I. Notes for Heat Transfer Portion of Thermal Physics PHYS 344-3

The next three lectures in this course are going to be about heat transfer. Heat transfer is not the same thing as thermodynamics (which is what you've been learning about in the course so far), but is closely related to it.

Three lectures is enough to get an overview of heat transfer, but not to cover the subject in depth. These notes will summarise the most important points; I will include references to textbooks which can provide a more detailed treatment should you ever need it. You can take the notes as defining the syllabus for this portion of the course. The lectures will not necessarily cover all the material in the notes; the aim of the lectures is to show how the methods of heat transfer can be applied in practice.

There is a problem set which you should attempt to verify that you have understood the material.

To introduce the subject of heat transfer, consider a thermodynamics problem:

A *solar pond* is a technology for capturing solar energy as heat. We take a body of salt water and put 1 m of fresh water on top of it. Absorbed solar radiation raises the temperature of the water, just as it does in a regular pond. In a regular pond, however, as soon as the sub-surface water gets hotter than the water at the surface, its density falls below that of the surface water and it rises to the surface, where it is cooled by evaporation. In a solar pond, the salt water always remains denser than the surface water, no matter how hot it gets. It can lose heat only by conduction through the surface water, which isn't a particularly good conductor.

After the solar pond reaches equilibrium, the top of the salt water may be close to boiling: 92°C, say. Now we can extract energy from the pond by running a heat engine between the hot water and cold water from the depths of the pond (or from a nearby non-solar pond or river.) Assuming the cold water is at 19°C, and assuming the heat engine we use can achieve 50% of Carnot efficiency, at what rate must we extract heat energy from the hot water to deliver 1 MW of mechanical power?

(Write your answer here: ....W)

That was a thermodynamics problem. To design a thermal-pond-based power station, we have to solve several other problems: for example, how do we get heat *into* our heat engine at the rate you've calculated? Thermodynamics is silent on this; it concerns only the fraction of heat that can be turned into work at given source and sink temperatures, but says nothing about the rate at which the heat can be transferred, or the mechanism via which it is transferred.

The solar pond provides examples of two major mechanisms of heat transfer covered in these notes: conduction and convection. It also includes a third important heat transfer mechanism, radiation, which we won't have time to study. (A full analysis would have to include evaporative cooling, which is an example of heat transfer with change of phase, and does not fit into any of the three mechanisms I've mentioned; we won't have time to study that either.)

Let me give another example of a problem which can only be solved using the methods of heat transfer: consider a semi-conductor device on a circuit board. This device is approximately a square, 1 cm on a side, and it dissipates 20 W of heat. What temperature is it going to run at?

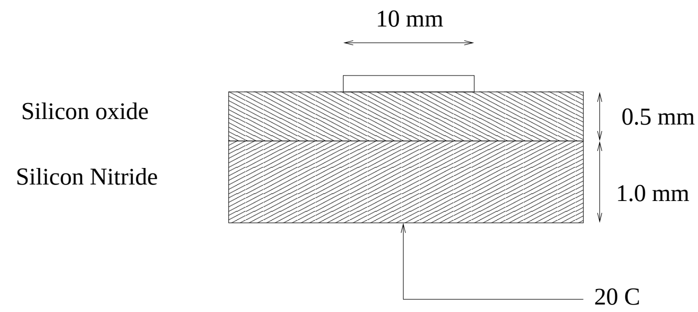

Fig. 1. Board-mounted Semi-Conductor

By the end of this week, you should be able to calculate an approximate solution to this problem.

A. *Conduction*

(This material will not be explicitly covered in the lectures; it is given here for your reference. In order to solve some problems of practical interest in class, I will need to assume that you have read through it at least once. You won't be examined on it, though.)

Conduction is the transport of heat through a material medium independently of the macroscopic motion of the medium. We won't be looking at what happens on the microscopic scale in any detail; we can note in passing that in fluids, the responsible mechanism is the exchange of energy in inter-molecular collisions, whereas in solids, there are two possible mechanisms: transport of energy as vibrations in the solid lattice; and, in conductors, transport of energy by the motion of electrons in the conduction band.

We can calculate everything we need to know about conduction from Fourier's Law of Heat Conduction:

$$Q = -kA\frac{\partial T}{\partial \eta} \tag{1}$$

in words, the rate of heat flow through an area A is proportional to the gradient of the change in temperature along the direction η normal to that area. The constant of proportionality, k, is the *conductivity* of the material.

Some things to note about this equation: firstly, it can only be applied when $\partial T/\partial \eta$ has the same value over the whole area A. Secondly, note the '$-$' sign in front of the differential; this tells us that heat flows in the opposite direction to the positive temperature gradient.

B. *Fourier's Law Applied to a Simple Problem*

Consider the circuit element discussed above. Assume for the present that its surroundings are as shown in Figure 2.

The circuit element dissipates $Q = 20$W of heat. The thickness of the silicon oxide is $d_1 = 0.5$mm, and that of the silicon nitride is $d_2 = 1$mm. We will assume that heat losses to the air are negligible. The lower surface of the silicon nitride substrate is held at $T_2 = 20°$C, and the circuit has been on long enough that it's reached steady state. The thermal conductivity of silicon nitride is $k_2 = 20$ W/m.K, and that of the silicon oxide is $k_1 = 1$ W/m.K. We want to find T_1, the temperature of the circuit element.

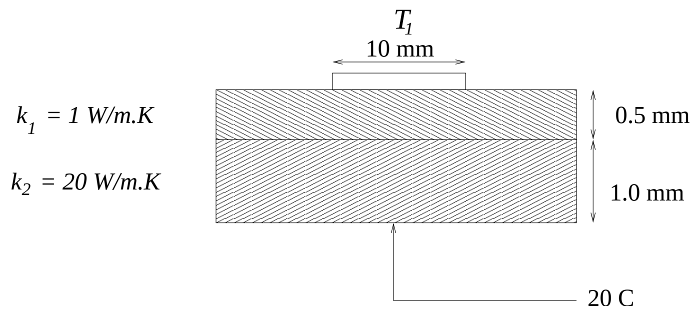

Fig. 2. Circuit Element and its Surroundings

1) Solution: Since the device is in steady state, the heat flow through any horizontal cross-section of the board must be the same, i.e., 20 W. The area of each cross-section is also constant. We can deduce that the temperature gradient, dT/dx, is constant within each material, though it will be steeper in one material than in the other.

To solve the problem, let us assign the temperature T_i to the interface between the materials. Then we can write the following equation for the silicon nitride:

$$Q = -k_2 A \frac{T_2 - T_i}{d_2} \tag{2}$$

$$\Rightarrow T_i = \frac{Qd_2}{k_2 A} + T_2 \tag{3}$$

$$\Rightarrow T_i = 30 \tag{4}$$

We use the same reasoning to find T_1, which turns out to be 130 °C.

2) Heat Flow in a Hollow Cylinder: Using Fourier's Law of Heat Conduction, we can deduce an expression for the temperature distribution and heat flow through a hollow cylinder. This will be useful for a number of practical problems.

Consider a hollow cylinder, or a pipe carrying a hot or cold fluid, as shown in Figure 3.

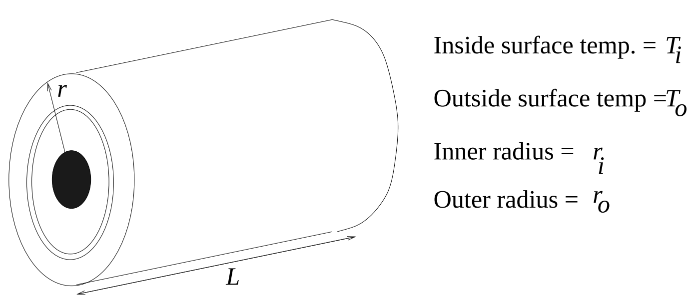

Fig. 3. Hollow Cylinder containing Hot Fluid

The cylinder is L meters long. Its inner radius is r_i meters, the outer radius r_o. Correspondingly, the temperatures of the inner and outer surfaces are T_i and T_o respectively. Let us suppose that the system is in a steady state. We can write down the FLHC:

$$Q = -kA_r \frac{dT}{dr} \tag{5}$$

Now, since the system is in a steady state, the heat flow through any cylindrical shell must be the same as that through any other. In particular, the flow through the shell at r must be the same as that through the shell at $r + \delta r$:

$$Q_r = Q_{r+\delta r} \tag{6}$$

But we can also write

$$Q_{r+\delta r} = Q_r + \frac{d}{dr} Q \delta r \tag{7}$$

Hence,

$$\frac{dQ}{dr} = 0 \tag{8}$$

We substitute $A_r = 2\pi r L$ into the FLHC to get

$$\frac{-2\pi k d}{dr}\left(\frac{r dT}{dr}\right) = 0 \tag{9}$$

So the relevant differential equation is

$$\frac{d}{dr}\left(\frac{r dT}{dr}\right) = 0 \tag{10}$$

subject to the boundary conditions that $T = T_i$ at $r = r_i$ and $T = T_o$ at $r = r_o$. It is straightforward to integrate this equation and obtain

$$T(r) = T_i - (T_i - T_o)\frac{\ln(r/r_i)}{\ln(r_o/r_i)} \tag{11}$$

Getting an expression for the heat flux through the cylinder is even easier. We start from FLHC, substitute $A_r = 2\pi r L$ and obtain

$$Q \int_{r_i}^{r_0} \frac{dr}{r} = -2\pi k L \int_{T_i}^{T_o} dT \tag{12}$$

which gives us

$$Q \ln\left(\frac{r_o}{r_i}\right) = -2\pi k L (T_o - T_i) \Rightarrow Q = \frac{2\pi k L (T_i - T_o)}{\ln(r_o/r_i)} \tag{13}$$

These formulae will come in useful in problems later on.

C. Conduction in Three Dimensions

We can use Fourier's Law of heat conduction to deduce an equation describing the temperature field in three dimensions. Consider a region of space bounded by the planes $x = x_0$, $x = x_0 + \Delta x$, $y = y_0$, $y = y_0 + \Delta y$, $z = z_0$, and $z = z_0 + \Delta z$. Heat is flowing into and out of this region, driven by local variations in temperature. The heat flowing in through the plane $x = x_0$ is given by

$$Q_{x,in} = -k\Delta y\Delta z \frac{\partial T}{\partial x}\bigg|_{x=x_0} \tag{14}$$

The heat flowing out through the plane $x = x_0 + \Delta x$ is given by

$$Q_{x,out} = -k\Delta y\Delta z \frac{\partial T}{\partial x}\bigg|_{x=x_0+\Delta x} \tag{15}$$

which, to first order, may be written as

$$Q_{x,out} = -k\Delta y\Delta z \frac{\partial T}{\partial x}\bigg|_{x=x_0} + \Delta x \frac{\partial}{\partial x}\left(k\Delta y\Delta z \frac{\partial T}{\partial x}\right) \tag{16}$$

So, subtracting what goes out from what comes in, we get

$$Q_{x,net} = \Delta x\Delta y\Delta z \frac{\partial}{\partial x} k \frac{\partial T}{\partial x} \tag{17}$$

It's possible that k could change with position in the material. This could happen, for example, if the chemical composition of the material varied, or if conductivity were a function of temperature. If k may be regarded as constant, we can write

$$Q_{x,net} = \Delta x\Delta y\Delta z k \frac{\partial^2 T}{\partial x^2} \tag{18}$$

We can get two similar equations for $Q_{y,net}$ and $Q_{z,net}$. Adding them altogether, we get

$$Q_{net} = \Delta x\Delta y\Delta z k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{19}$$

which we can write as

$$Q_{net} = \Delta V k \nabla^2 T \tag{20}$$

This influx of heat necessarily produces a change in the temperature of the material within the volume ΔV. In addition to the heat flowing in from outside, the temperature of the material might also be changed by internal heat generation – this might happen, for example, in a sub-critical mass of U_{235}, or a slice of pizza in a microwave oven. Suppose the internally generated heat amounts to q Watts per unit volume. Then the change in temperature over time produced by the inflowing heat and the internally generated heat is $\partial T/\partial t$, where

$$Q_{net} + q\Delta V = \Delta V \rho c \frac{\partial T}{\partial t} \tag{21}$$

where ρ is the material density and c is its specific heat. Substituting from Equation 20, we get

$$\Delta V k \nabla^2 T + q \Delta V = \Delta V \rho c \frac{\partial T}{\partial t} \tag{22}$$

Hence, dividing by $k\Delta V$,

$$\nabla^2 T + \frac{q}{k} = \frac{\rho c}{k}\frac{\partial T}{\partial t} \tag{23}$$

This is the General Heat Conduction equation. The quantity $k/\rho c$ is sometimes referred to as the *diffusivity* of the material and denoted by the symbol α. In the absence of internal heat generation, the GHC equation becomes Fourier's equation:

(24)

If we allow internal heat generation, but insist that the temperature not change with time, we get Poisson's equation:

$$\nabla^2 T + \frac{q}{k} = 0 \tag{25}$$

Lastly, if there is neither internal heat generation nor change with time, we get Laplace's equation:

$$\nabla^2 T = 0 \tag{26}$$

Now that we have this impressive array of equations, what can we do with them? As it turns out, we can only solve the equations analytically for rather simple problems – generally, those characterised by a high degree of symmetry. The best reference for analytic methods of solution is 'Conduction of Heat in Solids' by Carlslaw and Jaeger, Oxford University Press. If we have to deal with a problem involving complex geometry, or non-uniform boundary conditions, or materials whose properties depend on temperature, we must fall back on numerical methods of solution. Two of the most popular numerical methods are *finite differences* and *finite elements*. Covering these methods is beyond the scope of this course. Myers, 'Numerical Methods in Heat Transfer' is a good reference for both methods; Ferziger, 'Numerical Methods for Engineering Application', published by John Wiley, is good for the finite difference method.

II. Convection Heat Transfer

Convection is the transfer of heat by the macroscopic motion of a fluid. We know the fundamental equations describing the motion of fluids – the Navier-Stokes equations – and solving these equations would allow us to calculate the rate of convective heat transfer in any given situation. Unfortunately, the equations are very difficult to solve. For a few cases where the geometry of the problem is simple – for example, fluid flowing through a pipe or over a flat plate – solutions can be found, but most problems cannot be solved analytically at all. We cope with this by introducing semi-empirical relationships. These are equations with a limited domain of validity, derived either from lengthy algebraic analysis or from experiment. In the sections that follow, we will introduce several such equations and illustrate their use.

We distinguish two forms of convection: *natural* convection and *forced* convection. If a body and a fluid at different temperatures are brought into contact, the fluid immediately adjacent to the body exchanges heat with it, and consequently changes in density. Being at a different density than the fluid around it, it either rises or sinks, its place being taken by fresh fluid. In this way, a flow establishes itself in the fluid, the effect of which is to increase the rate of heat transfer between the body and the fluid. This is *natural* convection.

Alternatively, some outside agency, such as a fan, may *force* the fluid to flow at a given rate, again increasing the rate of heat transfer over what it would be for a stationary fluid. This is *forced* convection. Typically, heat transfer rates associated with forced convection are considerably higher than those associated with natural convection. Most of our discussion will be concerned with forced convection.

A. Newton's Law of Cooling

The theory of forced convection is founded on Newton's observation that the rate of cooling of a heated body exposed to a moving fluid is proportional to the temperature difference between the body and the fluid. We can represent this as

$$Q \propto \Delta T \tag{27}$$

where ΔT is the difference in temperature between the surface of the body and the bulk of the fluid. If we add the observation that the rate of cooling is also proportional to the exposed surface area A of the body, we can write this in a more useful form as:

$$Q = hA\Delta T \tag{28}$$

where h is a constant of proportionality known as the 'convective heat transfer coefficient' or 'film heat transfer coefficient' To solve any problem involving convection, then, the first thing we need to do is to find h. Unfortunately, this is not straightforward. h depends in a complicated way on the rate of flow, the geometry of the problem and the physical properties of the fluid, in particular, its conductivity, its specific heat, its density and its viscosity. It is *not* a property of the heated body, or of the fluid. To understand how we can get a value for h, we need to introduce some facts about fluid flow:

B. Basic Facts about Fluid Flow

All you would ever want to know about fluid flow is contained in the Navier-Stokes equations, which, as mentioned above, are insoluble for almost all real problems. For our present purposes, we can get by with some simple observations. We start with an experiment performed by Professor Reynolds of Manchester University, UK, and shown in Figure 4.

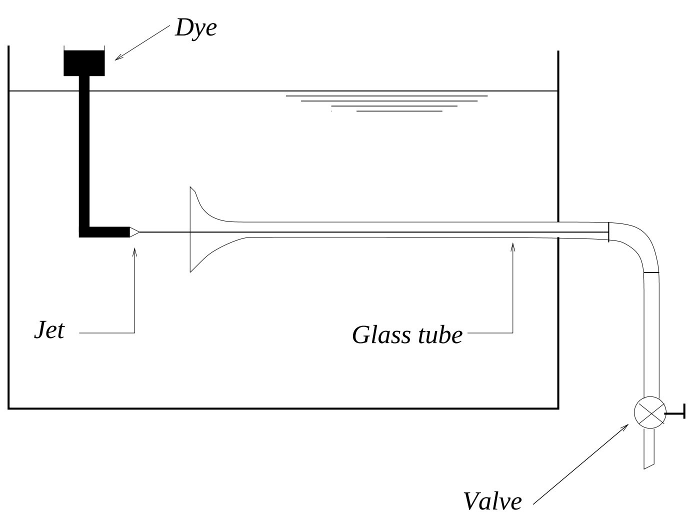

Fig. 4. Reynolds' Apparatus

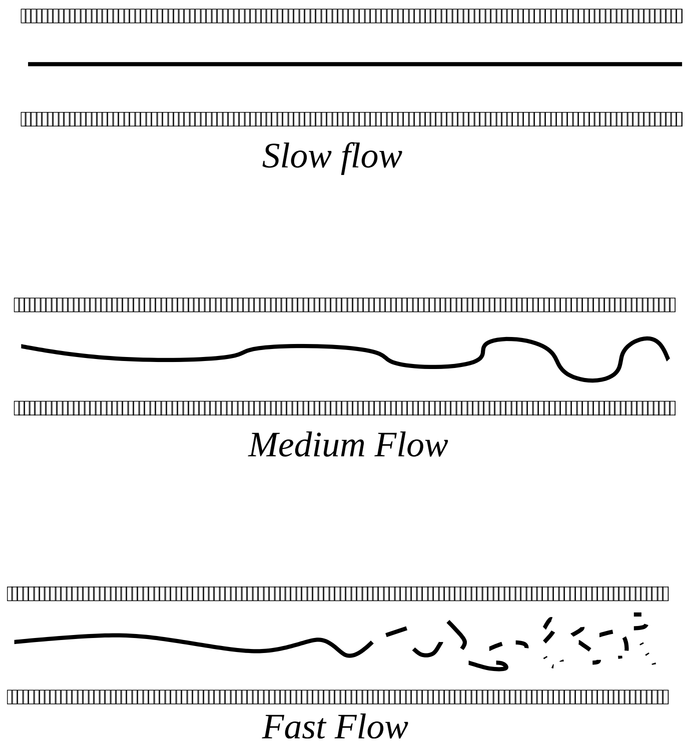

Fig. 5. Reynolds' Results

Having set up this apparatus, Reynolds opened the valve to various extents, and observed the pattern of flow through the pipe, as revealed by the dye. As he gradually increased the flow rate, he saw the series of transitions shown in Figure 5.

At the lowest flow rates, the dye would remain as a single thread in the water, no noticeable mixing taking place. As the flow rate increased, a value was reached at which the thread began to waver some distance down the tube. At higher flow rates, the fluctuations grew more intense, until at some point there was a position in the tube at which the dye mixed, thoroughly and quite suddenly, with the flowing water. Further increases in velocity led to the dye becoming mixed even more readily and closer to the tube inlet.

On the basis of this and similar experiments, we distinguish two kinds of fluid flow: *laminar* and *turbulent* flow. In the early stages of Reynolds's experiment, the fluid behaves as though it is made up of many thin films, or *laminae*, formed into nested cylinders concentric with the tube and slipping over each other. A particle in one lamina never passes into an adjacent lamina; thus the thread of dye never mixes with the rest of the fluid. Not all laminae move at the same speed; all fluids except superfluid He_4 have *viscosity*, which may be thought of as the 'stickiness' of the laminae. Thus the lamina closest to the tube wall is stuck to it, and doesn't move at all; the next one in moves a little faster, and the highest velocity is attained in the centre of the tube.

At a certain, rather sharply defined critical velocity, this picture ceases to apply. Mixing occurs between adjacent layers of the fluid; the fluid particles take on a random motion, superimposed on the flow of the fluid along the pipe. We say that the flow has become turbulent.

Which do you think would give a higher rate of heat transfer, laminar flow or turbulent flow?

1) Dynamic Similarity and Dimensionless Numbers: We might repeat Reynolds's experiment many times, using different diameters of tube, different flow velocities and fluids of different viscosity, and each time noting where the transition from laminar to turbulent flow occurred. We would like to find a law to summarise and predict the results of these experiments. The *fundamental* physical laws governing what's going on are captured by the Navier-Stokes equations, but solving these to predict how the flow will develop in each case is impossibly difficult. We can, nevertheless, achieve some generality by applying the principle of *dynamic similarity*.

Two situations are *dynamically similar* if the forces acting in each situation are in the same ratio. The transition from laminar to turbulent flow in a liquid occurs when the pressure forces, which are responsible for the fluid's motion, come to predominate over the viscous forces. For a given geometry, this transition always occurs when the forces are in a fixed ratio, independently of the scale of the experiment.

The ratio of pressure forces to viscous forces is given by *Reynold's number*. For fluid flowing in a pipe, this can be calculated to be

$$Re = \frac{\rho d u}{\mu} \tag{29}$$

where ρ is the fluid density, d is the tube diameter, u is the fluid velocity and μ is its viscosity (This is derived in rather more detail in the Appendix.). Re is a *dimensionless* number, since it is a ratio of two forces. Thus it will take the same value in a given situation no matter what system of units is used [as long as the system is consistent]. In a pipe, the transition from laminar to turbulent flow occurs at about $Re = 2300$. For other geometries, we may choose the characteristic dimension differently, and the transition to turbulent flow may occur at another value of Reynolds number. For example, if we are considering flow over a flat plate, we take the characteristic dimension as the distance from the leading edge of the plate. Using a Reynold's number based on this dimension, we find that the transition to turbulent flow occurs at a critical value of about 5×10^5.

At this stage, it is helpful to consider an example:

Example 2.1

Water is flowing through a pipe, inside diameter 2.3 mm. At what velocity does the flow become turbulent? How does the answer change if we consider a pipe 30 mm in diameter? How about if the pipe is carrying glycerin rather than water?

Solution: We assume everything is happening at room temperature, about 20°C. At this temperature the viscosity of water is $\mu = 0.001$Nsm^{-2} and its density is about 1000 kg.m^{-3}. The corresponding figures for glycerin are about 1.3 Nsm^{-2} and 1260 kg.m^{-3} respectively. Thus Reynold's number for water is given by

$$Re = \frac{1000 \times 0.0023 \times u}{0.001} \tag{30}$$

where u is the flow rate in ms^{-1}. Setting Re to the critical value of 2300 and solving for u, we find the transition occurs at a flow velocity of 1 ms^{-1}. For the wider-diameter pipe, the flow becomes turbulent at a much lower velocity, 0.1 ms^{-1}. For glycerin, the calculation tells us that flow in the narrow pipe remains laminar up to a flow velocity of about 1000 ms^{-1}. (Although if we actually tried to pump glycerin through a pipe at such a high velocity, fluid friction would rapidly raise its temperature, reducing the viscosity and invalidating our calculations.)

2) Fluid Flow and Heat Transfer: Before we can represent the relationship between flow and heat transfer, we have to find a suitable dimensionless expression for heat transfer rate. We use *Nusselt's number*:

$$Nu = \frac{hd}{k} \tag{31}$$

which represents the ratio between convective heat transfer and heat transfer by conduction through the fluid. We will also need one more dimensionless number, the *Prandtl number*:

$$Pr = \left(\frac{\rho c}{k}\right)\frac{\mu}{\rho} \tag{32}$$

which is the ratio of the kinematic viscosity of the fluid, μ/ρ, to its diffusivity, $\alpha = k/\rho c$. The Prandtl number is a property of the fluid. It may be thought of as a measure of how rapidly the fluid dissipates momentum compared with how rapidly heat diffuses through the fluid. A high value of Pr, such as oil ($Pr \approx 10,000$), favours heat transfer by convection, whereas a low value, as is found in liquid metals ($Pr \approx 0.01$), favours heat transfer by conduction. Pr for gases is typically about 1. Now all we need is some means of calculating the Nusselt number in any given situation.

For simple geometries, we can solve the governing equations to determine heat transfer rates. This involves lengthy calculation. For more complicated geometries, we can only determine heat transfer rates by experiment. Considerable effort is involved in either case. If we express our results as correlations between non-dimensional groups, we can use these correlations to solve classes of dynamically similar problems with minimal additional effort.

In this course, we will not derive any correlations ourselves. Such derivations can be found in, for example, *Engineering Heat Transfer* by Karlekar and Desmond, West Publishing Company, 1977, pp. 335-342. Instead, we will give examples of the correlations that have been derived for a few simple problems, and show how they are used.

Let us begin by considering flow through a pipe. For heat to be transferred from the pipe to the fluid, there must be a temperature difference between them. We could maintain such a temperature difference by several methods: for example, we could put the pipe at the focus of a single-axis solar concentrator. This would give us a constant heat flux into the pipe. Alternatively, we could heat the pipe by allowing steam to condense on it; this would keep the pipe surface at a constant temperature. The method we use makes a difference; the heat transfer equations have a different solution for a constant flux boundary condition than for a constant temperature boundary condition. Working through the derivation shows that, for laminar flow in a pipe, the Nusselt number is constant, being 4.364 for constant flux and 3.658 for constant wall temperature.

If the Nusselt number is constant, this means that the heat transfer rate is the same for any fluid velocity, up to the critical value of Re. In particular, it's the same when the fluid is moving as when it's stationary. But when the fluid is stationary, the only available mechanism for heat transfer is conduction. Hence we can say that, for laminar flow in a tube, heat is transferred only by conduction.

This seems a little confusing. We've said that Nu is constant, and that this shows that heat is being transferred only by conduction. Yet Nu is non-zero, and therefore h, the convective heat transfer coefficient, is also non-zero. But I introduced h as a measure of *convection*, not conduction. So is convective heat transfer taking place or not?

In fact, heat transfer from a solid body to a fluid must *always* be at least partially due to conduction, because the layer of fluid immediately adjacent to the body is always stationary with respect to the body. (A convincing demonstration of this is the dust on the blades of a fan. If you examine the blades of a fan, you will find a thin layer of dust particles resting on the surface. These particles remain in place even when the fan is rotating, because they lie within the stationary boundary layer.)

The parameter h includes *both* the heat transferred by conduction and the heat transferred by motion of the fluid, if any. As we shall shortly see, in turbulent flow the rate of heat transfer can be many times greater than that due to conduction alone. (Nevertheless, a fluid with zero conductivity could transfer no heat, however turbulent the flow; so in this sense, it is not entirely accurate to think of convection and conduction as independent heat transfer mechanisms.)

For turbulent flow in a pipe, the Nusselt number becomes a function of the flow rate. There are a variety of alternatives for the exact form of this correlation, some based on experiment, some based on approximate analytical solutions. For illustrative purposes, we shall just consider one correlation, suggested by Colburn[1]

$$Nu = 0.023 Re^{0.8} Pr^{0.33} \tag{33}$$

Qualitatively, this tells us that in turbulent flow the rate of heat transfer goes up in almost direct proportion to the flow rate.

We can now illustrate the use of these correlations with two examples:

In the example of the solar pond we considered above, you calculated that we needed to get 10 MW of heat into our heat engine to put out 1 MW of work. Now, our heat engine uses freon as a working fluid, and the freon flows through a plastic pipe suspended in the hot salt water. The pipe is 10^5 m long, 20 mm internal diameter, 21 mm external diameter, and the freon is flowing at 1 m/s. If we want heat to flow from the water into the freon, the freon must be cooler than the water. How big a temperature difference is required if we are to get 10 MW flowing into the freon? (Obviously, we want this difference to be as small as possible, since it's the freon temperature that corresponds to the hot reservoir temperature in our calculation of Carnot efficiency.)

Suppose the bulk temperature of the freon is T_f and the temperature on the inside of the plastic pipe is T_i. Then, since we have a steady state,

$$Q = hA(T_i - T_f) = \frac{2\pi kL(T_i - T_o)}{\ln(r_o/r_i)} \tag{34}$$

We will use these two equations to calculate the two unknowns, T_f and T_i. First, we use the formula for conduction of heat through a tube to find T_i, noting that the conductivity of the plastic tube is 0.15 W/mK:

$$T_i = T_o - \frac{Q\ln(r_o/r_i)}{2\pi kL} = 92 - \frac{10 \times 10^6 \times \ln(21/20)}{2\pi \times 0.15 \times 10^5} \approx 92 - 5 = 87 \tag{35}$$

Now we go on to find the bulk temperature of the freon. Before we can do that, though, we must calculate h... and in order to calculate h, we must calculate Nu...for which we need Re. We will also need to know the relevant physical properties of freon. They are as follows: $k = 0.073$ W/mK; $\rho = 1364$ kg.m^{-3}; $\mu = 0.273 \times 10^{-3}$ Nsm^{-2}; and $Pr = 3.5$. Now, the mass flow rate, G, is just the flow velocity multiplied by the density:

$$G = u\rho = 1 \times 1364 = 1364\mathrm{kg/m^2s^{-1}} \tag{36}$$

Hence,

$$Re = \frac{Gd_h}{\mu} = \frac{1364 \times 0.02}{0.273 \times 10^{-3}} \approx 100,000 \tag{37}$$

So the flow is turbulent and we can use Colburn's correlation:

$$Nu = 0.023Re^{0.8}Pr^{0.33} = 0.023 \times 100,000^{0.8} \times 3.5^{0.33} \approx 350 \tag{38}$$

From which we can get h:

$$h = \frac{kNu}{d} = \frac{0.073 \times 350}{0.02} \approx 1280\mathrm{W/m^2K} \tag{39}$$

Now at last we get to the answer:

$$T_f = T_i - \frac{Q}{hA} = 87 - \frac{10^7}{1280 \times 10^5 \times \pi \times 0.02} \approx 87 - 1 = 86^{\circ}\mathrm{C} \tag{40}$$

So the bulk temperature of the freon is 86°C. We could now go back and recalculate the Carnot efficiency, and hence recalculate Q and repeat the calculation we've just completed to get a slightly better estimate of T_f; in this case, the improvement would not justify the effort, so we won't bother.

C. Log Mean Temperature Difference

In the preceding example, we assumed that the freon was at a single temperature, T_f, all through the pipe. This can't be quite right – after all, the reason we have the freon flowing through the pipe in the first place is in order to *change* its temperature. So in fact it will go in at one temperature, T_{ci}, say, and come out at another temperature T_{co}. Now, the question is, what temperature do we substitute into the convective heat transfer equation in order to get the right heat flow rate? If we use T_{ci} we'll overestimate the heat flow, while if we use T_{co} we'll underestimate it. Obviously some kind of average is needed. Call this average T_a, where by definition

$$Q = hA(T_i - T_a) \tag{41}$$

Let's sketch the temperature profiles through the pipe, assuming the inner wall temperature is constant. The temperature rise of the freon isn't linear (in fact it's negative exponential), so a simple arithmetic average isn't going to be right.

Consider a small increment of tube length Δx.

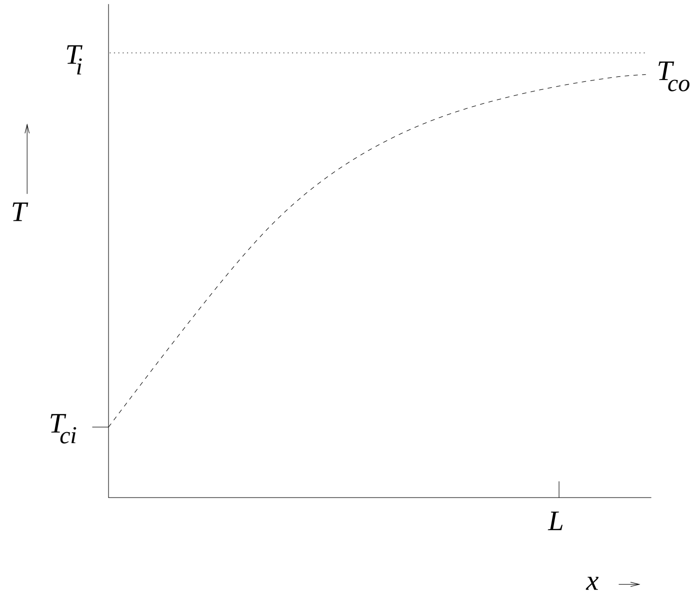

Fig. 6. Temperature Profile of Flow in Tube

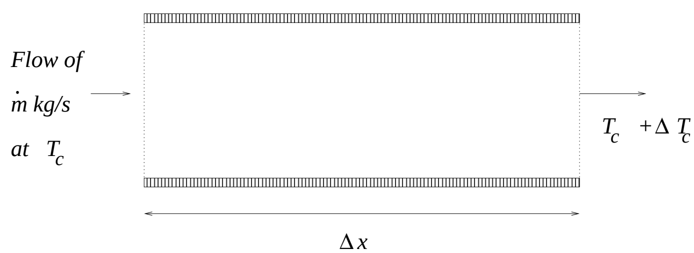

Fig. 7. Small Segment of Tube

Taking an energy balance over Δx, we have

$$hp\Delta x(T_h - T_c) = \dot{m}c_p\Delta T_c \tag{42}$$

$$\Rightarrow \dot{m}c\frac{dT_c}{dx} = hp(T_h - T_c) \tag{43}$$

$$\Rightarrow \frac{d(T_h - T_c)}{T_h - T_c} = \frac{-hp}{\dot{m}c_p}dx \tag{44}$$

$$\Rightarrow \ln\frac{T_h - T_{co}}{T_h - T_{ci}} = \frac{hLp}{\dot{m}c_p} = \frac{hA}{\dot{m}c_p} \tag{45}$$

If we take an energy balance for the freon, we have

$$Q = \dot{m}c_p(T_{ci} - T_{co}) \tag{46}$$

Using this to substitute for $\dot{m}c_p$, we get

$$Q = hA\frac{T_{co} - T_{ci}}{\ln((T_h - T_{co})/(T_h - T_{ci}))} \tag{47}$$

Comparing this with the definition of T_a gives us

$$T_h - T_a = \frac{T_{co} - T_{ci}}{\ln((T_h - T_{ci})/(T_h - T_{co}))} \tag{48}$$

This expression is known as the *log mean temperature difference*, and should be used for calculating heat transfer to a fluid flowing inside a duct of constant wall temperature. (If the wall temperature is also changing along the length of the duct, as it might in some heat exchangers, we can extend the above analysis to take a varying T_h into account too. The details of this extension are left as an exercise for the reader.)

D. Second Example: Cooling of Circuit Element

Reconsider the microcircuit element we examined while discussing conduction. The element is a square, surface area $A = 0.0001\text{m}^2$, dissipating $Q = 20$ Watts of heat. Let us now suppose that the element is mounted on a circuit board, 40 cm from the edge, and that a fan is circulating air at $T_3 = 15°\text{C}$ over the board at 50 m/s. We can recalculate the temperature of the element, T_1, taking convective cooling into account.

Reynolds number for a flat plate geometry is calculated in terms of the distance, x, from the leading edge of the board. (We will add the subscript x to Re to remind us of this fact.) Under laminar flow conditions, the Nusselt number a distance x from the leading edge is given by

$$Nu = \frac{hx}{k} = 0.331 Re_x^{0.5} Pr^{0.333} \tag{49}$$

The transition from laminar to turbulent flow occurs at $Re_x \approx 5 \times 10^5$. For turbulent flow conditions, the appropriate correlation is

$$Nu = 0.029 Re_x^{0.8} Pr^{0.43} \tag{50}$$

(It's worth noting that these are *empirical* relationships, obtained through curve-fitting to experimental measurements or by approximate solution of the governing equations; thus there may exist several alternative correlations for a given geometry, varying slightly in conditions of applicability. Where several alternatives exist, I have chosen the simplest.)

1) Solution:: We first calculate the value of Reynolds number for the air flow:

$$Re_x = \frac{\rho x u}{\mu} = \frac{1 \times 0.4 \times 50}{2 \times 10^{-5}} = 10^6 \tag{51}$$

So the flow is turbulent, and we can use the latter correlation to find Nu and h:

$$Nu = 0.029 \times (10^6)^{0.8} \times 0.7^{0.43} = 1570 \tag{52}$$

and hence

$$h = \frac{Nu \times k}{x} = \frac{1570 \times 0.03}{0.4} = 117 \quad \text{W/m}^2\text{.K} \tag{53}$$

To find T_1, the temperature of the element, we can now consider the following equivalent circuit:

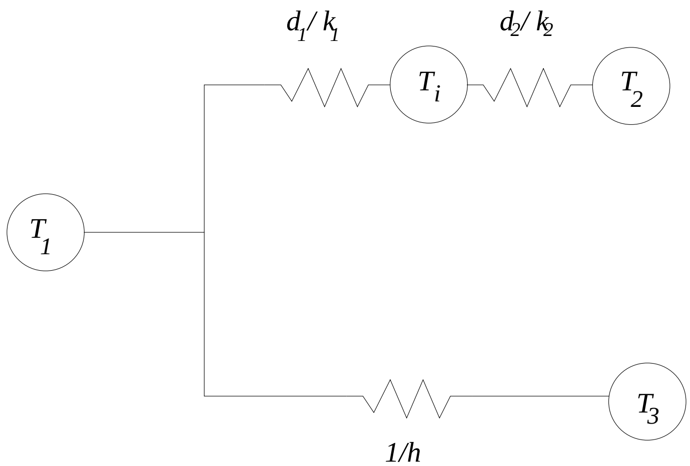

Fig. 8. Equivalent Circuit

This gives us a set of simultaneous equations,

$$Q_1 = -k_2 A \frac{T_2 - T_i}{d_2} = -k_1 A \frac{T_i - T_1}{d_1} \tag{54}$$

$$Q_2 = hA(T_1 - T_3) \tag{55}$$

$$Q_1 + Q_2 = Q = 20 \quad \text{Watts} \tag{56}$$

After some algebra, this gives us

$$T_1 = \left(Q/A + hT_3 + \frac{k_1 k_2 T_2}{k_1 d_2 + k_2 d_1} \right) \left(\frac{k_1 d_2 + k_2 d_1}{k_1 k_2 + h k_1 d_2 + h k_2 d_1} \right) \tag{57}$$

Substituting in values and evaluating gives us that $T_1 = 122$°C. Note that this is only 8° cooler than we achieved by pure conduction. We could cool more efficiently using a circulating liquid coolant in place of the cooling air; the Cray 2, for example, has its circuits immersed in circulating freon.

E. Problem Set : Conduction and Convection

1) Figure 9 shows a frustum, that is, a truncated cone, made of pure copper. The smaller end, A, is held at a temperature T_A, while the larger end, B, is held at a lower temperature T_B. The curved surfaces of the frustum are insulated. Which of the three sketches (i), (ii) and (iii) shows the approximate temperature distribution in the frustum?
2) Figure 10 shows a cylindrical rod of cavorite. One end, A, is held at a temperature T_A, while the other end, B, is held at a lower temperature T_B. The curved surfaces of the rod are insulated. If the thermal conductivity of cavorite increases with temperature, which of the three sketches (i), (ii) and (iii) shows the approximate temperature distribution in the rod?
3) Figure 11 shows a cylindrical rod of plutonium. The smaller end, A, is held at a temperature T_A, while the larger end, B, is held at a lower temperature T_B. The curved surfaces of the rod are insulated. q watts of heat are generated per cubic meter of plutonium as a result of radioactive decay. Which of the three sketches (i), (ii) and (iii) is *not* a possible temperature distribution in the rod?

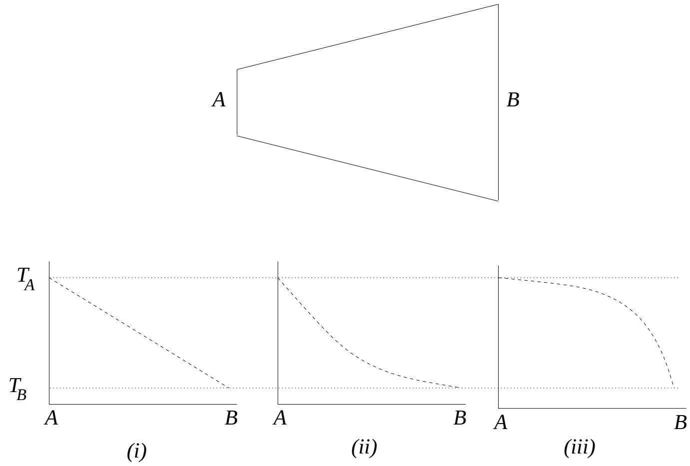

Fig. 9. Problem 1

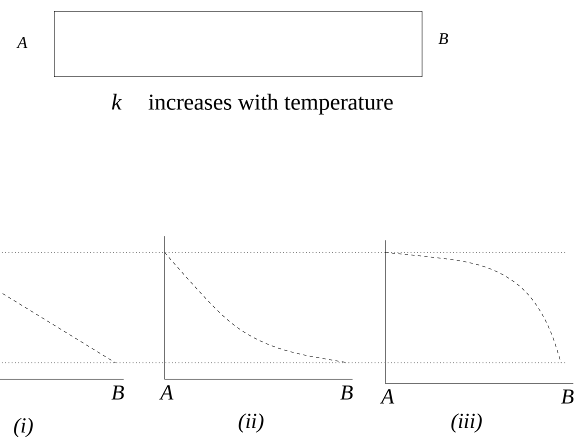

Fig. 10. Problem 2

4) An insulating wall is composed of 10 cm of a material having an thermal conductivity of 5 W/m.K, and an unknown thickness of cork, which has a thermal conductivity of 0.045 W/m.K. How thick must the cork be if the heat losses through the wall are to be less than 10W/m^2 when the inside temperature is 20°C and it's -20°C outside?
5) Water flows through a tube of 5 mm diameter. Determine the flow rate corresponding to a Reynolds number of 100. Is this flow turbulent or laminar? If the tube is heated at a rate of 10 W/m length, what is the appropriate correlation between Reynolds number and Nusselt number? What is the value of the Nusselt number, and what is the value of h, the convective heat transfer coefficient? If the water enters the tube at 5°C, how long must the tube be if the water is to exit at a temperature of 25°C? What is the average temperature difference between the tube walls and the water? (Density of water = 1000 kg/m^3; viscosity of water = 0.001 Nsm^{-2}, specific heat of water = 4,200 J/kg.K; conductivity of water = 0.628 W/m.K)
6) Water is flowing through a copper pipe. The flow is turbulent, and the inner wall of the copper pipe is maintained at 95°C. The water enters the pipe at about 5 °C, and leaves at about 6°C. If the flow rate of the water is doubled, will the exit temperature of the water increase, decrease or stay the same?
7) Water is flowing through a brass pipe. The flow is turbulent, and the inner wall of the brass pipe is maintained at 95°C. The water enters the pipe at about 5 °C, and leaves at about 6°C. If we put the same number of kg/s of water through a different brass pipe, having half the inner diameter but being in all other respects identical, will the exit temperature of the water increase, decrease or stay the same?

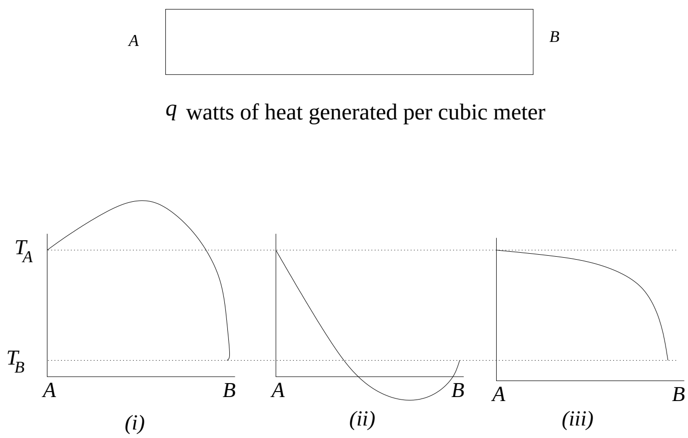

Fig. 11. Problem 3

8) Water is flowing through an iron pipe. The flow is turbulent, and the inner wall of the iron pipe is maintained at 95°C. The water enters the pipe at about 5 °C, and leaves at about 6°C. Suppose we now replace the water with oil, flowing at the same number of kg/s, but having a density of 800 kg/m^3, thermal conductivity of 0.132 W/m.K, viscosity of 0.0045 Nsm^{-2}, and specific heat of 300 J/kg.K. Assume that the oil is also in the turbulent flow regime. Compared with the exit temperature of the water, will the exit temperature of the oil be lower, higher, or exactly the same?
9) Air flows through a tube of 20-mm internal diameter at an average velocity of 1 m/s. The bulk temperature of the air is 10°C at the entrance to the tube. The tube wall is maintained at 250 °C. Estimate the length of tube necessary to heat the air to a bulk temperature of 50°C. (Density of air = 1 kg/m^3; viscosity of air = 2×10^{-5} Nsm^{-2}; conductivity of air = 0.03 W/m.K; specific heat of air = 1000 J/kg.K).
10) Heat exchangers are devices for transferring heat from one stream of fluid to another. They can, for example, take the form of concentric pipes, with one fluid flowing in the outer pipe, the other in the inner pipe.
    In a *counter-flow* heat exchanger, the hot fluid enters on the left and exits on the right, while the cool fluid enters on the right and leaves on the left. In a *parallel-flow* heat exchanger, both fluids enter at the left and exit on the right.
    For fixed flow rates and fixed entry temperatures, in which of these two designs will the initially cool fluid have the higher exit temperature?

III. Appendix: Deriving Reynold's Number

The explanation given in the main body of these notes glosses over several points. The following account is more rigorous.

Any body acted upon by n forces can be represented by a force polygon with $n+1$ sides. The first n sides of the polygon represent the magnitude and direction of each of the n imposed forces; the $n+1$-th side represents the *resultant* of these forces. By a slight abuse of terminology, we can refer to the resultant as the *inertia force*: if the body accelerates as a result of the forces acting on it, its acceleration is given by

$$F = ma \tag{58}$$

where F is the resultant of the forces, a is the acceleration of the body and m is its mass, or inertia.

The pressure forces may be of various kinds – for example, they may be imposed by gravity, or by a pump, or, in magnetohydrodynamics, by an imposed field – so it is convenient to instead concentrate on the ratio between the *inertia force* and the viscous force. The inertia force, from Equation 58, is given by the particle's mass multiplied by its acceleration. We will try to express this in terms of units taken from the problem, using the pipe diameter, d, as the unit of length, the fluid velocity, u, as the unit of velocity, and their ratio, $t = d/u$, as the unit of time. The particle's mass may be taken as proportional to the fluid density, ρ, multiplied by the characteristic volume, d^3. Its acceleration is the rate at which its velocity changes with time,

$$a = u/t = u^2/d \tag{59}$$

Thus the inertia force is

$$F = ma = (\rho d^3)(u^2/d) = \rho d^2 u^2 \tag{60}$$

The shear stress resulting from viscosity is given by the product of viscosity μ and the rate of shear; this is proportional to $\mu u/d$. This stress acts over an area proportional to d^2, giving a viscous force proportional to $\mu u d$. The ratio of forces is therefore

$$Re = \frac{\rho d^2 u^2}{\mu u d} = \frac{\rho d u}{\mu} \tag{61}$$

(If this explanation is *still* not sufficiently rigorous, or too confusing, I recommend the Feynman Lectures in Physics, Vol II, pp. 41-5 to 41-6.)

References

[1] Colburn, A.P., "A Method of Correlating Forced Convection Heat Transfer Data and a Comparison with Fluid Friction," *Trans. A.I.Ch.E.*, 29, p. 174, 1933