Phys 221

Lecture 20.

Assignment #6.

6.1, 3, 5, 11, 18.

Ref. 6-7*, 6-8, 6-9. 6-10.

Time-dep. Maxwell's equations.

- Faraday's law: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\nabla \times \vec{E} \neq 0$, because $\frac{\partial \vec{B}}{\partial t}$ contribute to $\nabla \times \vec{E}$.

Question: Does $\frac{\partial \vec{E}}{\partial t}$ contribute to $\nabla \times \vec{B}$ as well?

Maxwell guess: it should!

why?

$\nabla \times \vec{B} = \mu_0 \vec{J}$ (5.45) holds for magnetostatics but can't be right when fields and sources are time-dependent $\left(\frac{\partial \rho_v}{\partial t} \neq 0\right)$

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 \underbrace{\nabla \cdot \vec{J}}$$

$\Downarrow$ 0 (identity)

$\Downarrow$ $-\frac{\partial \rho_v}{\partial t} \neq 0 \Rightarrow \nabla \cdot \vec{J} \neq 0$. Since $\nabla \cdot \vec{J} + \frac{\partial \rho_v}{\partial t} = 0$ (charge conservation)

Then, (5.45) needs to be modified to allow $\frac{\partial \rho_v}{\partial t} \neq 0$.

How?

· idea: replace $\nabla \cdot \vec{J}$ with $\nabla \cdot \vec{J} + \frac{\partial \rho_v}{\partial t} = \nabla \cdot \vec{J} + \nabla \cdot \frac{\partial \vec{E}}{\partial t} \cdot \varepsilon_0$

$$\left(\because \ \nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon_0} \right)$$

i.e. $\nabla \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$.

In media: $\nabla \times \vec{B} = \mu \vec{J} + \varepsilon \mu \frac{\partial \vec{E}}{\partial t} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right)$

$\frac{\partial \vec{D}}{\partial t}$ — displacement current density

(Based on a mechanical model — Don't worry about it)

OR:

$$\left. \begin{aligned} \nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ \oint_C \vec{H} \cdot \vec{dl} &= \iint_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{dS} \end{aligned} \right\} \quad (6.4)$$

• Potential functions (V, A).

Now $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \neq 0$.

We can't define $\vec{E} = -\nabla V$ any more.

However, since $\nabla \cdot \vec{B} = 0$ is still valid.

we can still define $\vec{B} = \nabla \times \vec{A}$.

Then, $\nabla \times \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{A})$

i.e. $\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$

Now we can define V using.

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V$$

i.e. $$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}.$$

$\therefore$ $\vec{E}$ can result from the spatial variation of V and/or the time variation of $\vec{A}$.

$\vec{E}$ and $\vec{B}$ are related via $\vec{A}$.

- What equations will V and $\vec{A}$ obey now?

(NOT Poisson's equation any more)

Since $\nabla \times \vec{B} = \mu \vec{J} + \varepsilon\mu \frac{\partial \vec{E}}{\partial t}$. and $\vec{B} = \nabla \times \vec{A}$

$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$.

$$\nabla \times \nabla \times \vec{A} = \mu \vec{J} + \varepsilon\mu \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \vec{A}}{\partial t}\right).$$

$$\nabla(\nabla\cdot\vec{A}) - \nabla^2 \vec{A} = \mu \vec{J} - \varepsilon\mu \nabla \frac{\partial V}{\partial t} - \varepsilon\mu \frac{\partial^2 A}{\partial t^2}$$

i.e.,

$$\nabla^2 \vec{A} - \varepsilon\mu \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} + \nabla\left(\nabla\cdot\vec{A} + \varepsilon\mu \frac{\partial V}{\partial t}\right)$$

In vacuum: $$\nabla^2 \vec{A} - \varepsilon_0\mu_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \nabla\left(\nabla\cdot\vec{A} + \varepsilon_0\mu_0 \frac{\partial V}{\partial t}\right)$$

Remember, $\vec{A}$ is not uniquely defined (only require $\nabla\cdot\vec{B} = 0$)

Now instead of choosing the Coulomb Gauge, we can choose the Lorentz Gauge:

$$\nabla\cdot\vec{A} + \varepsilon_0\mu_0 \frac{\partial V}{\partial t} = 0.$$

Then: $$\nabla^2 \vec{A} - \varepsilon_0\mu_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

This is a wave equation (including 3 components), mathematically well studied.

Note: The Lorentz Gauge reduces to the Coulomb gauge when $\frac{\partial V}{\partial t} = 0$. (static case — for example).

- The wave equation for V.

Since $\nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon_0}$, $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$

$$\nabla \cdot \left(-\nabla V - \frac{\partial \vec{A}}{\partial t} \right) = \frac{\rho_v}{\varepsilon_0}$$

$$\nabla^2 V + \frac{\partial}{\partial t}\left(\nabla \cdot \vec{A} \right) = -\frac{\rho}{\varepsilon_0}$$

Lorentz Gauge: $\nabla \cdot \vec{A} + \varepsilon\mu \frac{\partial V}{\partial t} = 0 \Rightarrow \nabla \cdot \vec{A} = -\varepsilon\mu \frac{\partial V}{\partial t}$.

$$\therefore \quad \nabla^2 V - \varepsilon\mu \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon_0}$$

(when $\frac{\partial V}{\partial t} = 0$. Lorentz gauge becomes Coulomb Gauge: $\nabla \cdot \vec{A} = 0$.

- Both V and $\vec{A}$ can be described by wave equations.

$\vec{J}$ is the source of $\vec{A}$ wave,

ρ_v is the source of V wave.

Both waves travel at a speed of $c = \frac{1}{\sqrt{\varepsilon_0 \mu}}$

in vacuum: $c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 2.998 \times 10^8 \ m/s$

Note: It's the Lorentz gauge that separates $\vec{A}$ and V mathematically.

in

• Source-free space, $\vec{J}=0$, $\rho_v = 0$.

Wave Equations for $\vec{A}$ and V:

$$\nabla^2\vec{A} - \varepsilon\mu \frac{\partial^2\vec{A}}{\partial t^2} = 0$$

$$\nabla^2 V - \varepsilon\mu \frac{\partial^2 V}{\partial t^2} = 0.$$

General wave equation: for any quantity $f(x, y, z, t)$

$$\nabla^2 f - \frac{1}{u^2}\cdot\frac{\partial^2 f}{\partial t^2} = 0.$$ (when there is No source)

u — Speed of wave.

e.g.: a plane wave : $f(x,t) = A\cos(\omega t - kx)$

OR $= A\cos[k(x-ut)] = A\cos(kx-kut)$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} = -k^2 A\cos(\omega t - kx)$$

$$\frac{\partial^2 f}{\partial t^2} = -\omega^2 A\cos(\omega t - kx)$$

$$\therefore\ -k^2 A\cos(\omega t - kx) - \frac{1}{u^2}(-\omega^2)A\cos(\omega t - kx)$$

$$= A(\cos\omega t - kx)\left(\frac{\omega^2}{u^2} - k^2\right) = 0.$$

$$\left(\begin{array}{ll} \omega = 2\pi f = \frac{2\pi}{T}, & k = \frac{2\pi}{\lambda}. \\ u = \frac{\lambda}{T} = \frac{\omega}{k}. & \frac{\omega}{u} = k. \end{array}\right)$$