Phys 221

Lecture 21

Ref: 7-1, 7-2, 7-3, 7-4.

- $\vec{E}$ and $\vec{B}$ in source-free space ($\rho_v = 0$, $\vec{J} = 0$)

$$\left.\begin{aligned} \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} &= \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \end{aligned}\right\} \quad \vec{E}-\vec{B} \text{ coupling}$$

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

Then,

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t}(\nabla \times \vec{B}) = -\varepsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Downarrow$$

$$\underbrace{\nabla(\nabla \cdot \vec{E})}_{\Downarrow \; 0} - \nabla^2 \vec{E}$$

$$\left.\begin{aligned} \therefore \quad & \nabla^2 \vec{E} - \varepsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \\ \text{Similarly,} \quad & \nabla^2 \vec{B} - \varepsilon\mu \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \end{aligned}\right\}$$

① $\vec{E}$ and $\vec{B}$ obey "symmetric" equations.

② $\vec{E}$ and $\vec{B}$ obey wave equations!
(They propagate as waves!)

- Plane wave propagation

e.g. $g(x,t) = A\cos(\omega t - \beta x + \phi_0)$

$$= A\cos\left(\frac{2\pi}{T}\cdot t - \frac{2\pi}{\lambda}x + \phi_0\right)$$

A – amplitude

T – period. $\quad \omega = \frac{2\pi}{T}$ – angular frequency

$f = \frac{1}{T}$ – frequency $\quad (\omega = 2\pi f)$

λ – wavelength

$\beta = \frac{2\pi}{\lambda}$ – wavenumber
(phase constant)

ϕ_0 - reference phase

$(\omega t - \beta x + \phi_0) = \phi(x,t)$ – phase.

This wave ~~p~~ propagates along x-direction.

(if it were $\omega t + \beta x$, then it travells in $-x$ direction)

"plane" wave : g = const in y-z plane $= g(x,t)$
(No y- or z- dependence)

- Surface of constant phase:

$$(\omega t - \beta x + \phi_0) = \text{const.} \qquad (\text{y-z planes})$$

- phase velocity: The travelling speed of constant-phase surface.

$$\frac{d}{dt}(\omega t - \beta x + \phi_0) = 0$$

$$\omega - \beta \frac{dx}{dt} = 0$$

$$u_p = \frac{dx}{dt} = \frac{\omega}{\beta} = \frac{2\pi f \lambda}{2\pi} = f\lambda = \frac{\lambda}{T}$$

(It takes T seconds for the wave to travell λ metres

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$
)

- In a lossy medium:

$$y(x,t) = A e^{-\alpha x} \cos(\omega t - \beta x + \phi_0)$$

α — attenuation constant.

$e^{-\alpha x}$ — attenuation factor.

- phasor

— a convenient math ~~form~~ description for time-harmonic functions

└ sinusoidal

> Note: $e^{j\theta} = \cos\theta + j\sin\theta$
>
> $Re[e^{j\theta}] = \cos\theta$.
>
> $j = e^{j\frac{\pi}{2}}$.
>
> $\frac{1}{j} = -j$

If: $z(t) = Re[\tilde{z}\, e^{j\omega t}]$

Then $\tilde{z}$ — phasor ($\tilde{z}$ - doesn't have t).

~~$\tilde{z}$ has not~~

e.g.: $z(t) = A\cos\omega t$, $\quad \tilde{z} = A$

$z(t) = A\cos(\omega t + \phi_0)$, $\quad \tilde{z} = A e^{j\phi_0}$

$z(t) = A\cos(\omega t + \beta x + \phi_0)$ $\quad \tilde{z} = A e^{j(\beta x + \phi_0)}$

$z(t) = e^{-\alpha x}\cos(\omega t + \beta x + \phi_0)$, $\quad \tilde{z} = A e^{-\alpha x} e^{j(\beta x + \phi_0)}$

$= A e^{-\alpha x + j(\beta x + \phi_0)}$

$\frac{d}{dt}[z(t)]$ $\qquad j\omega\tilde{z}$ $\qquad$ (differentiation → multiplication $j\omega$)

$\int z(t)\,dt$ $\qquad \frac{1}{j\omega}\tilde{z}$. $\qquad$ (Integration → division by $j\omega$).

in time domain $\qquad\qquad$ in phasor domain.

- Time-harmonic function is very important in Solving ac circuits and waves.

  — if the source is a harmonic function of time with a frequency ω
  Then the steady-state solutions will be a harmonic function with the same frequency ω.

  — Any time-dependent quantity (e.g. source) can be expressed in terms of the sum of a set of time-harmonic functions (Fourier Series).

  — The wave equation and Maxwell's equations are linear equations. So the linear combination of different solutions is still a solution.
  (Therefore, we can deal with one frequency at a time, and ~~then~~ combine ~~them back~~ the solutions to get the final solution).

- Time-harmonic fields

$$\vec{E}(x, y, z, t) = Re\left[\tilde{\vec{E}}(x, y, z)\, e^{j\omega t}\right]$$

$$\vec{H}(x, y, z, t) = Re\left[\tilde{\vec{H}}(x, y, z)\, e^{j\omega t}\right]$$

Maxwell's equations: (for the phasors)

$$\begin{cases} \nabla \cdot \tilde{\vec{E}} = \tilde{\rho}_v / \varepsilon \\ \nabla \times \tilde{\vec{E}} = -j\omega\mu \tilde{\vec{H}} \\ \nabla \cdot \tilde{\vec{H}} = 0 \\ \nabla \times \tilde{\vec{H}} = \tilde{\vec{J}} + j\omega\varepsilon \tilde{\vec{E}} \end{cases}$$

(7.57)

p. 209.

- If $\tilde{\rho}_v = 0$, $\tilde{\vec{J}} = 0$

Then:

$$\left.\begin{aligned} \nabla \cdot \tilde{\vec{E}} &= 0 \\ \nabla \times \tilde{\vec{E}} &= -j\omega\mu \tilde{\vec{H}} \\ \nabla \cdot \tilde{\vec{H}} &= 0 \\ \nabla \times \tilde{\vec{H}} &= j\omega\varepsilon \tilde{\vec{E}} \end{aligned}\right\}$$

No charge, $\rho_v = 0$.

lossless. $\sigma = 0$

$(\vec{J} = 0)$.

- In a lossy medium, $\sigma \neq 0$, not perfect insulator

$\sigma \neq \infty$, not perfect conductor.

Then $\vec{J} \neq 0$ if $\vec{E} \neq 0$. Since $\vec{J} = \sigma \vec{E}$.

(The medium will consume energy)

Then: $$\nabla \times \tilde{\vec{H}} = (\sigma + j\omega\varepsilon)\tilde{\vec{E}}$$
$$= j\omega\left(\frac{\sigma}{j\omega} + \varepsilon\right)\tilde{\vec{E}}$$
$$= j\omega\left(\varepsilon - j\frac{\sigma}{\omega}\right)\tilde{\vec{E}}$$

can be written as:

$$\nabla \times \vec{H} = j\omega\varepsilon_c \tilde{\vec{E}} \qquad (7.63)$$

$$\varepsilon_c \equiv \varepsilon - j\frac{\sigma}{\omega}$$ — complex permitivity.

$$\equiv \varepsilon' - j\varepsilon''$$

$$\varepsilon' = \varepsilon$$
$$\varepsilon'' = \frac{\sigma}{\omega}.$$

For lossless medium: $\sigma = 0$, $\varepsilon_c = \varepsilon$