Phys 221

Lecture 25.

- Phasor equations in charge-free, lossy medium:

$$\left(\rho_v = 0, \quad \sigma \neq 0, \quad \vec{J} = \sigma \vec{E} \neq 0 \right)$$

$$\nabla \cdot \tilde{\vec{E}} = 0$$

$$\nabla \times \tilde{\vec{E}} = -j\omega\mu \tilde{\vec{H}}$$

$$\nabla \cdot \tilde{\vec{H}} = 0$$

$$\nabla \times \tilde{\vec{H}} = j\omega \varepsilon_c \tilde{\vec{E}}$$

$$\varepsilon_c = \varepsilon' - j\varepsilon'' \qquad \left\{ \begin{array}{l} \varepsilon' = \varepsilon = \varepsilon_0 \varepsilon_r \\ \varepsilon'' = \dfrac{\sigma}{\omega} . \end{array} \right.$$

(For lossless, $\sigma = 0$, $\vec{J} = 0$, we have $\varepsilon_c = \varepsilon$. i.e. $\varepsilon'' = 0$)

- Wave equations: for lossy medium.

$$\nabla^2 \tilde{\vec{E}} - \gamma^2 \tilde{\vec{E}} = 0 . \qquad (7.70)$$

$$\nabla^2 \tilde{\vec{H}} - \gamma^2 \tilde{\vec{H}} = 0 \qquad (7.71).$$

$$\gamma^2 = -\omega^2 \mu \varepsilon_c$$

- Plane wave in lossy medium.

  - Relationship between $\vec{E}$ and $\vec{H}$.

$$\tilde{\vec{H}} = \frac{1}{\eta_c} \hat{k} \times \tilde{\vec{E}}$$

($\vec{k}$ — wavevector : direction — propagation direction
magnitude — $k = \frac{2\pi}{\lambda}$.
λ — wavelength in the medium.)

η_c — intrinsic impedance

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon'}} \left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-\frac{1}{2}} \qquad (7.125)$$

- γ, α, β.

$$\gamma = \alpha + j\beta$$

γ — propagation constant.
β — phase constant
α — attenuation constant.

$$\gamma^2 = -\omega^2 \mu \varepsilon_c . \qquad \varepsilon_c = \varepsilon' - j\varepsilon''$$

Q: How do we determine α and β from ε', ε'', μ.?

$$\gamma^2 = -\omega^2\mu(\varepsilon' - j\varepsilon'')$$

$$= -\omega^2\mu\varepsilon' + j\omega^2\mu\varepsilon''$$

Also: $\gamma^2 = (\alpha + j\beta)^2$

$$= \alpha^2 - \beta^2 + j\alpha\beta\cdot 2.$$

$$\therefore \begin{cases} \alpha^2 - \beta^2 = -\omega^2\mu\varepsilon' \\ 2\alpha\beta = \omega^2\mu\varepsilon'' \end{cases}$$

Solve for α, β:

$$\left.\begin{aligned} \alpha &= \omega\left\{\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1\right]\right\}^{1/2} \\ \beta &= \omega\left\{\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1\right]\right\}^{1/2} \end{aligned}\right\} \quad (7.121)$$

p. 225

- plane wave solution (polarized along $\hat{x}$)

$$\tilde{\vec{E}} = \hat{x}\, E_{x0}\, e^{-\gamma z} = \hat{x}\, E_{x0}\, e^{-\alpha z} \cdot e^{-j\beta z}$$

$$\tilde{\vec{H}} = \hat{y}\, \frac{E_{x0}}{\eta_c}\, e^{-\alpha z}\, e^{-j\beta z}$$

Note: Since η_c is complex in lossy medium,

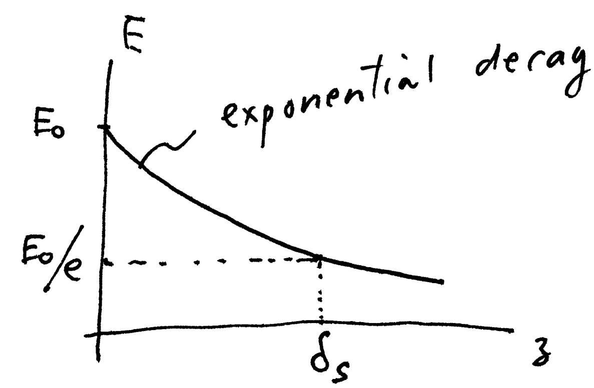
$\vec{H}$ and $\vec{E}$ are not in phase.

i.e. There is a phase shift between $\vec{H}$ and $\vec{E}$ in a lossy medium.

($\vec{H}$ and $\vec{E}$ are in phase in lossless media)

- Attenuation factor.

$$e^{-\alpha z}$$



Skin depth (penetration depth).

$$\delta_s = \frac{1}{\alpha}$$

at $z = \delta_s = \frac{1}{\alpha}$, $\quad E = E_0\, e^{-\frac{\alpha}{\alpha}} = E_0\, e^{-1} = \frac{E_0}{e}$.

$$= 37\%\ E_0.$$

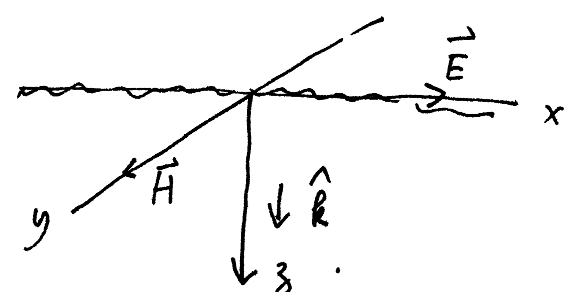
- Table 7-2 (p.228): α, β, η_c, u_p. ($\lambda = u_p/f$)

1. General (any medium)
2. lossless $\sigma = 0$.
3. low-loss $\varepsilon''/\varepsilon' \ll 1$. ($10^{-2}$ or smaller)
4. good conductor: $\frac{\varepsilon''}{\varepsilon'} \gg 1$. ($10^{2}$ or greater)

Note: $\varepsilon'' = \sigma/\omega$ — depends on frequency!

- e.g. plane wave in sea water.

$\varepsilon_r = 80$,
$\mu_r = 1$
$\sigma = 4$ S/m



$\vec{H}(0,t) = \hat{y}\, 100 \cos(2\pi \times 10^3 t + 15°)$ (mA/m)

(a). Find $\vec{E}(z,t)$ and $\vec{H}(z,t)$

(b) Find $z = d$ so that $\dfrac{\tilde{E}_0(d)}{\tilde{E}_0(0)} = 1\%$ (amplitude is 1% of its value at $z=0$)

(a). Since $\hat{k} = \hat{z}$, $\hat{H} = \hat{y}$.
$\hat{E} = \hat{x}$.

a). $\tilde{\vec{E}}(z) = \hat{x}\, E_{x0}\, e^{-\alpha z - j\beta z}$

$\tilde{\vec{H}}(z) = \hat{y} \frac{E_{x0}}{\eta_c} e^{-\alpha z} e^{-j\beta z}$.

Need to determine α, β, η_c:

Since $\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega \varepsilon} = \frac{4}{2\pi \times 10^3 \times \varepsilon_0 \varepsilon_r}$

$= 9 \times 10^{9} \gg 1$.

$\therefore$ Sea water is good conductor (at $f = 1\,KHz$).

(but, for $f = 10^{15}$ (visible light),

$\frac{\varepsilon''}{\varepsilon'} = 9 \times 10^{-16} \ll 1$.

low-loss!

Transparent!)

$\therefore\ \alpha = \sqrt{\pi f \mu \sigma} = 0.126$

$\beta = \alpha = 0.126$. ($\alpha = \beta$ for good conductors)

$\eta_c = (1+j)\frac{\alpha}{\sigma} = \sqrt{2}\, e^{j\frac{\pi}{4}} \frac{0.126}{4} = 0.044\, e^{j\frac{\pi}{4}}$.

Note: (1) E_{x0} is a complex constant. $E_{x0} = |E_{x0}| e^{j\phi_0}$.

(2) $e^{j\frac{\pi}{4}}$ in η_c means the phase shift between $\vec{H}$ and $\vec{E}$ is $45°$.

$$\therefore\quad \vec{E}(z,t) = Re\left[\hat{x}\, E_{x0}\, e^{-\alpha z}\, e^{-j\beta z} \cdot e^{j\omega t}\right]$$

$$= \hat{x}\,|E_{x0}|\,e^{-0.126z} \cos\left(2\pi \times 10^3 t - 0.126z + \phi_0\right)$$

$$\vec{H}(z,t) = Re\left[\hat{y} \cdot \frac{|E_{x0}|}{0.044\, e^{j\pi/4}}\, e^{-\alpha z}\, e^{-j\beta z}\, e^{j\omega t}\right]$$

$$= \hat{y}\,\frac{|E_{x0}|\, e^{-0.126z}}{0.044} \cos\left(2\pi \times 10^3 t - 0.126z + \phi_0 - \frac{\pi}{4}\right)$$

at $z=0$: $$\vec{H}(0,t) = \hat{y}\,\frac{|E_{x0}|}{0.044} \cdot \cos\left(2\pi \times 10^3 t + \phi_0 - 45^\circ\right)$$

$$= \hat{y} \cdot 100 \cos\left(2\pi \times 10^3 t + 15^\circ\right)$$

$$\therefore\quad \frac{E_{x0}}{0.044} = 100, \qquad \phi_0 - 45^\circ = 15^\circ$$

$$E_{x0} = 4.4\ (V/m) \qquad \phi_0 = 60^\circ$$

$$\therefore\quad \vec{E}(z,t) = \hat{x}\, 4.4\, e^{-0.126z} \cos\left(2\pi \times 10^3 t - 0.126z + 60^\circ\right)$$

$$\vec{H}(z,t) = \hat{y}\, 100\, e^{-0.126z} \cos\left(2\pi \times 10^3 t - 0.126z + 15^\circ\right)$$