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Identity, Agency, and Knowing in Mathematics Worlds

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INTRODUCTION

The number of people who choose to pursue mathematics within or beyond university is small. In the United States, as well as other countries across the globe, declining proportions of students are majoring in mathematics, with particularly small numbers of women and non-Asian minorities entering the discipline (Anderson, 1997; Gutierrez, 2000). These facts are assumed by many, particularly those in mathematics departments, to be owing to the cognitive challenge of the subject. Mathematics is regarded as difficult and attainable only by some. We will present new data in this chapter that challenges this view through a representation of learning as a process of identity formation in "figured worlds" (Holland, Lachicotte, Skinner, & Cain, 1998). The figured worlds of many mathematics classrooms, particularly those at higher levels, are unusually narrow and ritualistic, leading able students to reject the discipline at a sensitive stage of their identity development. Traditional pedagogies and procedural views of mathematics combine to produce environments in which most students must surrender agency and thought in order to follow predetermined routines (Boaler, 1997a; Doyle, 1988; Schoenfeld, 1988). Many students are capable of such practices, but reject them, as they run counter to their developing identification as responsible, thinking agents (Wenger, 1998). The application of thought and the development of agency (Holland et al., 1998) should be an intrinsic part of any learning environment, yet there is evidence that such practices are dismally represented for students in many mathematics classrooms (Boaler, 1997a; Cheek & Castle, 1981; Stigler & Hiebert, 1999).

Learning mathematics has traditionally been regarded as an individual, cognitive activity. Educators have focused on the knowledge and understanding that students develop, as a product both of individual student resources and the practices in which students engage. A supporter of traditional teaching methods, for example, may argue that a student's knowledge would be enhanced by working through a textbook, while a reform-oriented teacher may argue that a student's knowledge would be enhanced through the act of mathematical discussion. Both of these educators regard the activity of learning mathematics as a vehicle for acquiring mathematics knowledge, but essentially distinct from the knowledge that is eventually developed. More recent theories of mathematical knowledge (e.g., Kitcher, 1983; Tymoczko, 1986) and learning (Greeno & MMAP, 1998; Lave, 1988, 1993; Lave & Wenger, 1991) challenge this distinction, claiming that the *practices* of learning mathematics define the knowledge that is produced. Such theories are supported by Boaler's finding (1997a, 1998, 2000a) that students of mathematics who had predominantly worked through textbooks found it difficult to use their mathematics in new and varied situations that required a different set of practices. Students who had engaged in practices of negotiation and interpretation in the mathematics classroom were more able to use mathematics in different situations that required such practices. Both sets of students had learned how to form and solve equations, for example, but, consistent with sociocultural (Rogoff, 1990) or situative (Greeno & MMAP, 1998; Lave, 1988, 1993; Lave & Wenger, 1991) theories of learning, students from the different learning environments had qualitatively different forms of knowledge, mediated by the beliefs that students developed about mathematics and learning in response to different teaching methods. Their knowledge was co-constituted by the practices of their learning and therefore differentially useful in real world situations (Boaler, 1997a). But situated theories do not only illuminate the discontinuity of mathematical practice that is recorded between sites in different research studies (Lave, 1988). Their focus on the patterns of participation that constitute learning gives insights into the nature and extent of identification and belonging that students develop as they learn to *be* mathematics learners (Dowling, 1996; Wenger, 1998).

We propose in this chapter that broadened perspectives of mathematics learning provide considerable insight both into students' mathematical understanding, as well as the choices they make about life and work. We consider knowing and understanding mathematics as aspects of participation in social practices, particularly discourse practices, in which people engage in sense-making and problem solving using mathematical representations, concepts, and methods as resources. Calling these "social practices" does not exclude activities of individuals who work alone, using and developing mathematical representations, concepts, and methods that they have encountered by participating in classrooms or by reading texts. An important implication of this idea is that students' learning of mathematics can be considered as a *trajectory of participation* in the practices of mathematical discourse and thinking. This view goes beyond recognizing that social practices provide a context for learning mathematics—instead, according to this view, participation in social practices is what learning mathematics is. The social practices of a community provide an environment in which students can participate, and their ways of participating are adaptations to the constraints and affordances of the environment (Greeno & MMAP, 1998).

In this chapter, we make use of a practice-based interpretation of mathematics learning in our analysis of interviews with 48 high school students of calculus. We interpret the results of these interviews using a concept of *ecologies of participation*. We find it useful to consider ecologies of participation in terms developed by Holland and associates (1998). This group of anthropologists discussed social systems in terms of *figured worlds*, *positioning*, and *authoring*. Figured worlds (Holland et al., 1998, p. 52) are places where agents come together to construct joint meanings and activities. A mathematics learning environment could be regarded as a particular figured world because students and teachers construct interpretations of actions that routinely take place there. Figured worlds are socially and culturally constructed realms "of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others" (1998, p. 52). The importance of this label for researchers of mathematics education resides in the characterization of a mathematics classroom as an interpretable realm, in which people fashion their senses of self. Figured worlds draw attention to interpretations by actors—students and teachers, for example—and to the rituals of practice. The mathematics classroom may be thought of as a particular social setting—that is, a figured world—in which children and teachers take on certain roles that help define who they are.

Holland and associates (1998) use the term "positional identity" to refer to the way in which people comprehend and enact their positions in the worlds in which they live. This builds on their theory that identities develop in and through social practice. They acknowledge that identities are centrally related to structural features of society such as ethnicity or gender but draw attention to the specific practices and activities situated in "worlds" such as academia, romance, or local politics. "Positional identities have to do with the day-to-day and on-the-ground relations of power, deference, and entitlement, social affiliation and distance—with the social-interactional, social-relational structures of the lived world" (Holland et al., 1998, pp. 127–128). Another aspect of identity they describe is the "space of authoring," which is encapsulated by the notion that "the world must be answered—authorship is not a choice" (1998, p. 272). This idea is concerned with the responses individuals give, with human agency, and with improvisation. The possibility and forms of authoring that are created in different mathematics environments, among learners who are often conceived as "receivers" of education (Corbett & Wilson, 1995), is an important question that will be pursued in this chapter.

We consider students' talk about their mathematics learning in their interviews with us as reports of their perceptions and understandings of the figured social worlds of mathematics education in which they participated as learners. The students' descriptions may also be taken to indicate their positionings in the ecologies of participation in practices of mathematics education and reflections of their authoring of identities as learners and performers of mathematics. The figured worlds of mathematics learning environments are not all alike, and our results illustrate two kinds of figured world that differ in an important way. In interpreting differences between the two kinds of learning environments these students described, we use concepts developed by Belenky, Clinchy, Goldberger, and Tarule (1986), and extended by Clinchy (1996) and Tarule (1996). Based on interviews with individuals about their beliefs and understandings of knowing and learning, these researchers developed distinctions that they referred to as *ways of knowing*. The typology that these researchers developed included the following:

- *Received knowing,* in which the individual considers her knowledge as primarily dependent on and derivative from an authoritative source other than herself
- *Subjective knowing,* in which the individual considers her knowledge as primarily a result of her affective reactions to information and ideas
- *Separate knowing,* in which the individual considers her knowledge as primarily being constructed to comply with rules that establish validity and to be defensible against challenges based on rules for validating knowledge
- *Connected knowing,* in which the individual considers her knowledge as primarily being constructed in interaction with other people (either directly, in conversation, or indirectly, through interacting with texts or other representations of others' knowledge and thinking), in a process that depends on understanding others' experiences, perspectives, and reasoning, and incorporates this understanding into the individual's knowing and understanding[1]

Belenky and associates' (1986) interviews were concerned with quite general aspects of knowing and learning, and they used their ways of knowing to characterize individuals. Our interviews dealt more specifically with students' experiences and beliefs regarding their learning of mathematics, and we consider that the different ways of knowing are characteristics of students' adaptations to their mathematics learning environments. Indeed, the interviews we report include considerable evidence of the different ways of knowing that students are required to accept, negotiate, or oppose in mathematics classrooms, compared with other school subjects.

RESEARCH METHODS

As part of a research project investigating the nature of mathematical confidence, researchers[2] interviewed 8 students from each of 6 Northern Californian high schools, 48 students in total. The students were all attending advanced placement (AP) calculus classes. The six schools are all part of the public system and serve diverse populations of students. The proportion of nonwhite students at the schools ranges from 37 to 61 percent, while the proportion of students classified as being eligible for free school meals ranges from 4 to 13 percent. The schools are located in a relatively affluent part of the United States, and they all include high proportions of middle and upper-middle class students. All six schools are popular with parents and contain large numbers of college-bound students. Researchers interviewed two girls and two boys that the teacher of AP calculus identified as mathematically confident, and two girls and two boys that the teacher identified as lacking confidence, in each of the six classes. The eight students in each school were interviewed in single-sex pairs. The teachers of the six classes, half of whom are female, are experienced and well respected in their departments. As the students were taken from AP calculus classes, they may all be regarded as successful students of mathematics, having all chosen to take mathematics into a fourth year, at an advanced level. Indeed, the success of the students and their self-selection into an advanced mathematics class meant that the students we interviewed were well placed to choose mathematics as a field of study. All of the interviews were coded using a system of open coding (Miles & Huberman, 1994). The different themes that emerged were then combined into broader categories, which are reported in this chapter. The interviews were semistructured, enabling the interviewer to pursue directions raised by the students. Students were asked to describe mathematics lessons; they were asked about lessons they particularly liked and disliked, the extent of discussion in mathematics, and the nature of mathematical confidence. In this chapter, we will consider the students' representation of their mathematics classroom environments and their subsequent beliefs about mathematics. In discussion, we will consider the implications of these different experiences and beliefs for the nature of mathematics knowing, identification, and participation.

RESEARCH RESULTS

We present our results in three sections. First, we describe students' reports of their perceptions of the mathematics classroom environments in which they worked and the characteristics of mathematics learning that they believed. These findings provide a picture of the figured social worlds of mathematics learning that the students experienced. Second, we describe students' reports of their beliefs about their places in the figured worlds of mathematics learning, their understandings of their positioning in these learning ecologies. Third, we present students' reports about their affective reactions and identifications toward their participation in mathematics learning, in the present and future. This provides information about the identities they authored regarding knowing and learning mathematics. Implications of the students' views for their future as mathematics learners and the future of mathematics education will be discussed.

The Figured Worlds of the Mathematics Classroom

All 48 of the students we interviewed were asked to describe their AP-calculus mathematics lessons, and interviewers engaged students in conversation about the different features they described. The students all described teachers reviewing homework, explaining methods at the board, and assigning questions to be completed. However, the descriptions of lessons at two of the schools differed in an important way. Teachers at two of the high schools, both women, encouraged students to work on questions collaboratively. When teachers explained methods to students, they encouraged student discussion, and when students worked on problems, they did so in groups. Students of the other four teachers described mathematics classes as individual environments in which their role was to practice and repeat the procedures teachers demonstrated. The nature of the different classroom environments will be explored briefly in the pages that follow.

Ecologies of Didactic Teaching

The pedagogical practices students reported in the four individualistic environments, which we call Apple, Cherry, Lemon, and Lime schools, were remarkably consistent, with student after student portraying an image of mathematics teaching that afforded received knowing. The teachers presented procedures that students were supposed to learn to perform. The students' characterizations of these classroom practices are illustrated by the following remarks:

> Basically, throughout my experience, we go to class and the teachers lecture, go over the material and show us exactly how to do the problems, cover the subjects that they're teaching and after the teacher's finished teaching if we ask questions and sort of like clear up anything that we don't know and then homework will be assigned to us that day, then we go home and do it. (Brad, Cherry school)

Students from these schools reported that mathematics classes always followed the same pattern—of reviewing homework and then working through exercises—and that even the questions in the exercises were similar:

> It's calculus! Everything is the same. It's all derivatives, and somehow you gotta use it somehow. I never liked derivatives or integrals, but the whole book so far has been the same thing, derivatives. (Khir, Apple school)

The students seemed to accept the lack of variety they reported in mathematics lessons, not because they enjoyed the lessons, but because they thought that was the way mathematics *had to be*. Few of the students had experienced anything different:

> K: I'm just not interested in, just, you give me a formula, I'm supposed to memorize the answer, apply it and that's it.
>
> Int: Does math have to be like that?
>
> B: I've just kind of learned it that way. I don't know if there's any other way.
>
> K: At the point I am right now, that's all I know. (Kristina and Betsy, Apple school)

The mathematics textbooks that the schools used all presented the fundamental theorem of calculus, expanded on the different concepts underlying the domain, and demonstrated procedures that could be used to solve problems. The books then

introduced a series of questions that required students to practice the different procedures. Students reported that they worked through their textbooks every lesson, and they were encouraged to attend carefully to the words and practices of their teacher. Students were not expected nor encouraged to discuss the mathematics they were learning:

> There isn't much between students because you can't spend the time, like talking, if you want to pay attention and listen to what they're explaining. So unless like, when somebody next to me has a question, then I'll lean over because it's a little thing I can ask, I think it's more with the teacher. (Janet, Lemon school)

One of the students laughed for a long time when we asked him whether he and other students were encouraged to interact or talk to each other in mathematics lessons. He then gave the following response:

> Unless Mr. Bond says the marvelous sentence, "you have the period to yourself," we will *never* interact with each other directly. We go (whispering) "pass me your book"—we will never interact directly, not even with Mr. Bond because it's like our little cubicle, we have to do it. (Chris, Cherry school)

The figured worlds that these students portray, and that will be given more depth in the next section, are highly ritualized. Students come to class, watch the teachers demonstrate procedures, and then practice the procedures—alone. The ways in which students position themselves in such a world and form relationships with mathematics and each other is the focus of the next section. First, we offer a similarly brief characterization of the two other mathematics classes in which students were required to play a different role.

Ecologies of Discussion-based Teaching

Students from Grape and Orange schools painted a markedly different picture of mathematics lessons. The students talked about the time they spent discussing the different questions, as a class, and in student groups. They described being positioned as active agents in their classes, and their role involved contributing to the shared understanding of ideas that developed among the class. Their classroom practices afforded growth of connected knowing, with development of mathematical understanding that the students constructed and shared. The students from Grape and Orange schools were generally more positive about their experiences of calculus than students in the other schools, which they attributed to the relaxed nature of lessons, the positive relationships they formed with the teacher and other students, and the chance to derive meaning through discussion. When the students described their mathematics lessons, they gave discussion a central role and they talked about the increased access to understanding that collaboration provided:

> V: Classes are social.
>
> J: We work in groups most of the time.

V: Ms. Green works really hard on making it social and not just by yourself.

J: I think the groups are the best situation just because you can talk to the kids in your group and see if they can figure out the problem too. (Veena and Jazz, Orange school).

J: The teacher gives us something and has us work on a work sheet and then work with people in our group on the work sheet, because if I understand something, then I can explain it to the group members or if I don't understand it the group members may explain it to me. Whereas if she teaches the lesson and sends us home with it, I'm not really that confident because I haven't put like things together. (Jacob, Orange school)

The students appreciated the opportunity to discuss their work, partly because their discussions gave them deeper insights into the mathematics they met, but also because their discussions changed the nature of the classroom environment. When the students talked about AP calculus, they emphasized relationships—between the different aspects of mathematics as well as the people in the class. Their figured worlds did not center around individualized procedure repetition, but rather around collaboration of ideas within a community of learners (Lave & Wenger, 1991):

D: Yeah, this is my favorite class this year because the environment is so like family and you can just go there and talk about math or any problems you have.

B: She gives you time when she's not teaching, she lets you work on the problems, so she kind of walks around the room and it could be like socializing if you want. So it's like even when we socialize we still get math.

D: Yeah we always socialize about math. Weird but it happens.

B: I've definitely done better in this class than any other math class.

D: I think it's the relationships you have with other people in the class and with the teacher. (Debbie and Becky, Grape school)

In discussion-oriented figured worlds, connections between learners are emphasized as students are positioned as relational agents who are mutually committed and accountable to each other for constructing understanding in their discourse (Mclaughlin & Talbert, 1999; Wood, 1999). Students are expected to be co-authors, with their teachers, of their understanding of mathematical principles and procedures.

This brief section has presented a summary of the students' presentations of figured worlds. One group of students presented their worlds as structured, individualized, and ritualized, the other group as relational, communicative, and connected. The difference between the students' reported experiences in classrooms that did and did not encourage discussion was unexpected, as we knew little about the classrooms before we went in to interview the students. But it became clear to us, through our discussions with the students, that such differences had a significant impact on the students' positioning as learners in the two versions of figured worlds, as we shall now explore.

Students' Places in the Figured Worlds of Mathematics Learning

Positioning within Didactic Teaching

In order to consider students' roles as learners in their figured worlds, we engaged them in conversation about the nature of the knowledge they encountered and their roles as interpreters of that knowledge. We were interested to know whether students in the more traditional classes played an *active* role considering and interpreting the meaning of the procedures they encountered, which may have led to a broad, conceptual understanding of mathematics, or a *passive* role as receivers of predetermined knowledge that appeared unavailable for discussion or negotiation. Through conversation with the students from the more didactic classrooms, it became clear that their role in the mathematics classroom was narrowly defined:

There's only one right answer and you can, it's not subject to your own interpretation or anything it's always in the back of the book right there. If you can't get it you're stuck. (Susan, Cherry school)

Some students believed that mathematics lessons did not require them to think in the way that other subject lessons did because of the closed nature of the problems they encountered:

There's definitely a right answer to it. The other subjects like English and stuff that really have no right answer so I have to think about it. (Kim, Apple school)

Some of the students regarded thinking practices to be an unnecessary part of their mathematical experiences, in a similar way to the students' taught in traditional classes in Boaler's previous study (1997a), as one of the students from that study reported:

L: In maths you have to remember; in other subjects you can think about it. (Louise, Year 11, set 1) (Boaler, 1997a, p. 36)

The idea that learning mathematics requires no or little thought, as students are only required to reproduce procedures, suggests that students are engaging in ritualistic acts of knowledge reproduction rather than thinking about the nature of the procedures and the reasons why and when they may be applied. That thinking practices are limited, even at such advanced levels of mathematics, seems both incredible and worrying. Further evidence for the limited role of thinking was provided by the students' answers to the following question, posed to them in the interviews: Is mathematics more about understanding concepts or memorizing procedures? Students from three of the four schools give these answers:

G: I think it's very procedural. Different chapters they have this blue section, theorems, just memorize theorems. (Greg, Apple school)

K: Procedures. Because you have to learn this to learn this to understand this. I think of it like that. (Karen, Cherry school)

L: It's all about the formulas. If you know how to use it then you've got it made. Even if you don't quite understand the concept, if you're able to figure out all the parts of the formula, if you have the formula then you can do it. (Lori, Lime school)

P: I'd say it's a little of both. I don't think it's conceptual. Like, there are big concepts, and it's derivative and everything, things like that. But you have to remember all the little tricks and rules you have to follow. (Paul, Lime school)

This student suggests that though he appreciates that "big concepts" underlie the mathematics he is learning, the "little tricks and rules" dominate his perception. Other student reflections supported the idea that the practice of working through multiple procedures reduced the opportunity for a broader understanding:

A: If we actually went into detail about certain, like the stronger concepts that we'll maybe use later, I think we would remember more than just bombarding us with all these different things on a weekly basis. It gets to the point where you're doing so much you don't see the relationship, you're just doing so many problems. (Anthony, Lime school).

A: Concepts I think are second priority and I'll spend more time trying to learn the procedure.

B: The book does the worst job of explaining it. You might as well just get those notes down. Who knows how to use them, but somehow learn the procedure. (Arnetha and Barbara, Lime school).

Students of both sexes, different confidence levels, and various levels of attainment in the didactic classes told us that mathematics was a closed, rule-bound subject. The students related breadth and openness with thought. In contrast, mathematics questions requiring one answer, which could be achieved by following a standard procedure, required little thought. Despite the range of initiatives aimed at reforming school mathematics in the United States (Fennema & Nelson, 1997; National Council for Teachers of Mathematics, 1989), there is evidence that traditional pedagogies dominate mathematics classrooms, particularly at higher levels (Boaler, 1997c; Rogers 1995). Thus, teachers of mathematics frequently expect students to spend the majority of their time in mathematics lessons working through exercises practicing procedures (Schoenfeld, 1988). The aim of such work, presumably, is that students will become conversant in the use of the different procedures and be able to use them in a range of mathematical situations. However, the act of practicing procedures appears to become sufficiently dominant for many students that it obscures the meaning of the subject, and takes students' thoughts away from the concepts that are intended to be exemplified by the procedures (Mason, 1989; Perry, 1991). Many students described the act of working through procedures as oppositional to understanding the big ideas in the domain:

C: I see it more as procedures and solving one problem at a time. It's hard for me to see how it relates to everyday things, so I don't really get the big picture a lot of the time. (Cathy, Lemon school)

The students suggested that the procedural presentation of mathematics they encountered forced them to become passive receivers of knowledge—with a narrowly defined role that was one of memorization:

K: This is AP so it's definitely going to be harder, but I feel as long as I can memorize the formulas and memorize the derivatives and things like that, then I should be pretty well off. (Kim, Apple school)

V: You have to memorize these little steps, there's always an equation to solve something and you have to memorize stuff in the equation to get the answer and there's like a lot of different procedures. (Vicky, Lime school)

In the four schools in which students worked through calculus books alone, the students appeared to view the domain of mathematics as a collection of conceptually opaque procedures. The majority of students interviewed from the traditional classes reported that the goal of their learning activity was for them to memorize the different procedures they met. Such a figured world of didactic teaching and learning rests on an epistemology of received knowing. In this kind of figured world, mathematical knowledge is transmitted to students, who learn by attending carefully to teachers' and textbook demonstrations. Ball and Bass (2000) offer supporting evidence when they reflect that "students often receive mathematical knowledge in school that is justified by little else than the textbook's or the teacher's assertion. By default the book has epistemic authority: Teachers explain assignments to pupils by saying, 'This is what *they* want you to do here,' and the right answers are found in the answer key" (Ball & Bass, 2000). Students' positioning in this kind of ecology is that of receiving and absorbing knowledge from the teacher and textbook. This knowledge consists of the ability to select and perform procedures of symbol manipulation, thereby producing sequences of symbols that are correct, according to specifications taken from authoritative mathematics (Povey, Burton, Angier, & Boylan, 1999). The students' responses to their positioning as received knowers in this highly ritualized figured world will be the focus of our final section. First, we will explore the students' positioning in the discussion-oriented classes.

Positioning within Discussion-based Teaching

The students in Grape and Orange schools used the same, or similar, textbooks as students in the other four schools, but they did not work through the exercises producing answers that were supported or invalidated by the teacher or textbook. Instead, they were asked to discuss the different questions and consider the meaning of possible solutions with each other, thus engaging in the process of validation alongside the teacher. The students at Grape and Orange schools engaged in acts of negotiation and interpretation that appeared to lead to their distinctly more progressive views of mathematics as a discipline. The students did not describe mathematics as an abstract, closed, and procedural domain, but as a field of inquiry that they could discuss and explore. Concomitantly, they regarded themselves as active learners whose role went well beyond memorization:

> L: I can get more into it, not just like "oh this kind of problem" but "why is this the way you do it?"
>
> M: Yeah you want to figure out the problem, you want to understand the concept. (Lori and Melissa, Grape school)

Some of the students in Grape and Orange schools contrasted mathematics with English, as the students in some of the other schools had done, but they did so to illustrate the depth of thinking required in mathematics, rather than the procedural nature of the subject:

> M: I don't know, it just seems like math is more important. In my English class, I can just kind of flow, and whatever's going on, write an essay about whatever, it's not a lot, well, in my case, it's not a lot of deep thinking. Not a lot under the surface.
>
> Int: Is there in math—deep thinking?
>
> M: Yeah. Yeah because the thing, being conceptual, and that's a lot harder than just like memorizing formulas, definitely. (Melissa, Grape school)

> T: I guess I've liked math overall because it's a lot better than English or social studies, just because I don't like to memorize just a bunch of stuff. It's a lot of solving the problems, not like looking over past stuff, it's a lot of new stuff you're covering. (Tom, Orange school)

The students in the discussion-oriented classes regarded their role to be learning and understanding mathematical relationships, they did not perceive mathematics classes to be a ritual of procedure reproduction. When the students described their figured worlds, they centralized relationships between people. Debbie and Becky, cited earlier, described their mathematics class as a family environment; Angier and Povey (1999) received similar responses from students they interviewed who engaged in mathematical discussions in class. Such descriptions suggest that the relationships students form in their classes are central to the learning that takes place (McLaughlin & Talbert, 1999), rather than a casual by-product of a change in pedagogy. The following student relates the attainment of conceptual understanding to the relationships that are formed with teachers and classmates:

> D: I don't know, I guess there's a feeling of more, it's kind of more laid back, I mean we get a lot of work done, we have people get 100 percent on the tests and things like that. We have people who don't understand it, they still get a grasp of the general concept and it's not like we're sitting there with our hands on our desk, like. We're allowed to make, we're allowed to make jokes, to be out of whack sometimes. We have fun with the teacher, and when we get to work we get to work. With Mr. Cain, and our other teachers, it's kind of like, there's not that relationship. (David, Grape school)

In interviews at Grape and Orange schools, the students were extremely positive about the discussions of mathematics that took place in class, which they related, among other things, to the opportunities to learn through student language, the accessibility of other students' ideas, and the formation of strong, personal relationships that enhanced their learning. One of the students who was in one of the more didactic environments reflected on the discussions he had with friends when he was completing homework, concluding that discussions were helpful as they helped him consider mathematics from another person's perspective:

> S: You can't always understand every problem. Like if you go through a test or homework or an assignment, like it's hard. Sometimes your mind doesn't always click on what you have to do for this certain problem, you have to approach it in a different way so you have to kind of get everyone's point of view. You have to get everybody's take on how to do it. So it helps if someone else could be looking at it in a different way, then they would have seen something a little bit different. That definitely helps. (Seth, Lime school)

This student's reflection seems to encapsulate the spirit of connected knowing, with discussions of mathematics offering him the occasion to consider other people's representations of knowledge. He valued the opportunities that homework discussions provided for such insights, whereas the students at Grape and Orange schools were afforded occasions for connected knowing on a daily basis, the implications of which will be considered now.

Students' Authored Identities in Their Different Mathematics Worlds

Didactic Teaching and Received Knowing

For the students in Lemon, Lime, Apple, and Cherry schools, mathematics was presented as a series of procedures that needed to be learned, as the students have described. For students to be successful in such classes, they needed to both assume the role of a received knower and develop identities that were compatible with a procedure-driven figured world. We assert that compatibility with forms of knowing, and identification with pedagogical practices, are both crucial aspects of mathematical success and we expand on both of these points next. It is critical to our analysis that when we interviewed students in the didactic classes about the nature of success in mathematics classrooms, they did not prioritize "ability," the cognitive demand of the discipline, or even effort. Instead they prioritized students' willingness to accept a particular form of knowing and to build identities that give human agency a minimal role.

The following students, from three of the four didactic classes, were asked what it takes to be successful in mathematics. The interviewers of these students expected to hear about interest, effort, or even talent, and were surprised by the students' replies:

A: Patience.

Int: Patience?

V: Yeh motivation and just wanting to do it. Perseverance. Wanting to do it over and over again. (Vicky and Amy, Lime school)

A: Obedience.

Int: Obedience?

A: Obedient. I've known these people like 4 years of my life and long enough to know that no matter, even if they didn't like what they're doing you would feel like what we're doing is completely ridiculous, they're not going to raise a fuss about it, they're not going to speak their mind about it. They'll just do it because it's required and that's what the teacher wants. The teacher would rather you do it and not hear your thoughts on the thing, than have you contest what you're doing and like—"I don't understand this." . . . I can't sit there for hours, just, I just can't follow directions when I see people doing something completely irrational. Or like if I don't agree with like the question that he wants us to answer or whatever. (Anthony, Lime school)

T: You have to be willing to accept that sometimes things don't look like—they don't seem that you should do them. Like they have a point. But you have to accept them. (Tom, Lemon school)

R: I guess it depends on how you take frustration. (Rick, Apple school)

These students suggest that success in their calculus classes required a form of received knowing, in which obedience, compliance, perseverance, and frustration played a central role. There seemed to be considerable consensus for this perspective among the students in the didactic classes, even though the students were divided in their responses to the form of knowing with which they apparently needed to comply. We asked all 32 of the students in the didactic classes whether they enjoyed mathematics; 18 said that they did (56 percent). Thirty of the students were asked whether they intended to take any other mathematics classes; 14 said that they did (47 percent). Those who disliked mathematics and had decided to cease their study of mathematics (generally the same students) were not unsuccessful in class; indeed, some of them were extremely able mathematics students. But the students resented the lack of opportunities they received to develop a deep, relational, and connected understanding of mathematics, as this student describes:

K: We knew HOW to do it. But we didn't know WHY we were doing it and we didn't know how we got around to doing it. Especially with limits, we knew what the answer was, but we didn't know WHY or how we went around doing it. We just plugged into it. And I think that's what I really struggled with is—I can get the answer, I just don't understand why. (Kate, Lime school)

That such responses were unrelated to "ability" is not surprising. In Boaler's previous study of students working in traditional classrooms (1997b, 1997c), the students who became most alienated, and ultimately unsuccessful, were at one time the highest attaining mathematics students in the school; their attainment progressively deteriorated as their mathematics teaching became more procedural. The alienated students in that study—most of whom were girls—were capable of practicing the procedures they were given and gaining success in the classroom and on tests, but they desired a more connected understanding that included consideration of "why" the procedures they used were effective. The alienated students in this study seemed to be rejecting mathematics for the same reasons. With a remarkable degree of consistency, the students who stated that they liked mathematics (11 of the 18 were boys) gave reflections that suggested they did so precisely because they wanted to be received knowers, with minimal requirement to interpret knowledge or think about connections within and across the mathematical domain:

Int: Why do you like math?

S: Because I think with so many of the other classes—what they teach and how they teach it, they're opinionated and political and it all depends, it's never the same, you can never depend on it. But with math, it's pretty constant. (Seth, Lime school)

R: For me, it's one of my strongest subjects and for me it's something I'm happy about and feel good in. Again, it's that methodology of mathematics that leads to the one answer that you can get, that there's no answer other than that. (Rich, Cherry school)

J: I always like subjects where there is a definite right or wrong answer. That's why I'm not a very inclined or good English student. Because I don't really think about how or why something is the way it is. I just like math because it is or it isn't. (Jerry, Lemon school)

It seems striking that the students in didactic classes who liked mathematics did so because there were only right and wrong answers, and because they did not have to consider different opinions or ideas, or use creativity or expression. Jerry states that he likes mathematics because he does not have to "think about how or why," the implications of which will be pursued in the conclusion to this chapter.

Belencky and associates describe "received knowers" in the following way: "For those who adhere to the perspective of received knowledge, there are no gradations of the truth—no gray areas. Paradox is inconceivable because received knowers believe several contradictory ideas are never simultaneously in accordance with fact. Because they see only blacks and whites but never shades of grey" (1986, p. 4). The three students quoted previously appear to exemplify this position. Seth describes other classes as "opinionated and political and it all depends," which to him was the anathema of the knowing he wanted. The following student reflects on the students who liked mathematics in his class, with a description that is strikingly similar to that of Belencky and associates' (1986) received knowers:

T: There's definitely a certain type of person who's better at math. Generally, if you're better at English they seem to be more social. And the math people. I don't know, they're just as social, but in a different way. They express themselves differently, they like to see things in black and white. They don't see the colors and greys between. (Tom, Lemon school)

Tom suggests that a certain type of person is attracted to mathematics and a certain type rejected. The differences align, not with capability, but with the ideas of knowing available.

Another striking aspect of the students' reflections on their response to mathematics, concerned the primacy they gave to their developing identities. A large proportion of the students interviewed appeared to reject mathematics because the pedagogical practices with which they had to engage were incompatible with their conceptions of self. Unlike the people Belenky and associates (1986) characterized as received knowers, for whom received knowing was a general pattern in their lives, these students considered themselves as constructive knowers in other school subjects. They understood themselves as received knowers in the limited circumstances of the mathematics classes in which the learning practices available to them required that they acquire specified procedures with no opportunity that they perceived to be thoughtful or creative about what they needed to learn to do. The students' descriptions particularly centered around the need they perceived for occasions to be creative, use language, and make decisions, for example:

Int: Why wouldn't you major in math?

C: I think I'm a more creative person, I can do it and I can understand it but it's not something I could do for the rest of my life and I think if I had a job I'd like one that let me be a little more creative.

Int: Math isn't creative . . . ?

C: No. (Cathy, Lemon school)

S: Well it's not that I don't understand it, when I understand concepts I like doing it because it's fun. I'm more of a language/history person, kind of. And also there's only one right answer and you can, it's not subject to your own interpretation or anything. (Susan, Cherry school)

Int: Do you like math?

V: No, I hate it.

Int: Why do you hate it?

V: It's just too, I'm into the history, English. . . . It's like too logical for me, it always has to be one answer, you can't get anything else BUT that answer. (Vicky, Lime school)

One of our calculus students suggested that women, in particular, needed to identify with subjects that allowed them to explore rather than receive knowledge:

I think women, being that they're more emotional, are more emotionally involved and math is more like concrete, it's so "it's that and that's it." Women are more, they want to explore stuff and that's life kind of like and I think that's why I like English and science, I'm more interested in like phenomena and nature and animals and I'm just not interested in just you give me a formula, I'm supposed to memorize the answer, apply it and that's it. (Kristina, Apple school)

A range of studies support the idea that girls and women are particularly likely to reject subjects that preclude deep, connected understanding (Becker, 1995; Burton, 1995). This suggests that the procedural presentations of knowledge that dominate within many higher level mathematics classes are likely to be a major factor in the under-representation of women at high levels. This seems partly to be due to the desire for connected understanding that is evident among many girls and women (Becker, 1995; Boaler, 1997c), and partly due to the need to pursue subjects that fit with developing identities. For many girls, mathematics appears too alien, otherworldly, and "weird" to be a major part of their lives:

B: I used to love math, but now I think, it's like I'm going to make sure that I don't major in math or anything because it's starting to be like too much competition, it's so weird. When it came to calculus and precalculus, I just kind of lost interest. I care more about science and English, stuff that makes sense to me where I think I'm learning morals and lessons from this, where I can apply it to something. (Betsy, Apple school)

L: I think that math is the lowest priority in my life. I don't have a favorite subject but math is the least important to me. It's OK and if I don't understand something I'm not going to die, I just don't think it's that applicable to what I want to do in life, and I don't even know what that is. (Lori, Lime school).

When students talked about their rejection of mathematics, their reasons went beyond cognitive likes and dislikes, to the establishment of their identities. They talked not about their inability to do the mathematics, but about the kinds of person (Schwab, 1969) they wanted to be—creative, verbal, and humane. Unfortunately for the students, there was a distinct inconsistency between the identities that were taking form in the ebb and flow of their lives and the requirements of AP-calculus classrooms. The students did not want to be told what to do and do it—when it was "completely irrational" (Anthony, Lime school); they were not prepared to give up the agency that they enjoyed in other aspects of their lives, or the opportunities to be creative, use language, exercise thought, or make decisions. The disaffected students we interviewed were being turned away from mathematics because of pedagogical practices that are unrelated to the nature of mathematics (Burton, 1999a, 1999b). Most of the students who had rejected mathematics in the four didactic classrooms—nine girls and five boys, all successful mathematics students—did so because they wanted to pursue subjects that offered opportunities for expression, interpretation, and agency.

Discussion Based Teaching and Connected Knowing

At Grape and Orange schools, 15 of the 16 students said that they enjoyed mathematics (94 percent), and 8 out of the 10 students asked (80 percent) stated that they planned to continue with other mathematics courses. Two of the girls interviewed stated that they planned to major in mathematics. Veena, cited in the following passage, was one of them:

> Sometimes you sit there and go "it's fun!" I'm a very verbal person and I'll just ask a question and even if I sound like a total idiot and it's a stupid question I'm just not seeing it, but usually for me it clicks pretty easily and then I can go on and work on it. But at first sometimes you just sit there and ask—"what is she teaching us?" "what am I learning?" but then it clicks, there's this certain point when it just connects and you see the connection and you get it. (Veena, Orange school)

One of the interesting aspects of Veena's statement is her description of herself as a "verbal person," which was one of the common reasons students gave for rejecting mathematics in the four didactic classes. Indeed, it seems worrying, but likely, that Veena may have rejected mathematics if she had been working in one of the four other classrooms in which the discussions and connections she valued were under-represented. It seems clear from Veena's statement that she valued both connected understanding as well as the opportunities she received to express her thinking, and that both of these were part of her mathematics world.

At Lemon, Lime, Apple, and Cherry schools, almost half of the students reported negative identifications with mathematics, in some cases contrasting this with their more positive identifications with other subjects, such as English, where they could be thoughtful or creative. Many of those students who reported positive identifications did so because mathematics allowed them to passively receive knowledge. Students at Grape and Orange schools, in contrast, identified more positively with mathematics and many of them did so because they were able to be thoughtful and to develop connected, relational understanding.

DISCUSSION AND CONCLUSION

In California, as well as other parts of the world, important decisions about mathematics education are being made by university mathematicians (Becker & Jacob, 2000). The majority of the mathematicians who involve themselves in matters of K-12 education appear to be outspoken opponents of nontraditional teaching methods. Mathematicians frequently argue that students in schools should be taught abstract mathematical procedures through repeated practice of the procedures, in order that they reach university conversant in the range of methods that they will need to use and apply there. But their assumptions about the importance of procedure repetition contain two important flaws. First, it is assumed that by practicing procedures out of context, students will be able to use and apply procedures in the future—a number of studies provide evidence that this is often not the case (Boaler, 1997a; Lave, 1988). Second, they overlook the fact that students do not just learn mathematics in school classrooms, they learn to *be*, and many students develop identities that give negative value to the passive reception of abstract knowledge. It is probable that many able students who could become world-class mathematicians

leave mathematics because they do not want to author their identities as passive receivers of knowledge. This has been the subject of this chapter.

We have found it useful to consider mathematics classrooms in terms of "figured worlds" in which people are not only regarded as mathematics learners, assuming the cognitive order of the discipline, but people negotiating a sense of self. Figured worlds seem to extend Bourdieu's (1986) notion of habitus, which includes the norms and practices of a place, to the interpretations that people make and the significance that is attached to certain acts. Thus, what actually happens in mathematics classrooms matters less within representations of figured worlds than the teachers' or students' perceptions of what happens. This has been useful in this chapter in helping us understand students' responses to didactic pedagogy in which memorization and procedure repetition are central practices. Corbett and Wilson talk about the exclusion of students from conversations about reform, with adults dictating to students the "conditions of their participation" (1995, p. 15). In didactic mathematics classrooms, students' participation is defined by textbooks, rules, and procedures—they are excluded from the negotiation or development of procedures; they are restricted in their application of selves; and their ideas, inventiveness, and general agency do not appear to be valued. Becker (1995) has proposed that "connected teaching," in which teachers share the process of mathematical problem solving with students rather than presenting neatly solved problems and procedures, would enable connected knowing, making mathematics more equitably accessible, and also encouraging larger numbers of students to pursue mathematics as a career. She asserts that "mathematics needs to be taught as a process, not as a universal truth handed down by some disembodied, non-human force" (Becker, 1995, p. 168). Stephen Ball (1993) also talks about the "curriculum of the dead," describing the inclination of right-wing politicians to support curriculum that are composed of remote facts, in which students have no role to play other than receivers of those facts. Becker and Ball both highlight the nonhuman or nonliving characteristic of traditional curricula. Data from this study suggest that many students find the narrowly defined roles they are required to play within such curricula incompatible with their developing identities.

The type of participation that is required of students who study in discussion-oriented mathematics classrooms is different. Students are asked to contribute to the judgment of validity, and to generate questions and ideas. Students in this study described their involvement within such environments in terms of community participation and family relationships. The students in the discussion-based environments were not only required to contribute different aspects of their selves, they were required to contribute *more* of their selves. In the discussion-based classrooms students were, quite simply, given more agency. To be a successful participant of a traditional classroom, students need to give up their choice and decision making, which is reflected in the students' comments about obedience and compliance. The act of surrendering their thoughts and ideas is difficult for many students, including those who could make significant contributions to the discipline of mathematics.

Belencky and associates (1986) have presented different characterizations of "knowers," suggesting that some people need to make connections as they learn, either directly through interactions with people, or indirectly through interactions with texts and other representations of knowledge. Others prefer to receive knowledge that is derived solely from a separate authoritative source. For the students studying calculus in the four didactic classes, there appeared to be few, if any, opportunities for connections, and students were forced to become received knowers. We do not regard the categories that Belencky and associates offer as stable characteristics of the students we interviewed, as these learners seemed able to move in and out of different "forms of knowing" in different circumstances. Indeed, some learners enjoy the chance to receive knowledge in some circumstances as a change from others in which they are thinking and making connections. Nevertheless, it seemed clear that the mathematics environments at Lemon, Lime, Apple, and Cherry schools encouraged a particularly passive and received form of knowing that alienated many learners.

We have analyzed interviews with 48 high-attaining students in this chapter, 32 of whom were taught in traditional classes, 16 in discussion-oriented classes. Seventeen of the 48 students reported that they hated or disliked mathematics, 16 of these students (94 percent) were taught in the traditional classes. These are small numbers of students, but the students' reflections in interviews give meaning to this percentage. The students who were planning to leave the discipline wanted the opportunity to think, negotiate, and understand the procedures they encountered. When mathematicians oppose nontraditional pedagogies, they argue that they do not want the discipline to be "watered down" and they want standard procedures to be available for those students who choose to major in mathematics. But by emphasizing the drill and practice of procedures, they create a rite of passage that is attractive only for received knowers. This reduces the numbers of students who want to study mathematics at advanced levels to a critical minimum (at Stanford University, for example, approximately seven students per year choose tc major in mathematics); it also eliminates creative, divergent thinkers from the discipline. The elimination of such learners may be extremely damaging for mathematics and out of place with our time (Noss, 1991). In years gone by, students may not have expected to challenge or negotiate ideas in school and procedural mathematics practices were less distinctive. Now students are offered choices, they expect to have their ideas valued, they enjoy being treated as responsible young adults, and many do not choose mathematics.

Burton (1999a, 1999b) conducted an important study, one of the first to give us insight into the practices of university mathematicians. She interviewed 70 research mathematicians to find out about the nature of their work, as well as their understanding of knowing. She found that the mathematicians emphasized the importance of intuition, uncertainty, and connectivity. Not surprisingly, they did not talk about the procedural nature of mathematics, but rather about the creativity of the enterprise with which they were engaged. They spoke about the euphoria they experienced when solving problems and the fun and excitement of mathematics, thus offering a sharp contrast to the views of many of the students interviewed in this study. Even those students we interviewed who enjoyed mathematics in didactic classes did not relate their enjoyment to the pleasure of problem solving, but to the structure and limits of the discipline as they experienced it. While the mathematicians Burton interviewed emphasized the uncertainty of their explorations, the students in didactic classes who liked mathematics emphasized the certainty of their work. This suggests that narrow mathematical practices within school are problematic, not only because they disenfranchise many students, but because they encourage forms of knowing and ways of working that are inconsistent with the discipline. Thus, school mathematics, as noted by Burton (1999a) and others, is unlike the mathematics encountered in life or university (Boaler 1997a; Noss, 1991).

Burton's results remind us that that the excitement of mathematical inquiry and discovery is not always experienced as a product of social interaction. Indeed, the prevailing image of mathematical work is that of an individual struggling, like Rodin's *The Thinker,* to find coherence in a deep conceptual problem, and experiencing near ecstasy if s/he is able to achieve an elegant solution. In our discussion here, we have opposed an individualistic version of received knowing of procedures with socially connected knowing that has conceptual depth. Those are the kinds of knowing that were reported by the students whom we interviewed, but we acknowledge that the coupling of social and conceptual aspects is not necessary or universal. Skills that are mainly procedural can be learned in socially cooperative environments (consider learning the steps of a traditional Gaelic dance in Britain or a square dance in the United States), and individuals can and do explore conceptual issues beyond the boundaries of the current understandings that members of their community support (consider the familiar examples of Galois and Ramanujan in the history of mathematics). We acknowledge that in mathematics classrooms, activities can be organized so the students engage in enjoyable social interaction without achieving significant mathematical learning—either conceptual or procedural. And there are some students who perceive even quite productive class discussions as "clutter" that distracts them from the concentrated individual attention to mathematical concepts and methods that they prefer. The conceptual framework that we have used to interpret the data of our interviews would need to be extended to accommodate examples of engaged conceptual knowing that is only weakly supported by discourse interactions in the individual's immediate learning community. We believe that such an extension could be quite important for mathematics education. It could involve hypothesizing a form of connected knowing that emphasizes the knower's being connected with the contents of a subject-matter domain.

We believe that the distinction between separate and connected knowing, discussed by Belenky and associates (1986) and Clinchy (1996) mainly in relation to other people, can also be used to understand different ways in which learners can relate to the things, ideas, and representations of a subject matter. Belenky and associates used Elbow's (1973) phrases, "the doubting game" and "the believing game," to convey an important difference between learners' ways of considering other people's experience and opinions in their distinction between separate and connected knowing. We consider these phrases as distinguishing between attitudes that can be directed toward things, texts, and ideas that a person studies, as well as toward other people with whom one communicates. One way that this distinction applies is in the relation of scientists to the things or people that they study. For example, many ethnographers have made the point that their main task is to understand the people they study in their own terms, the so-called "emic" stance. To take this ethnographic stance toward the people being studied is a version of "the believing game," in which the researcher assumes that the practices of other people are sensible, and the ways in which they are sensible can be learned by the researcher if he or she succeeds in their study. To succeed, the researcher needs to incorporate the ways of sense-making of the studied people into her or his ways of understanding. We consider this a profound kind of connected knowing, in which the knower-researcher has come to know the people he or she studied through an openness to their ways of knowing and understanding.

The attitude of openness—of understanding things on their own terms—also is expressed by scientists whose subject matter is not human social activity. A famous example is Barbara McClintock's attitude toward the plants she studied, expressed in the title of her biography, *A Feeling for the Organism* (Keller, 1983). Keller's characterization of McClintock's work and thinking includes phrases that identify her openness to learning by interacting sympathetically with the plants that she attended to so carefully. "McClintock's feeling for the organism is not simply a longing to behold the 'reason revealed in this world.' It is a longing to embrace the world in its very being, through reason and beyond" (Keller, 1983, p. 199). "Over the years, a special kind of sympathetic understanding grew in McClintock, heightening her powers of discernment, until finally, the objects of her study have become subjects in their own right; they claim from her a kind of attention that most of us experience only in relation to other persons" (Keller, 1983, p. 200).

We also contend that "the believing game" characterizes some people's learning relationship with conceptual domains. Indeed, the kind of study and knowing that characterized McClintock's scientific work is not just intimate and sympathetic incorporation of the subject's character into the scientist's way of understanding the world. It also includes finding an orderly and coherent explanatory account in the conceptual resources of the scientific discipline. Learning these conceptual resources can be approached either in the "believing" or the "doubting" game. More accurately, any conceptual learning includes a mixture of these attitudes, but we believe that people—including students—differ in their basic expectations regarding the prospects of a subject matter to provide productive sense-making resources in return for the effort of incorporating them into one's understanding. Engaging in "the believing game" with the concepts and methods of a subject-matter domain is relational, involving positive expectations both about the subject matter and about one's self as a learner of that subject.

The connection of a successful scientist with both the phenomena and the explanatory concepts of the subject-matter domain supports her or his expectation that there is order to be discovered in the phenomena and that the conceptual re-

sources of the discipline can provide a way of representing that order and thereby explaining the phenomena. The moments of drama that scientists report often involve situations in which the scientist was faced with phenomena that he or she hoped or expected to be able to explain with the resources of her or his discipline, but seemed unable to; then achieved the understanding that was needed. An example in McClintock's biography occurred during a visit to Stanford as a relatively young, albeit quite well-established scientist, where she had been invited in the hope that she could solve a problem of working out the cytology of mutations and enzyme deficiencies in the bread mold *Neurospora.* At first, things did not go well. In Keller's telling:

> By her own account, her confidence had begun to fail even before setting out. "I was really quite petrified that maybe I was taking on more than I could really do." She went, set up the microscope, and proceeded to work, but after about three days, found she wasn't getting anywhere. "I got very discouraged, and realized that there was something wrong—something quite seriously wrong. I wasn't seeing things, I wasn't integrating. I wasn't getting things right at all. I was lost." Realizing she had to "do something" with herself, she set out for a walk.
>
> A long winding driveway on the Stanford campus is framed by two rows of giant eucalyptus trees. Beneath these trees, she found a bench where she could sit and think. She sat for half an hour. "Suddenly I jumped up, I couldn't wait to get back to the laboratory. I knew I was going to solve it—everything was going to be all right."
>
> She doesn't know quite what she did as she sat under those trees. She remembers she "let the tears roll a little," but mainly, "I must have done this very intense, subconscious thinking. And suddenly I knew everything was going to be just fine." It was. In five days, she had everything solved. (Keller, 1983, p. 115)

We interpret this as an example of reasoning that depended on McClintock's knowing of biological concepts, which supported her intuitive confidence in the potential validity of an explanatory scheme that she then went back and worked out.

We recognize that mathematics is unlike empirical sciences in that it lacks a domain of empirical phenomena that concepts and principles are used to explain. However, the natural histories of mathematical discoveries contain the same kinds of drama as they do in other domains. Here is an example, from Aczel's recounting of the proof of Fermat's Last Theorem. The situation was subsequent to Andrew Wiles's presentation of his analysis, which he believed to be a proof, at Cambridge and the discovery by others that his argument was flawed.

> When more than a year passed since his short-lived triumph in Cambridge, Andrew Wiles was about to give up all hope and to forget his crippled proof.
>
> On Monday morning, September 19, 1994, Wiles was sitting at his desk at Princeton University, piles of paper strewn all around him. He decided he would take one last look at his proof before chucking it all and abandoning all hope to prove Fermat's Last Theorem. He wanted to see exactly what it was that was preventing him from constructing the Euler System. He wanted to know—just for his own satisfaction—why he had failed. Why was there no Euler System?—he wanted to be able to pinpoint pre-

cisely which technical fact was making the whole thing fail. If he was going to give up, he felt, then at least he was owed an answer to why he had been wrong.

> Wiles studied the papers in front of him, concentrating very hard for about twenty minutes. And then he saw exactly why he was unable to make the system work. Finally, he understood what was wrong. "It was the most important moment in my entire working life," he later described the feeling. "Suddenly, totally unexpectedly, I had this incredible revelation. Nothing I'll ever do again will . . ." at that moment tears welled up and Wiles was choking with emotion. What Wiles realized at that fateful moment was "so indescribably beautiful, it was so simple and so elegant . . . and I just stared in disbelief." Wiles realized that exactly what was making the Euler System fail is what would make the Horizontal Iwasawa Theory approach he had abandoned three years earlier *work.* Wiles stared at his paper for a long time. He must be dreaming, he thought, this was just too good to be true. But later he said it was simply too good to be *false.* The discovery was so powerful, so beautiful, that it *had* to be true. (Aczel, 1996, pp. 132–133)

And it was, as the mathematics community soon verified when Wiles circulated copies of his new argument to several close colleagues and successfully submitted his paper, co-authored by Richard Taylor, as a correction to the paper that Wiles had presented at Cambridge.

When mathematicians decide that a proof is "true," their judgment does not rest on empirical observations of the kind that biologists or physicists use. Instead, their verification depends on the outcome of applying accepted procedures of proof and computation. Pickering's (1995) analysis of agency in mathematical and scientific work provides a helpful framework for considering this.[3] In Pickering's terms, an advance in mathematics involves three processes, called *bridging, transcription,* and *filling.* Bridging involves a proposal for making some extension of a base model—that is, a set of accepted concepts and methods (establishing a "bridgehead"). Transcription involves transferring components of the base model analogically to the bridgehead—that is, attempting to treat the contents of the new topic with methods that are previously accepted. Filling involves providing additional definitions of terms in the new domain or modifying (preferably by generalization) methods from the base model. In the process of transcription, the mathematician performs procedures that he or she is not free to vary; Pickering refers to this as *agency of the discipline.*[4]

It is in bridging and filling that the agency of mathematical work resides with the human mathematical thinkers. In Pickering's words,

> As I conceive them, bridging and filling are activities in which scientists display choice and discretion, the classic attributes of human agency. . . . Bridging and filling are free moves, as I shall say. In contrast, transcription is where discipline asserts itself, where the disciplinary agency just discussed carries scientists along, where scientists become passive in the fact of their training and established procedures. Transcriptions, in this sense, are disciplined forced moves. Conceptual practice therefore has, in fact, the familiar form of a dance of agency, in which the partners are alternately the classic human agent and disciplinary agency. (1995, p. 116)

Interpreting the example of Fermat's Last Theorem in these terms, Wiles's initial bridgehead was a proposal to prove the theorem by constructing an Euler System, and he believed that he had accomplished that transcription successfully, but his colleagues discovered that he had not. He revised the bridgehead to use the Horizontal Iwasawa Theory. When he said, in recollection, "The discovery was so powerful, so beautiful, that it *had* to be true," he expressed his conviction that the methods of that theory could be transcribed successfully for the proof of Fermat's Last Theorem. And this conviction proved correct.

We find Pickering's distinctions helpful in understanding differences like those between the four didactically procedure-oriented classrooms and the two more discussion- and conceptually oriented classrooms of the students interviewed for this study. We consider that learning mathematics is like doing mathematics in at least one important respect. At any stage of learning mathematics, learners have some concepts and methods that they already know and understand. Their next learning extends what they already know. We can think of a learning episode, then, as one that includes bridging and transcribing, and possibly filling, so that some new topic is included in, and integrated with, some of their previous mathematical knowledge. In didactic, procedure-driven learning, students are shown constituents of the mathematical discipline to absorb so they can apply them. In learning that focuses more on conceptual discussion, they can participate in bridging and filling, and experience the functional significance of transcription. In other words, by including students in processes of meaning-making, they can experience and learn those aspects of mathematical thinking—bridging and filling—in which human agency is significant. If their opportunities for learning are limited to acquiring procedures, then their understanding and perception of mathematics can easily be limited to the aspects of mathematical thinking in which the human agent is relatively passive.

As we know, a few students develop identities of significant mathematical agency even in didactic learning environments that mainly present the parts of mathematical practice that are performed passively. In our conceptual framework, the explanation for these exceptional students probably arises from their authoring of identities that overcome deficiencies of the environment. Holland and associates (1998) described some women in the southern American colleges they studied who resisted the general pressure to adopt mainly passive positions in their figured social worlds that were primarily concerned with romantic attractiveness. These women authored identities with significant social and intellectual agency that departed from the norms that most women complied with. It is not surprising, then, that there are some students in figured social worlds of didactic mathematics teaching who author identities of individual agency in which they construct meaningful understanding and capabilities to formulate questions, conjectures, and arguments that provide satisfying conceptual coherence in their practices of mathematical knowing.

We believe that careful study of mathematical learning environments could provide important understanding of the development of learning identities with significant agency. We hypothesize that many didactic classrooms are organized to promote as "gifted" one or two students whose relation to the subject matter of mathematics is extraordinarily agentive. We expect that teachers and students in these classrooms probably attribute these students' lack of conformity to some combination of mathematical "talent" and "interest" that both motivates and supports their unusual participation in learning practices. It may also include being at least slightly "weird" (Boaler, 2000b). We also expect that identification of students to fill the few available "gifted" slots involves interesting interactions in which parents and others play significant roles, and support for these slots probably is biased in favor of boys from relatively affluent families.Our data support the conjecture that when mathematics learning practices place students in positions with more significant conceptual agency, it is much easier for many of them to author their identities as learners with that kind of agency. The positional identities that students in the discussion-oriented classrooms expressed had agency that did not require them to resist the prevailing expectations and become identified as especially "gifted."

Mathematicians often complain about the dependency of undergraduate students, and one of Burton's research subjects characterized the common concern: "One of the things I find about students, undergraduates in particular, is that they seem to have very little intuition. They are dependent upon being spoon-fed" (cited in Burton, 1999b, p. 37). But whereas many mathematicians who are critical of school practices link such problems with the "reform movement," this study suggests that lack of intuition and over dependency are more likely to be a product of narrow pedagogical practices in traditional classrooms. The students who like mathematics because they believe it is abstract and definitive, with only one correct answer, are engaging in a distressingly limited version of the discipline of mathematics (Schoenfeld, 1988). The certainty they have come to enjoy, and on which they will make decisions about future subjects, appears to be inconsistent with the mathematics with which they would engage at the highest mathematical levels. Students who choose mathematics as their main field of study, based upon the idea that the subject is structured, certain, and nonnegotiable, may encounter significant problems as the mathematics they learn at university becomes more advanced. Those who rise to the top of their undergraduate classes and eventually become mathematicians must surely be those who have a deep, conceptual understanding of the material, and students such as Jerry, cited earlier, who do not like to think about "how or why something is the way it is" may be limited in the understanding that they develop. It is unusual for undergraduates who excel in mathematics to become school teachers of mathematics, which means that the mathematics majors who choose teaching are sometimes those who have preferences for "received knowing" that were developed in school and that have limited their attainment. Thus, we perpetuate a cycle of received knowers, teaching received forms of knowing. This is a highly speculative interpretation that we nevertheless offer as a hypothesis to help explain the recurrence of traditional mathematical practices in schools (Cohen, 1990; Fennema & Nelson, 1997).

There is evidence that knowledge presented in an abstract, decontextualized way is more alienating for girls than boys (Becker, 1995; Belencky et al., 1986; Boaler, 1997c)—and for non-Western than Western students (Banks, 1993). This suggests that traditional pedagogical practices will maintain inequality in the attainment and representation of mathematics students, particularly at the highest levels, even as stereotypical societal expectations diminish. University mathematics will continue to be a white, male preserve and large numbers of girls and students from particular minority groups will be excluded from a subject at which they could excel. By giving the final word to equity, we hope to communicate our commitment to a different world in which classroom mathematical practices support the development of thinking, responsible agents and mathematical identification and knowing becomes a possibility for a broader, more diverse group of students.

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NOTES

1. Belenky and associates (1986) also characterized *silent knowing,* in which the individual considers herself powerless to know, and *constructed knowing,* which combines the stances of separate and connected knowing and emphasizes the knower's agency in the process of achieving and legitimizing knowledge. We use the term *constructive knowing* to refer to separate knowing, connected knowing, and constructed knowing. We depart, therefore, from Belenky and associates' use of the term *procedural knowing* for separate and connected knowing, mainly because "procedural" is a term that the community of mathematics education research uses to refer to a specific form of knowledge, consisting mainly of knowing how to perform procedures.

2. The interviews were conducted by Jo Boaler and Megan Staples.

3. Pickering developed these distinctions for mathematics with the example of Hamilton's construction of quaternions. He also discussed an example from physics, in which he analyzed agency as being divided by scientists and material systems; one example was Davis's development of the bubble chamber, and he attributed *material agency* to the functioning of apparatus that interacts with the agency of humans who construct apparatus in the hope that the material will behave in ways that support the construction of scientific findings related to theoretical issues.

4. To quote Pickering: "Think of an established concdptual practice—elementary algebra, say. To know algebra is to recognize a set of characteristic symbols *and how to use them*. . . . Such uses are [what I call] detached disciplines. . . . they are machinelike actions, in Harry Collins's terminology. Just as in arithmetic one completes '3 + 4 = ' by writing '7' without hesitation, so in algebra one authomatically multiplies out '*a*(*b* + *c*)' and '*ab* + *ac*.' Conceptual systems, then, hang together with specific disciplined patterns of human agency, particular routinized ways of connecting marks and symbols with one another. Such disciplines—acquired in training and refined in use—carry human conceptual practices along, as it were, independently of individual wishes and intents. The scientist is, in

this sense, passive in disciplines conceptual practice. . . . I want to redescribe this human passivity in terms of a notion of disciplinary agency. It is, I shall say, the agency of a discipline—elementary algebra, for example—that leads us through a series of manipulations withing an established conceptual system.

The notion of discipline as a pereformative agent might seem odd to those accustomed to thinking of discipline as a constraint upon human agency, but I want (like Foucault) to recognize that discipline is productive. There could be no conceptual practice without the kind of discipline at issue; there could be only marks on paper" (Pickering, 1995, p. 115).

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8

"Cracking the Code" of Mathematics Classrooms: School Success As a Function of Linguistic, Social, and Cultural Background

Robyn Zevenbergen

INTRODUCTION

The role of language in the teaching and learning of mathematics has been given increasing recognition over recent years. Much of this attention has been inspired by constructivist epistemologies that have placed aspects of language central to the learning process. These literatures have shown how language is inextricably bound to learning. It provides the medium through which communication of ideas is made possible, and negotiation of ideas and concepts is delivered. These literatures has alerted educators to the mismatch of language between experts (teachers) and novices (students) and suggested that a more appropriate level of language and communication is made possible through dialogue among the students. This chapter extends this work by drawing attention to the political nature of the language used in classrooms. Drawing on Bourdieu's notion of cultural capital, or more particularly, linguistic capital, it is argued that some students will have greater or lesser access to the modes of communication in a classroom, and hence have more or less access to the mathematics inherent in such communications.

Three common communicative strategies found in mathematics classrooms form the basis of this chapter. The first is the type of questions commonly found in texts and tests. These represent the register of mathematics that I argue is very structured and that students must come to learn in order to be able to participate in a productive and effective manner. The second communicative strategy is that of classroom talk, which has its own internal rules that are not made explicit to students but form the basis for communication in the classroom. The third and final example is that of what comes to constitute legitimate knowledge in the classroom, and this is bound to the contexts used to embed mathematical tasks.