Phys 344.

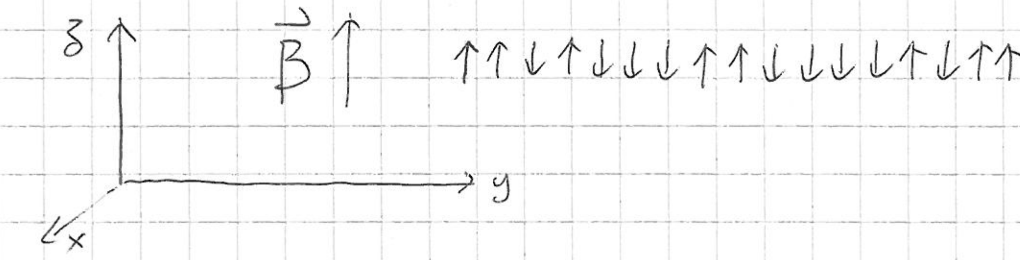
Lecture 12. Paramagnetism   Ref. 2.1, 3.3.

- The system (Model). One object with N dipoles.

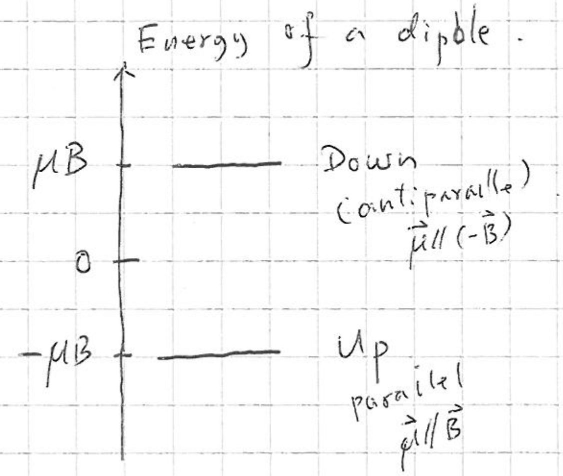
  N magnetic dipoles (each dipole is a "spin-$\frac{1}{2}$" particle)
  └ Quantum mechanics term.

  In a constant magnetic field $\vec{B}$.

  No interaction between dipoles.



Only 2 states for each dipole: up or down.
(quantum mechanics).



μ — dipole moment.

lower energy: when a dipole is aligned with the field $\vec{B}$

Energy of flipping a dipole: $2\mu B$.

$N_\uparrow$ — # of up dipoles

$N_\downarrow$ — # of down dipoles.

$N = N_\uparrow + N_\downarrow$.

Total energy of system: $U = \mu B (N_\downarrow - N_\uparrow) = \mu B(2N_\downarrow - N)$

$= \mu B (N - 2N_\uparrow)$

Magnetization. $(\vec{M} = \Sigma \vec{\mu})$

(z-component). $M = \mu (N_\uparrow - N_\downarrow) = -\dfrac{U}{B}$.

Question? $U(T) = ?$ $M(T) = ?$

① $\Omega = ?$

② $S = k \ln \Omega$. $S = S(U, N, N_\uparrow, \text{etc.})$

③ T. $\left(\dfrac{\partial S}{\partial U} = ? \right)$. Then. $U(T)$.

④ $M(T)$, $U(T)$.

⑤ C_B

$$\Omega = \binom{N}{N_\uparrow} = \frac{N!}{N_\uparrow!\, N_\downarrow!}$$

$$S/k = \ln \Omega \qquad \left(\begin{array}{l} \text{stirling :} \\ \quad \ln N! \approx N \ln N - N \end{array} \right)$$

$$= \ln N! - \ln N_\uparrow! - \ln N_\downarrow!$$

$$= N \ln N - N - N_\uparrow \ln N_\uparrow + N_\uparrow - N_\downarrow \ln N_\downarrow + N_\downarrow$$

$$= N \ln N - N_\uparrow \ln N_\uparrow - N_\downarrow \ln N_\downarrow \qquad (N = N_\uparrow + N_\downarrow)$$

$$= N \ln N - N_\uparrow \ln N_\uparrow - (N - N_\uparrow) \ln (N - N_\uparrow)$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_{N,B} = \frac{\partial S}{\partial N_\uparrow} \frac{\partial N_\uparrow}{\partial U} = -\frac{1}{2\mu B} \frac{\partial S}{\partial N_\uparrow}$$

$$\left(\frac{U}{\mu B} = N - 2N_\uparrow, \quad N_\uparrow = \frac{N}{2} - \frac{U}{2\mu B} \right)$$

$$\frac{\partial S}{\partial N_\uparrow} = \left[-\ln N_\uparrow - 1 + \ln (N - N_\uparrow) + 1 \right] k$$

$$= k \ln \frac{N - N_\uparrow}{N_\uparrow} = k \ln \frac{N - N_\uparrow}{\frac{N}{2} - \frac{U}{2\mu B}}$$

$$= k \ln \frac{2N - 2N_\uparrow}{N - \frac{U}{\mu B}} \qquad \left(\begin{array}{l} 2N - 2N_\uparrow = N + N - 2N_\uparrow \\ \qquad\qquad\quad = N + U/\mu B \end{array} \right)$$

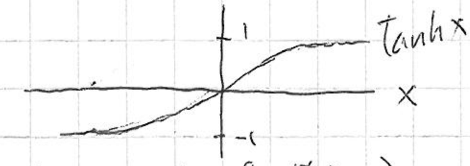
$$= k \ln \frac{N + U/\mu B}{N - U/\mu B}$$

$$\therefore \quad \frac{1}{T} = \frac{k}{2\mu B} \ln \frac{N - U/\mu B}{N + U/\mu B}$$

$\sinh(x) = \frac{1}{2}(e^{x} - e^{-x})$

$\cosh(x) = \frac{1}{2}(e^{x} + e^{-x})$

$\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$



(Hyperbolic function)

$$\ln \frac{N - U/\mu B}{N + U/\mu B} = \frac{2\mu B}{kT}$$

$$e^{\frac{2\mu B}{kT}} = \frac{N - U/\mu B}{N + U/\mu B}.$$

Solve for U: $\quad N e^{\frac{2\mu B}{kT}} + \frac{U}{\mu B} e^{\frac{2\mu B}{kT}} = N - \frac{U}{\mu B}.$

$$\frac{U}{\mu B} + \frac{U}{\mu B} e^{\frac{2\mu B}{kT}} = N - N e^{\frac{2\mu B}{kT}}.$$

$$U = \frac{N\left(1 - e^{\frac{2\mu B}{kT}}\right)\mu B}{1 + e^{2\mu B/kT}} = N\mu B \tanh\left(\frac{\mu B}{kT}\right).$$

$$M = N\mu \tanh\left(\frac{\mu B}{kT}\right)$$

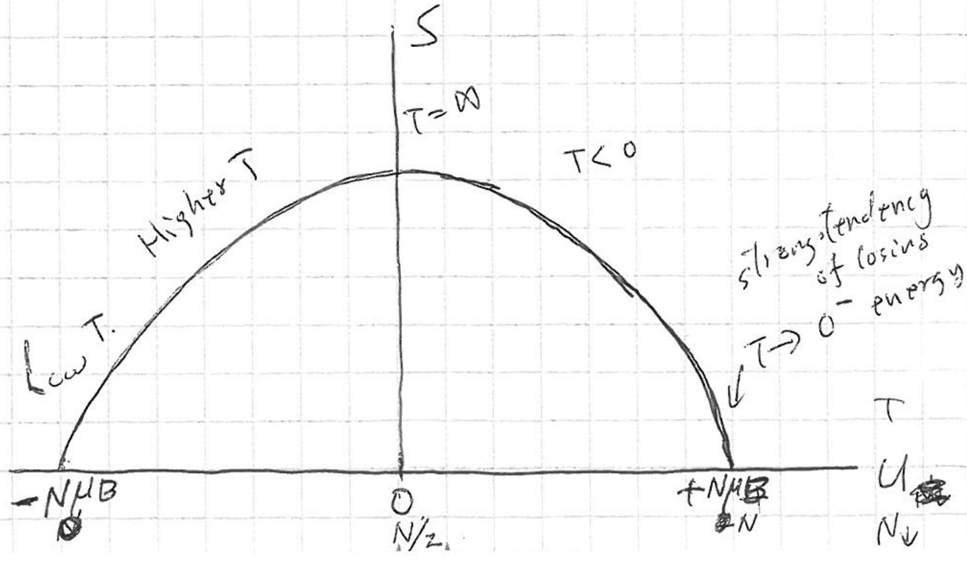
$$C_B = \left(\frac{\partial U}{\partial T}\right)_{N,B} = \frac{Nk\left(\frac{\mu B}{kT}\right)^2}{\cosh^2(\mu B/kT)}$$

Numerical results.

Table 3.2.

Figure 3.8

$\frac{1}{T} \propto$ slope.



- From left to right.
  - starting from $U = U_{min} = -N\mu B$.

    $\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$ All up.

    S vs. U — very steep, positive slope.

    $\frac{1}{T}$ large. T — low.

    strong tendency of receiving energy.

    (small gain in U leads big increase in S.)
  - as $U \uparrow$. (~~at~~ on the left half).

    S vs U slope gets shallower. T — higher.

    ~~system still tends to receive energy~~.

    Tendency of receiving energy decreases.
  - When $S = S_{max}$. $(U = 0)$

    S vs U: slope $= 0 = \frac{1}{T}$, $T \to \infty$.

    gain in U won't increase S.

    (No tendency of receiving energy).
  - as $U \uparrow$ further. in the right half.

    S vs. U — negative slope! $T < 0$.

    giving away energy increases S.

    Tendency of losing energy.

$\therefore$ Negative temperature means "hotter" than $T=\infty$.

i.e., It tends to lose energy with it is in contact with any object with a positive T.

- How come we can have negative temperature ?

$\left\{\begin{matrix}\text{energy gain} \\ \text{entropy loss}\end{matrix}\right\}$ go together.

negative S vs U slope $\left(\frac{\partial S}{\partial U}\right)$

Because there is a limit in the total energy U.

When U reaches its limit, $U = U_{max}$
$= N\mu B$.

all the dipoles are down $\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow$.

$\Omega = 1$, $S = 0$

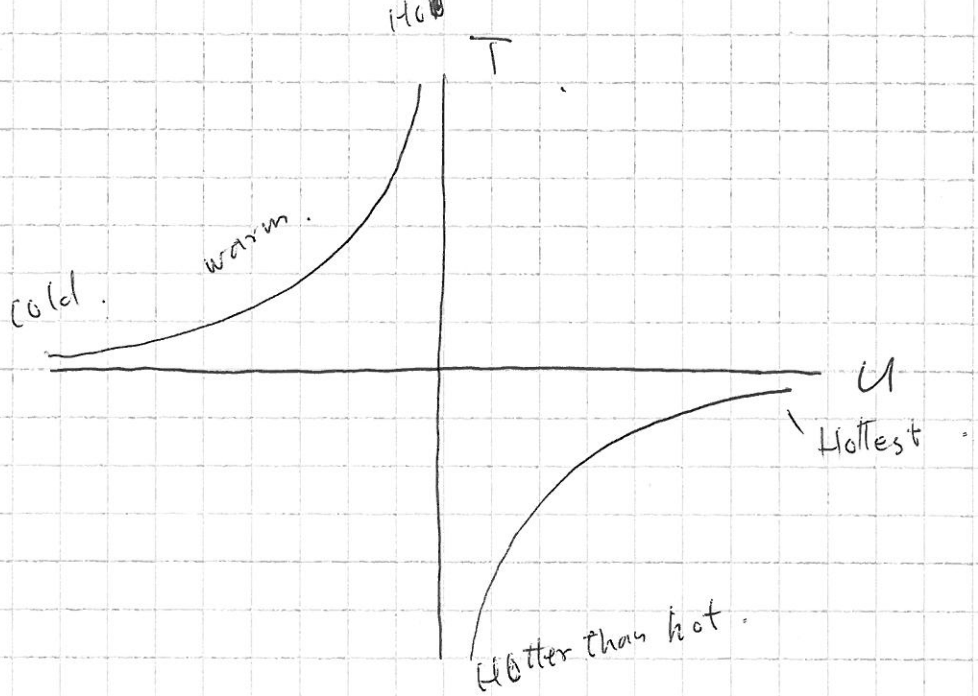
unique microstate.

$\therefore$ As the system approaches U_{max}. (~~on the right half~~)

$S \to 0$ from S_{max}., (U is increasing)
Ω decreases to 1.

On the right half, $U\uparrow$ leads to $S\downarrow$.

Less and less microstates.



- Example. LiF. (Pound, Purcell, Ramsey 1951).

(Lithium Fluoride) dipole — nucleus.

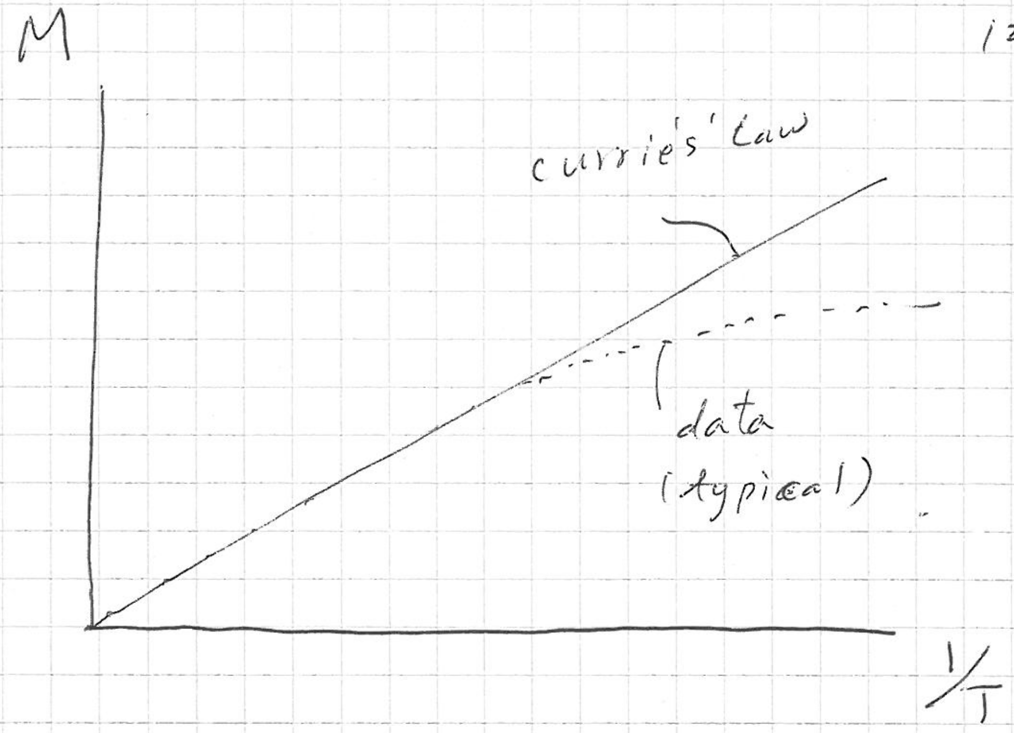
- Curie's law:

$$M \propto \frac{1}{T} \quad \text{(experimentally)}$$

for $\frac{\mu B}{kT} \ll 1$ (strong B-field, low temperature)

$$M = N\mu \tanh\left(\frac{\mu B}{kT}\right) \approx N\mu \cdot \frac{\mu B}{kT} = \frac{N\mu^2 B}{kT}.$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \xrightarrow[\text{as } x \to 0]{} \frac{1+x-(1-x)}{1+1} = x.$$



- Heat capacity.

$$C_B = \left(\frac{\partial U}{\partial T}\right)_{N,B} = Nk \frac{(\mu B/kT)^2}{\cosh^2(\mu B/kT)} \approx \frac{N}{k}\left(\frac{\mu B}{T}\right)^2.$$

For small x. $\left(x = \frac{\mu B}{kT} \to 0 \right)$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \to \frac{1}{2}(1+1) = 1.$$

$$\therefore \quad C_B \propto \frac{1}{T^2}$$