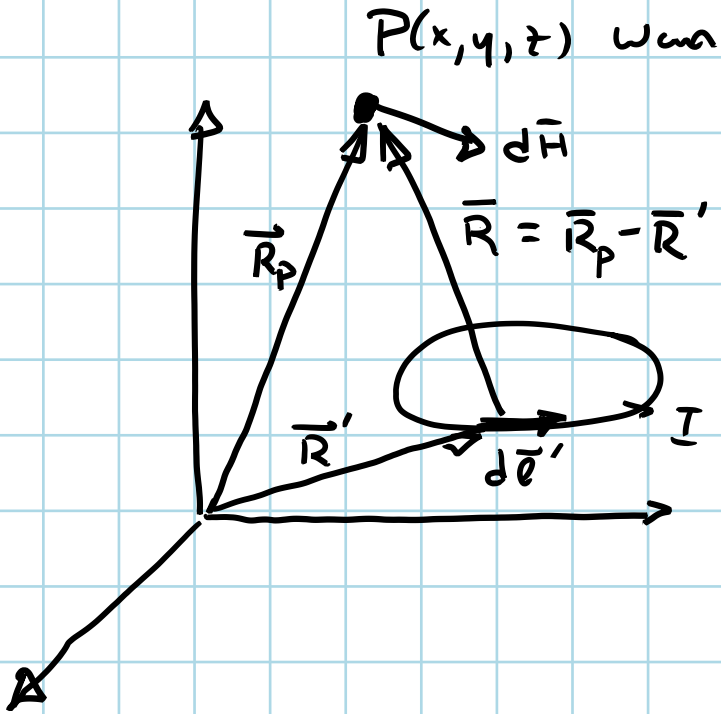
⊗ Magnetic field intensity $\vec{H}$ due to a steady current:

$$\vec{H} = \frac{1}{4\pi} I \int \frac{d\vec{\ell}' \times \hat{R}}{R^2}$$

in isotropic linear medium ($\Rightarrow \vec{B} = \mu \vec{H}$)



Magnetic fields $\vec{H}$, $\vec{B}$:

$\vec{H}$: "magnetic field intensity"
$[H] = A/m$

$\vec{B}$: "magnetic flux density"
$[B] = T$ tesla

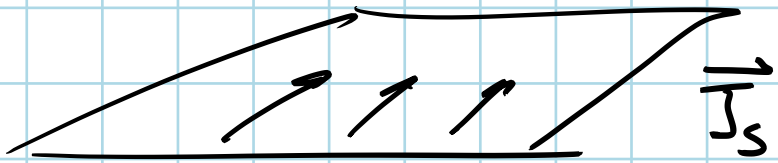
⊗ Similar to Coulomb's law in electrostatics.

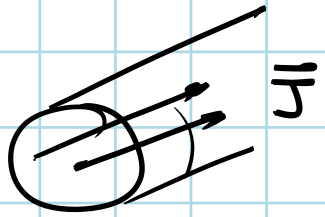
⊗ Generalize to surface and volume current distributions

$$I d\vec{\ell} = \vec{I}\, d\ell = \vec{J}_s\, ds = \vec{J}\, dV$$

So $$\vec{H} = \frac{1}{4\pi} \int_S \frac{\vec{J}_s(\vec{R}') \times \hat{R}}{R^2}\, ds' \qquad \text{surface current}$$

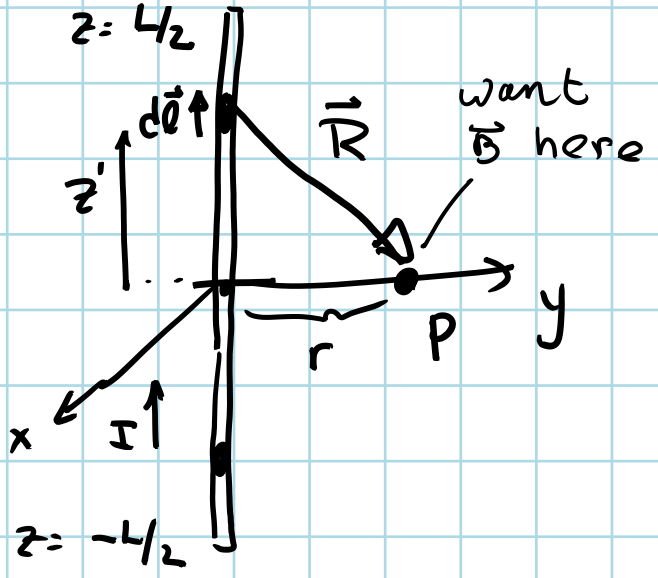
$$= \frac{1}{4\pi} \int_V \frac{\vec{J}(\vec{R}') \times \hat{R}}{R^2}\, dV' \qquad \text{volume current}$$





⑧ Biot-Savart Examples

Example 5-2



$$I(z) = I \qquad -L/2 < z < \frac{L}{2}$$
$$= 0 \qquad \text{otherwise}$$

$B(z=0) = ?$ in free space $(\mu = \mu_0)$

By symmetry $\vec{H}(z=0) = \vec{H}(r)$ magnitude indep of ϕ

$$\vec{H} = \frac{1}{4\pi} \int I \frac{d\vec{\ell}' \times \hat{R}}{R^2} = \frac{1}{4\pi} \int I \frac{d\vec{\ell}' \times \vec{R}}{R^3}$$

$$R = \left(r^2 + (z')^2\right)^{1/2} \qquad \vec{R} = -z'\hat{z} + r\hat{r}$$

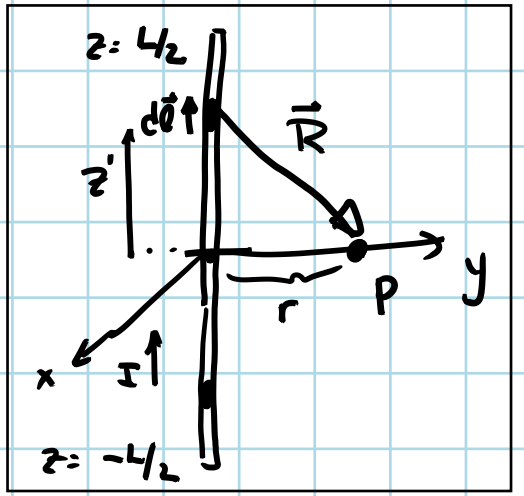
$$\therefore \; d\vec{\ell}' \times \vec{R} = \left(dz'\hat{z}\right) \times \left(-z'\hat{z} + r\hat{r}\right)$$

$$= r\,dz' \underbrace{\hat{z} \times \hat{r}}_{\hat{\phi}}$$

$$\therefore \quad \vec{H} = \hat{\phi}\,\frac{1}{4\pi} I \int_{-L/2}^{L/2} \frac{r\,dz'}{\left[r^2 + (z')^2\right]^{3/2}}$$

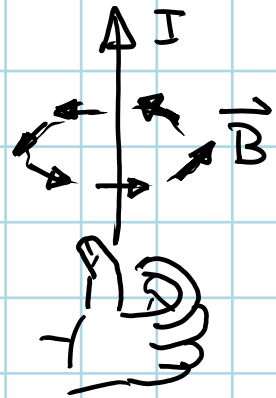
$$\frac{r z'}{r^2\left[r^2 + (z')^2\right]^{1/2}}\Bigg|_{-L/2}^{+L/2} = \frac{L}{r\left(r^2 + \frac{L^2}{4}\right)^{1/2}}$$

$$\boxed{\vec{B} = \mu_0 \vec{H} = \hat{\phi}\,\frac{\mu_0 I}{4\pi}\,\frac{L}{r\left(r^2 + \frac{L^2}{4}\right)^{1/2}}}$$



∞ wire : $L \to \infty$?

$$\vec{B} = \hat{\phi}\,\frac{\mu_0 I}{4\pi}\,\frac{L}{r\,L/2} = \boxed{\hat{\phi}\,\frac{\mu_0 I}{2\pi r}} \propto \frac{1}{r}.$$



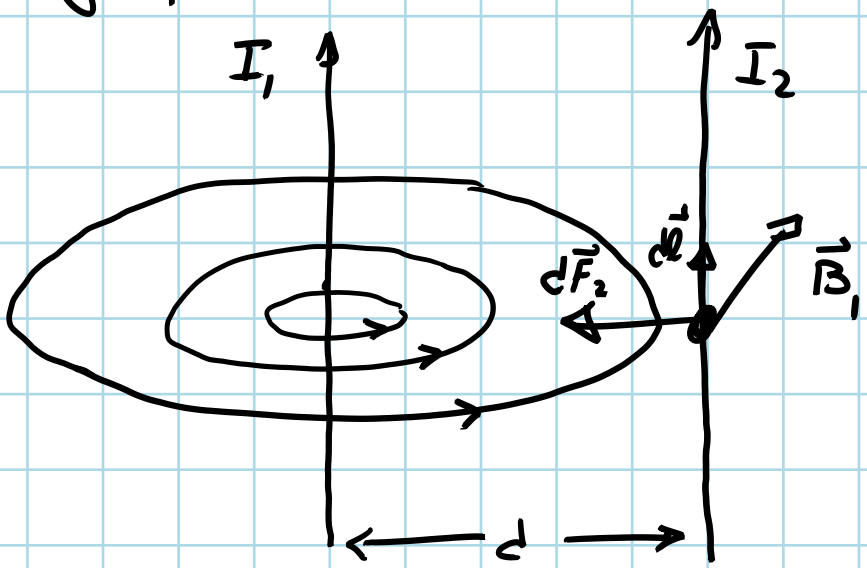
⊗ Magnetic force between 2 long parallel wires.

$\vec{B}$ due to wire 1:

$$\vec{B}_1 = \frac{\mu_0 I_1}{2\pi r}\hat{\phi}$$

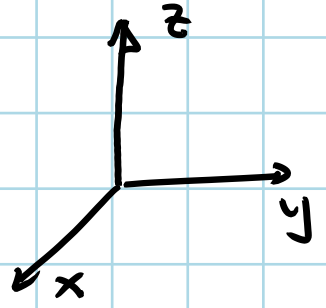
@ wire 2:

$$\vec{B}_1 = \frac{\mu_0 I_1}{2\pi d}(-\hat{x})$$



$$d\vec{F}_{2,1} = I_2\, d\vec{\ell}' \times \vec{B}_1 = \frac{\mu_0 I_1 I_2}{2\pi d} dz' \underbrace{\left(\hat{z}\times(-\hat{x})\right)}_{-\hat{y}}$$

on 2 due to 1



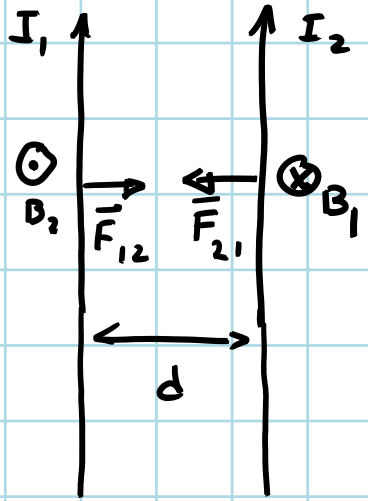
$$\therefore \quad \vec{F}_{21} = -\hat{y}\,\frac{\mu_0 I_1 I_2}{2\pi d}\underbrace{\int dz'}_{\ell}$$

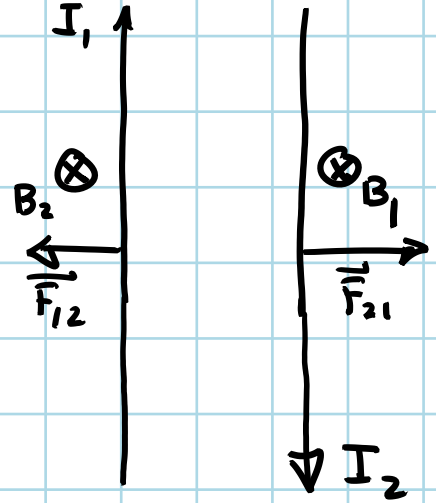
Force per unit length : $\overline{F_{21}}' = \frac{F_{21}}{l} = \underbrace{-\hat{y}}_{\text{attractive}} \frac{\mu_0 I_1 I_2}{2\pi d}$.

⊗ Check that $\overline{F_{12}} = +\hat{y}\frac{\mu_0 I_1 I_2}{2\pi d} = -\overline{F_{21}}$

($\overline{F_{12}}$: on 1 due to 2)

(in agreement with Newton's 3rd law).





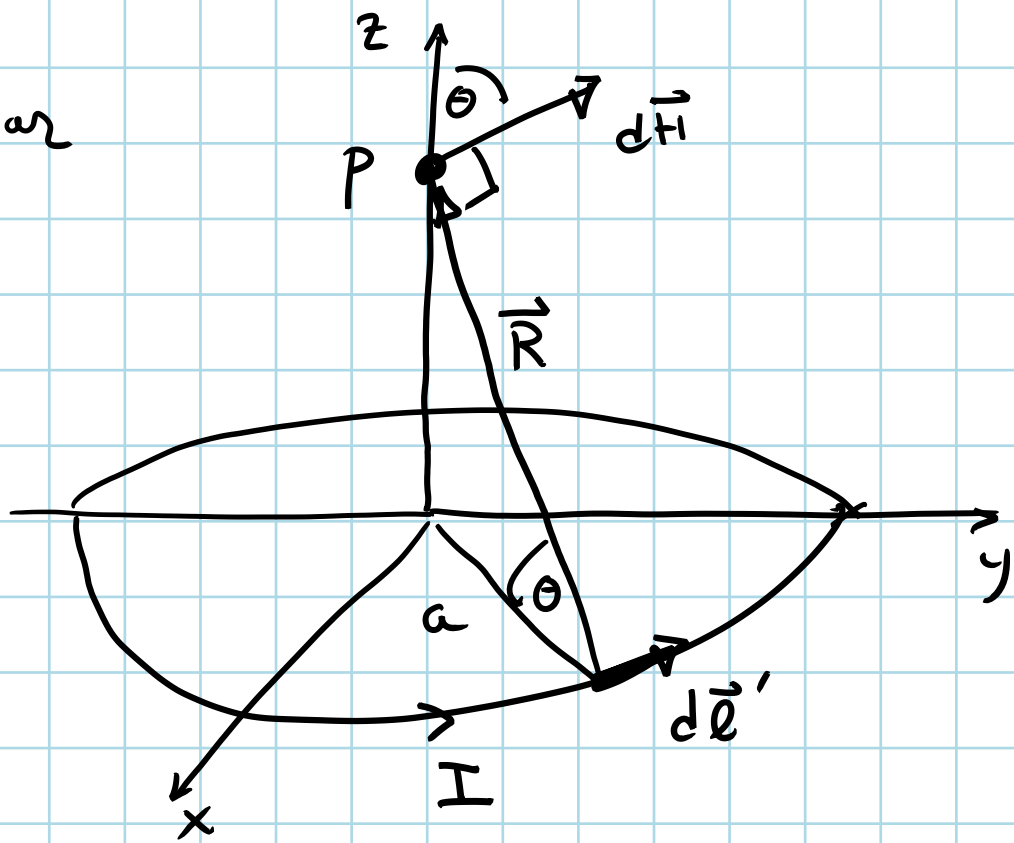
⊗ If currents opposite, force repulsive..

⊗ Ex 5-4 !

Magnetic field $\bar{H}$ of a circular loop at pt P on axis

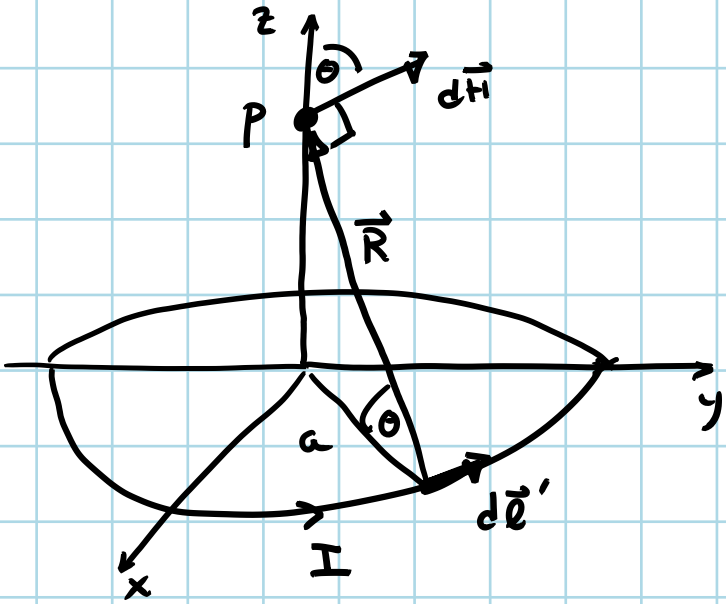
- radius a
- centered at (0,0,0)
- current I

$$\vec{H} = \frac{1}{4\pi}\int \frac{I\, d\vec{\ell}' \times \hat{R}}{R^2}$$



⊗ By symmetry, only $\hat{z}$ component of $\bar{H}$ survives integral around loop (H_z)

$$d\vec{H} = \frac{I}{4\pi} \frac{d\vec{\ell}' \times \hat{R}}{R^2}$$



$d\vec{\ell}' \perp \hat{R}$ everywhere

$$\therefore dH = \frac{I}{4\pi} \frac{d\ell'}{R^2}$$

$\hat{z}$ component : $dH_z = dH \cos\theta$

$$\therefore H_z = \frac{I}{4\pi} \int \frac{d\ell' \cos\theta}{R^2}$$

$$R^2 = a^2 + z^2 = \text{Const}$$

$$d\ell' = a\, d\phi$$

$$\therefore H_z = \frac{I}{4\pi} \frac{a\cos\theta}{R^2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} = \frac{I}{2} \frac{a\cos\theta}{R^2}$$

$$\cos\theta = \frac{a}{R} \quad \therefore \quad \boxed{\vec{H} = H_z \hat{z} = \hat{z} \frac{I}{2} \frac{a^2}{(a^2+z^2)^{3/2}}}$$

⊗ Some limits :

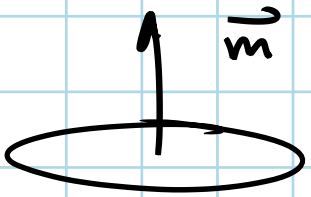
① Center of loop $z = 0$

$$\vec{H} = \hat{z}\,\frac{I}{2}\,\frac{a^2}{a^3} = \hat{z}\,\frac{I}{2a}$$

② Far away $|z| >> a$

$$\vec{H} \approx \hat{z}\,\frac{I}{2}\,\frac{a^2}{|z|^3} \propto \frac{1}{(\text{dist})^3}$$

⊗ Can rewrite in terms of the loop's magnetic moment :



$$m = AI = \pi a^2 I.$$

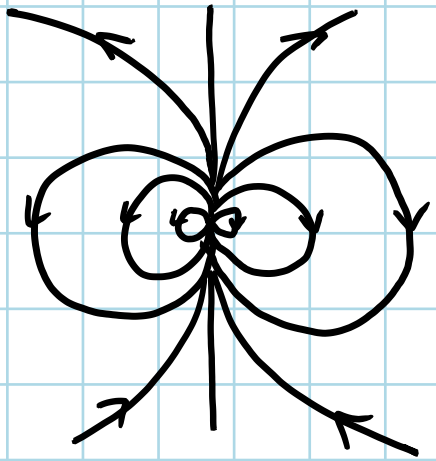
$$\vec{H}_{\text{on axis}} = \hat{z}\,\frac{m}{2\pi}\,\frac{1}{|z|^3} \propto \frac{1}{(\text{dist})^3}$$

⊗ In general, it can be shown that for a small dipole ($a \to 0$ with $m = \text{const.}$) or far away:

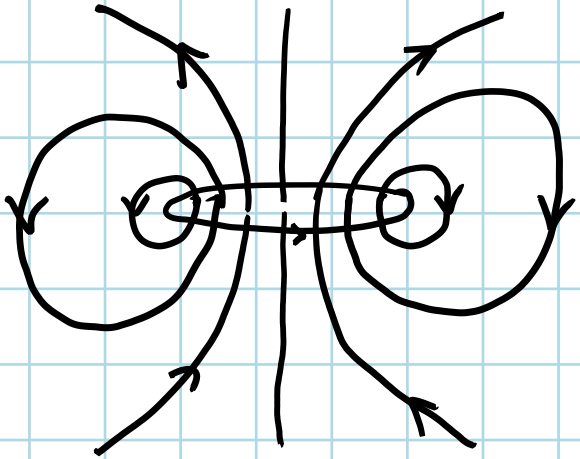
$$\vec{H} = \frac{1}{4\pi} \frac{m}{R^3} \left(2\cos\theta\, \hat{R} + \sin\theta\, \hat{\theta} \right)$$

ideal magnetic dipole.

Ideal Dipole

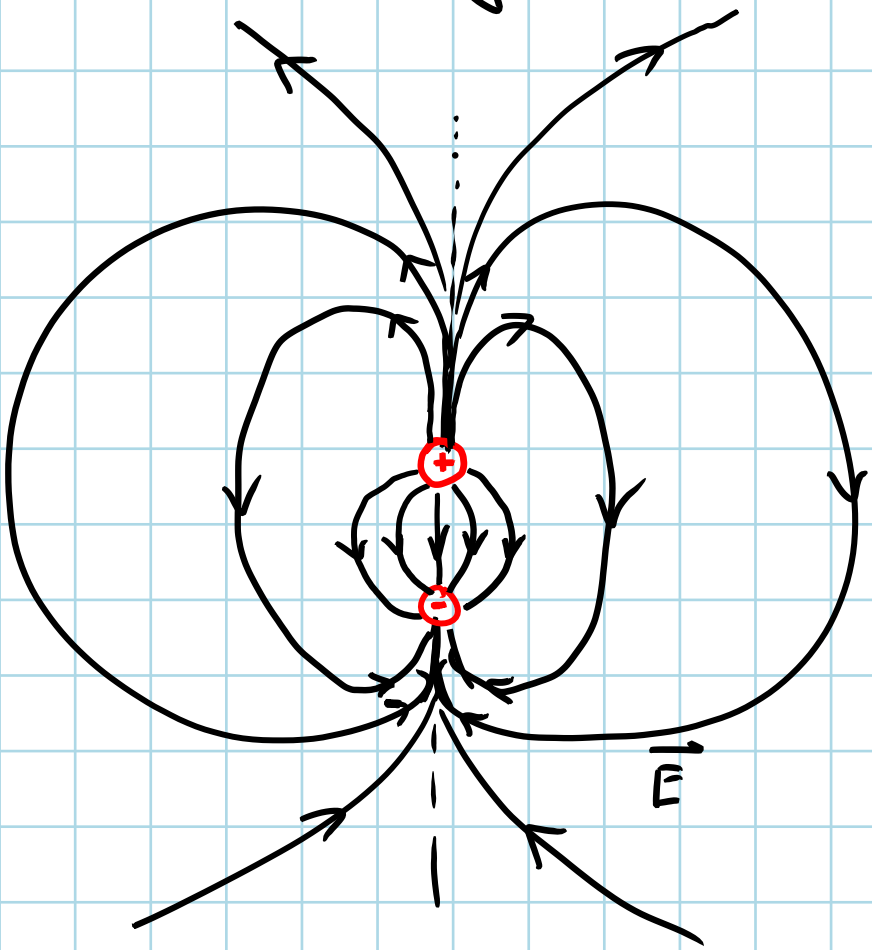


Real dipole.

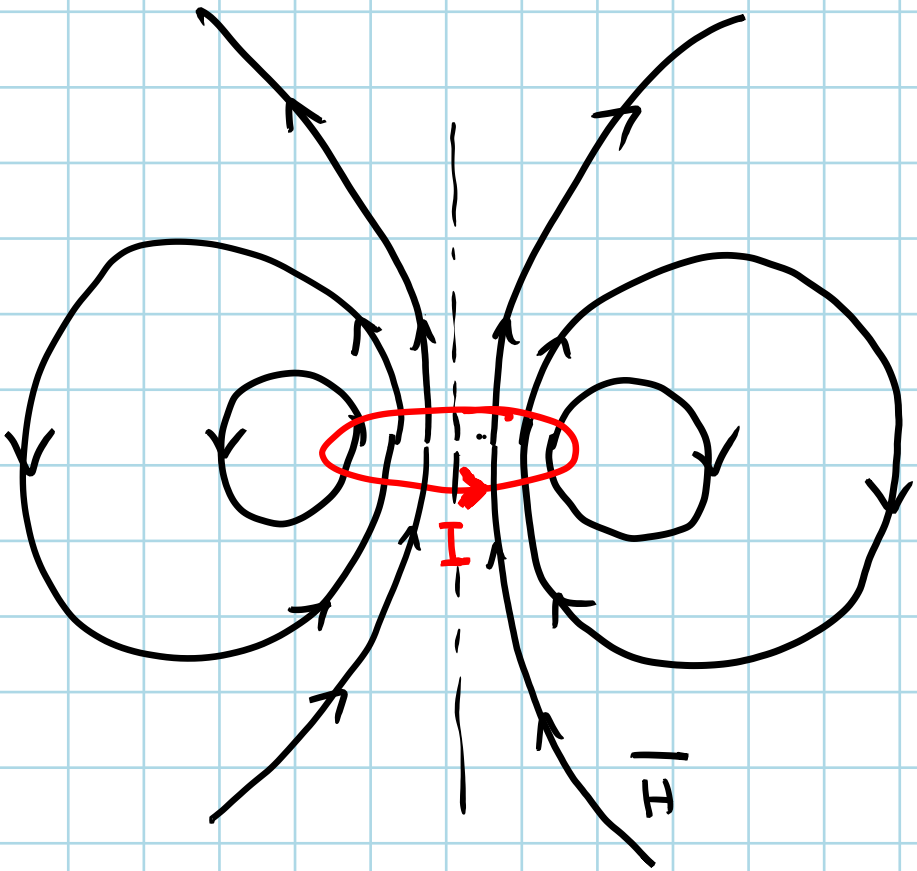


⊗ For enough away, the $\vec{B}, \vec{H}$ fields match.

⊗ Pictorially, compared to electric dipole.



electric dipole



magnetic dipole.

⊗ similar shape far away

(In close, different e.g. $\vec{E}$ & $\vec{H}$ pt. opposite inside)