D}_1 - \vec{D}_2) = \rho_s$

Table 4-3: Boundary conditions for the electric fields.

Field Component	Any Two Media	Medium 1 Dielectric ε_1 / Medium 2 Dielectric ε_2	Medium 1 Dielectric ε_1	Medium 2 Conductor
Tangential **E**	$E_{1t} = E_{2t}$	$E_{1t} = E_{2t}$	$E_{1t} = E_{2t} = 0$	
Tangential **D**	$D_{1t}/\varepsilon_1 = D_{2t}/\varepsilon_2$	$D_{1t}/\varepsilon_1 = D_{2t}/\varepsilon_2$	$D_{1t} = D_{2t} = 0$	
Normal **E**	$\hat{\mathbf{n}} \cdot (\varepsilon_1 \mathbf{E}_1 - \varepsilon_2 \mathbf{E}_2) = \rho_s$	$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$	$E_{1n} = \rho_s/\varepsilon_1$	$E_{2n} = 0$
Normal **D**	$\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$	$D_{1n} - D_{2n} = \rho_s$	$D_{1n} = \rho_s$	$D_{2n} = 0$

Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.

Example 4-10 Application of Boundary Conditions

The x–y plane is a charge-free boundary separating two dielectric media with permittivities ε_1 and ε_2, as shown in Fig. 4-19. If the electric field in medium 1 is $\mathbf{E}_1 = \hat{\mathbf{x}}E_{1x} + \hat{\mathbf{y}}E_{1y} + \hat{\mathbf{z}}E_{1z}$, find (a) the electric field $\mathbf{E}_2$ in medium 2 and (b) the angles θ_1 and θ_2.

Solution: (a) Let $\mathbf{E}_2 = \hat{\mathbf{x}}E_{2x} + \hat{\mathbf{y}}E_{2y} + \hat{\mathbf{z}}E_{2z}$. Our task is to find the components of $\mathbf{E}_2$ in terms of the given components of $\mathbf{E}_1$. The normal to the boundary is $\hat{\mathbf{z}}$. Hence, the x and y components of the fields are tangential to the boundary and the z components are normal to the boundary. At a charge-free interface, the tangential components of $\mathbf{E}$ and the normal components of $\mathbf{D}$ are continuous. Consequently,

$$E_{2x} = E_{1x}, \qquad E_{2y} = E_{1y},$$

and

$$D_{2z} = D_{1z} \qquad \text{or} \qquad \varepsilon_2 E_{2z} = \varepsilon_1 E_{1z}.$$

Hence,

$$\mathbf{E}_2 = \hat{\mathbf{x}}E_{1x} + \hat{\mathbf{y}}E_{1y} + \hat{\mathbf{z}}\frac{\varepsilon_1}{\varepsilon_2}E_{1z}. \tag{4.98}$$

(b) The tangential components of $\mathbf{E}_1$ and $\mathbf{E}_2$ are $E_{1t} = \sqrt{E_{1x}^2 + E_{1y}^2}$ and $E_{2t} = \sqrt{E_{2x}^2 + E_{2y}^2}$. The angles θ_1 and θ_2 are then given by

$$\tan\theta_1 = \frac{E_{1t}}{E_{1z}} = \frac{\sqrt{E_{1x}^2 + E_{1y}^2}}{E_{1z}},$$

$$\tan\theta_2 = \frac{E_{2t}}{E_{2z}} = \frac{\sqrt{E_{2x}^2 + E_{2y}^2}}{E_{2z}} = \frac{\sqrt{E_{1x}^2 + E_{1y}^2}}{(\varepsilon_1/\varepsilon_2)E_{1z}},$$

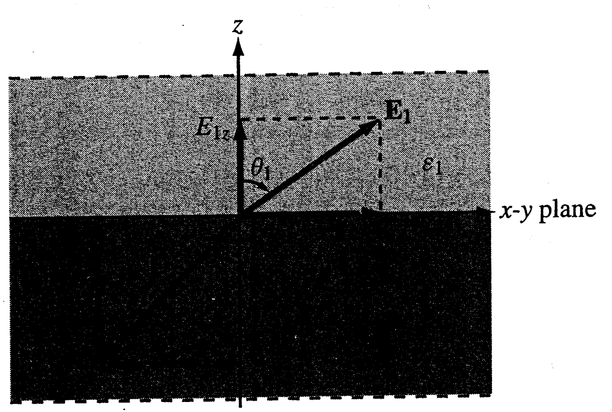
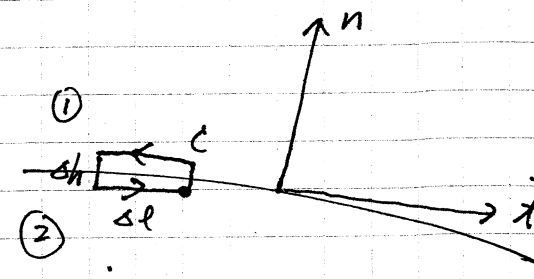


Figure 4-19: Application of boundary conditions at the interface between two dielectric media (Example 4-10).

- Why $E_{1t} = E_{2t}$?

Because $\oint_C \vec{E}\cdot d\vec{l} = 0$



closed path C: $\Delta h \to 0$

$\Delta l \sim$ small.

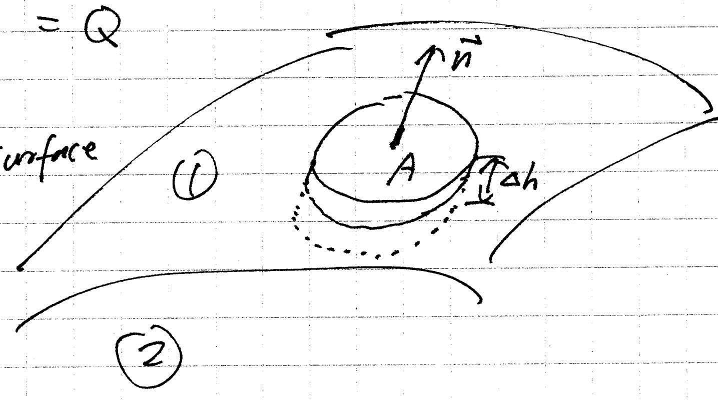
so that $\vec{E} \simeq$ const. along Δl.

$$\oint_C \vec{E}\cdot d\vec{l} = E_{2t}\cdot\Delta l - E_{1t}\,\Delta l = 0$$

$$\therefore \quad E_{2t} = E_{1t}.$$

- why $D_{1n} - D_{2n} = \rho_s$?

because $\oint_S \vec{D}\cdot d\vec{s} = Q$



Small pill-like Gaussian surface

(very thin, $\Delta h \to 0$)

A — small so that $\vec{D}$ = const.

$$\oint_S \vec{D}\cdot d\vec{s} = D_{1n}A - D_{2n}A = \rho_s A.$$

$\Delta h \to 0$
D — finite.
$\therefore$ lateral flux $\to 0$

$$\therefore \quad D_{1n} - D_{2n} = \rho_s$$

note: Definition of $\hat{n}$: from medium ② to medium ①.

Note: All the other boundary conditions in table 4-3 are just examples of

$$\begin{cases} E_{1t} = E_{2t} \\ D_{1n} - D_{2n} = \rho_s \end{cases}$$

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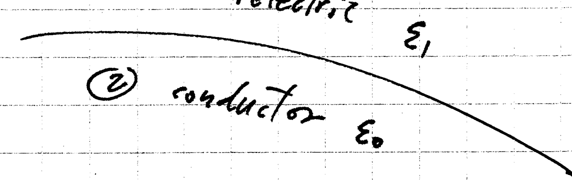
e.g $\begin{cases} ① - \text{dielectric} \\ ② - \text{conductor} \end{cases}$



Since $\vec{E}_2 = 0$, $E_{2t} = 0$ $D_{2t} = 0$ $(\because \vec{D}_2 = \varepsilon_0 \vec{E}_2)$

$E_{1t} = 0$ $D_{1t} = 0$ $(\because \vec{D}_1 = \varepsilon_1 \vec{E}_1)$

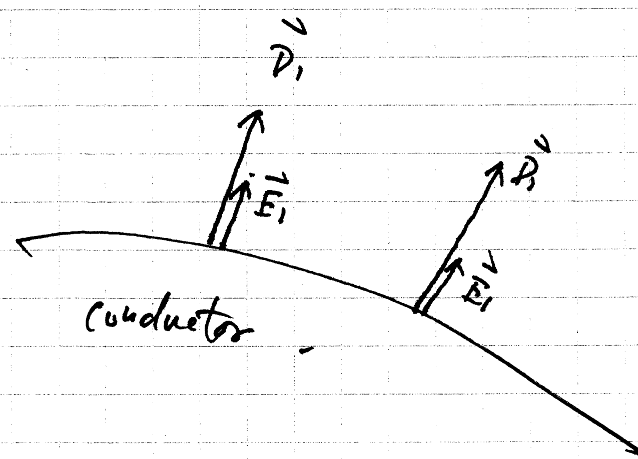
$(\because E_{1t} = E_{2t})$.

Also, since $\vec{E}_2 = 0$, $E_{2n} = 0$, $D_{2n} = 0$.

$$D_{1n} - D_{2n} = \rho_s$$

$$\therefore \quad D_{1n} = \rho_s$$

$$E_{1n} = \frac{\rho_s}{\varepsilon_1}$$



$\therefore$ $\vec{D}$, $\vec{E}$ are perpendicular to the surface of conductor.

Assignment 4: 4.45.

good exercise for using B. Cs.

but, ~~its~~ there are conceptual problems in the solution! what are they?

① Maxwell's eqs.
② conservative.