DRAGON PLAY: MICROWORLD DESIGN IN A WHOLE-CLASS CONTEXT*

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ABSTRACT

We propose a whole-class view of microworld design, in which the total ecology of students, technology, and teacher is considered by the decisions and choices of the design. Through a case-study analysis of a classroom implementation, we illustrate a dynamic geometry microworld designed using such an approach and attempt to identify both the effects such an approach has on student affect and participation and the roles that educational technology fills in such a social context. We argue that whole-class microworld design not only has practical benefits in terms of classroom time and management, but also develops social interactions conducive to educative learning experiences.

What makes for effective microworld design? Perspectives from the mathematics education community identify many important objectives: plurality of linked representations, curricular topicality; more generally, reification of learning hypotheses and cognitive models (Hoyles & Noss, 1992; Steffe & Olive, 1996;
Steffe & Wiegel, 1994). Many of these design objectives focus on the close relationship between student and technology. This relationship might be from the perspective of the functions and features a microworld makes available to its user, or from the analysis and interpretation of students’ decisions and actions “inside” the microworld. These emphases attest to the monopoizing physical presence of the machine—one’s hands fully engaged with mouse and keyboard, one’s attention rapt before the video screen. And yet, in cultivating such attention, such emphases perhaps underattend the social context in which students’ mathematics is deeply embedded, and the evidence supporting the cognitive and affective benefits of classroom environments fostering interaction, transaction, and reflection among participants (Boaler, 1997; Cobb, Boufi, McClain, & Whitenack, 1997; Noddings, 1989; Yackel & Cobb, 1996). Said another way, from a lay perspective, these design objectives might be better considered to describe one’s perception of a microworld’s function (how we use a tool) than its design (how we feel about how we use a tool; and how we perceive the contexts and environments in which we find ourselves using one). We take an object to be functional if it accomplishes the primary objectives intended for its use by its creator. On the other hand, an object—be it a microworld or a chair—strikes us as “well designed,” only if it reaches beyond functional imperatives, and delivers its functional potential to its user within some matrix of delight, promise, surprise, and satisfaction (Smith & Tabor, 1996). We use functional objects perhaps without thought. By contrast, we are inevitably pleased to find ourselves using well-designed ones. And we identify well-designed objects because we continuously attend to their particular consequences—that is, to our own curiosity, anticipation, and pleasure—as we strive to accomplish our (own, or externally imposed) functional objectives. Thus effective design—beyond effective function—provokes a dual form of reflection: upon the object, as it accomplishes its function; and simultaneously upon ourselves, in the contexts in which we manipulate the object.

The proximity of these conceptions—of how an object’s design mediates not only its function, but also our conscious affective relationships to the object itself, and to ourselves and our context as we wield it—to Dewey’s (1938) emphasis on reflection and interaction as the foundations of an educative experience, motivates this study. To explore this relationship, we propose a view of microworld design in which the total ecology of students, technology, and teacher is taken into account; and we argue that this expanded and socially-contextualized view of the microworld may satisfy not only many functional goals of effective microworlds, but also affective and discursive goals.

In particular, we are interested in the apparent dichotomy of use characterizing today’s prevalent Dynamic Geometry technologies. Programs such as The Geometer’s Sketchpad (Jackiw, 1991, 1995) and Cabri Géomètre (Baulac, Bellemain, & Laborde, 1992, 1994) have tremendous currency in today’s classrooms: basal textbooks, policy frameworks, and adoption policies exhort their use
(Alexander, Canton, Harrison, McLeish, Nielsen, & Sinclair, 1999; Hollowell, Schultz, Ellis, & Wade, 1997; NCTM, 2000); and within the United States, mathematics teachers rate Sketchpad as the “most valuable software for students” (Becker, Ravitz, & Wong, 1999). Yet, in our experience working with Sketchpad-using teachers, we find these technologies’ most frequent classroom application is to reinforce textbook lessons through supplemental computer-based activities, usually in the form of step-by-step geometric construction directives leading to a “guided-discovery” conclusion which teachers assess through (off-line) written responses.¹ In effect, these conventional modes of use contrast starkly with the active and provocative life of Dynamic Geometry outside the classroom—in the imaginations and investigations of mathematicians, researchers, and educators who pursue mathematics recreationally or professionally to satisfy curiosity, develop understanding, or—more simply—enjoy themselves mathematically. Schattschneider and King (1997) collect a representative portfolio of these enthusiastic voices. Thus, where the conventional classroom diligently pursues a functional application of Dynamic Geometry, outside of that classroom we see teachers and researchers energized by “great interest and enthusiasm” (Goldenberg & Cuoco, 1998, p. 351) for the tool—in other words, by the apparent affective dimensions of effective design.

We take as given that the conventional scripted dynamic geometry activity has achieved such prominence because it serves two eminently practical goals: advancement of curricular content objectives (that is, these activities tell us “what to do”) and sufficient technology training or instruction to allow most students to attain these objectives in a self-directed fashion within a pre-defined amount of class or lab time (on their way to “what to do,” they tell us “how to do it”). Whether such activities lead also only to scripted conclusions depends on the quality of the activity; some seek to enable the conditions of relatively “open” discovery; others march toward fixed, numerical answers. In the process of reaching their eventual denouements, these activities structure and constrain students’ software interactions to a degree such that students rarely sense the potential versatility or evocative fertility of dynamic geometry, and may miss entirely the volubility of mathematical pursuits in these compelling environments. But more importantly, the strictly functional goals of these activities tend to mitigate possibilities for socially-oriented interactions.

Researchers have identified several strategies that can mediate the types of social interactions, transactions, and reflections that we aim to include in our microworld design, for example: small group discussions; whole class presentations, explanations, and act as administrators of internet forums for Sketchpad-using teachers in the United States and Canada.

¹ We inservice 400-500 secondary-level mathematics teachers annually, and act as administrators of internet forums for Sketchpad-using teachers in the United States and Canada.
and negotiations; and, shared constructions of objects and meanings. How can such strategies be deployed in the prevalent computer-based environments that promote individual reflection, activity, and meaning-making? (We are thinking here of the computer lab environment with its individualized work-space set-up and its “guided-discovery” activities, where teachers often spend much time putting out technical fires be they related to the computer or the software.) More specifically, how can microworlds be designed to explicitly encourage those socially-based behaviours conducive to educative learning experiences?

**TOWARD A SOCIAL DEFINITION OF “MICROWORLD”**

We call microworlds that are designed to accommodate these socially-oriented roles of technology microworlds designed through a whole-class approach. We propose that the whole-class approach can create a public space that allows students to ponder, articulate, and develop understandings collaboratively and consensually. This consensuality will be a frequent characteristic of the whole-class approach, but by no means a requirement. The attempt to develop a whole-class culture should not be confused with an attempt to homogenize, or to identify a greatest common denominator of individual student needs. While consensus-building is part of the pedagogic strategy, it is consensus-building—to whatever degree that consensus can be built—rather than consensus-brokering, where all participants stall at some imaginary negotiating table until, exhausted, they finally accept the terms of the deal (or fall excluded from the process). Put another way, the forms of knowledge emerging from whole-class social interactions are characterized more importantly by an agreement on terms and meanings rather than by agreement on conclusions or “facts.” In this context, dissent may be as powerful a voice as consent. Thus while positioning both mathematics and its technology in a public discourse keeps them open for negotiation, elaboration, and construction, we do not necessarily expect that each student will adopt the same orientation and approach to them.

Applying a whole-class approach to computational technologies implies that the meaning and relevant applications of such technology be collectively negotiated by the class rather than teacher driven. Of course, the teacher continues to play a strategic role in introducing potentially-useful new technologies, just as within a non-technology rich mathematical situation, the teacher remains responsible for ensuring that potentially-useful mathematical constructs are made available at moments when the need for them becomes established by a situation’s evolving dynamic.

The emphasis on the social and cultural practices that engender and legitimate such an environment focuses our attention not so much on the individual learner and her construction of knowledge, but on the whole classroom, the sources and growth of knowledge within it, and the interactions between its members. Such a perspective draws on an interactionist theoretical framework. For the
interactionist, as Sierpinska and Lerman (1996) suggest, “meanings are elaborated through negotiations whereby the group comes to agree on certain conventions in the interpretations of signs, situations, and behaviours” (p. 491). These negotiations have emergent properties: we can ask how individual contributions may add up to something that nobody in particular (teacher or student) has thought about or anticipated. This framework provides insight into the shaping and understanding of a whole classroom in which a rich mathematical situation can lead to the collaborative construction of a shared body of knowledge, remarkable both in breadth and depth.

ELEMENTS OF A WHOLE-CLASS MICROWORLD

Whereas the design of traditional microworlds has not paid explicit attention to the socially-oriented dimensions of a learning environment, leaving them instead in the hands of the teacher, we describe the design of a microworld design that attempts to incorporate these dimensions. The conception of microworld design we use here extends beyond decisions about a single software objects, and even beyond a collection of such objects (as in Battista, 1997) and (Bowers, 1995); instead, it includes decisions about the ways in which the software objects are introduced, perceived, and used in the classroom setting. The elements of this microworld design include:

- the introduction of Sketchpad in the context of an ongoing exploration—as a communication medium—rather than as a subject in its own right.
- the delay of the moment at which students might work directly with computers (one-on-one or in pairs) so the whole-class approach keeps the role and purpose of the technology open in a public space, and thus subject to class-wide discussion and improvisation.
- a rich mathematical situation (as described in Higginson, 1973) that: provides multiple levels of entry for students of varying ability and background; is fertile both in the number of branches it leads to and in the mathematical importance of these branches; enables students to make links between the situation and their previous experiences from both inside and outside the classroom; naturally leads to symbolization and generalization.
- the use of Sketchpad in multiple student configuration modes including individual or partner work, small group work, and classroom work.
- the use of Sketchpad in multiple modes, viz. as a vehicle for exploration, documentation, and inspiration.
- the emergent use of sketches as classroom artefacts that can help explore student questions
and problems, provide cognitive amplification, and mediate the move to mathematization.

We point out that the design features described above do not restrict themselves to the creation of the computer-based component of the learning environment; this in turn makes it more difficult to describe in precise, well-delineated terms the actual microworld. Moreover, one can imagine the actual character of the microworld varying with different students and teachers. Where directed educational research on individual learning can be well illustrated by vignettes that demonstrate a single learner’s response to, or position within, a pedagogical situation, such emphasis comes at the cost of de-emphasis of the social context.

In its relocation of mathematical meaning-making from the individual to the collective, the whole-class methodology is better understood through illustrations that focus on the global interactions between and among the classroom players and their environment. Similarly, a global perspective develops a sense of a classroom’s history, atmosphere, and possibilities. As such we have chosen to illustrate one possible instantiation of the microworld through a classroom narrative of a case-study implementation. This method of presentation also allows us to analyze some of the ways in which a whole-class approach to investigation and discovery permits students to develop and to reinforce both their technological and mathematical conceptions collectively and socially, and how in this process students may develop a confidence in—and orientation toward—dynamic geometry software that enables them to develop relevant, personal articulations of their process of mathematical learning and discovery. We emphasize that this study is exploratory: our objective is to refine the principles of whole-class microworld design by inquiring into the ways our tentative principles operate within the complex, local domain of classroom experience.

**DESCRIPTION OF THE CASE STUDY**

During Fall 1999, author Sinclair (with co-researcher Bill Higginson) conducted one class a week in a secondary school (grades 9–13) of approximately 1300 students located near Queen’s University, in Kingston. (Author Jackiw collaborated with Sinclair—throughout the semester but outside of the classroom—in lesson planning, activity design, and evaluation.) The 29 grade 9 students we work with are in a semestered mathematics course, running from September to January. The school, Kingston Collegiate and Vocational Institute, is considered one of the more “academic” high schools in the Kingston area, with a high rate of students continuing to post-secondary education. Classes meet every morning for 75 minutes. Prior to our study, the regular classroom teacher had been teaching both mathematics and science for five years at the
secondary level and had shown much interest in using new technologies in the classroom as well as in problem-based learning. He provided advice and guidance on our proposed activities, and acted as both an observer and assistant during the research classes. Our research activities are intended to complement a province-wide curriculum.

This research is a part of a larger project being conducted in a grade 9 classroom in Kingston, Ontario, that aims to develop activities reflecting the humanistic, aesthetic aspects of mathematics—dimensions that are often over-looked in our traditional utilitarian curricula and teaching practices. A primary research interest is to determine whether working in rich mathematical situations (Higginson, 1973) can encourage students to act as cognizers and creators of mathematics, to experience mathematics as a diverse, powerful, and evolving discipline with significant links to broad aspects of human experience (Higginson & Flewelling, 1996). Our explorations were not intended to address hypotheses framed at the level of curriculum deployment or policy—the current focus of our work is not on issues of scalability, and the new curriculum itself does not particularly mandate pursuit of rich mathematical situations for their own sake.

NARRATIVE AND THEMES

In the following sections, we describe the whole-class environment of our investigation in a chronological narrative drawn from author Sinclair’s annotated teaching journal. We briefly interrupt the narrative at various points in order to highlight specific design features and outcomes that are most pertinent to this study. In addition, we separate the narrative into four sections, interweaving them with more global, extended analyses of the narrative. For ease of reading, the narratives are set apart from comments and analyses by indentation. Following the opening narrative, we discuss the role of Sketchpad in the whole-class context. Following the second narrative installment we characterize the effect of the whole-class approach on students’ mathematical creativity. These two sections together address the place of public, or communal, applications of technology, and the enaction of the rich mathematical situation—each an essential element in binding disjoint student activity (socially and mathematically) into a coherent, shared discourse. In the third section, we consider the students’ interaction with technology while in the fourth section we return to the theme of whole-class learning. These last two sections turn particular attention to the specific dynamic geometry technology of our study (in terms of how

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2 Co-investigators for this research project (Tomorrow’s Mathematics Classroom – Grade 9) are Bill Higginson and Nathalie Sinclair. See http://tilp.educ.queensu.ca/faculty/tmc/ for a more in-depth description of the project.
students react to and appropriate Sketchpad within their mathematical narratives); and to a methodological reflection on the insights afforded by a whole-class perspective on individual and group student learning.

NARRATIVE I: ENTER THE DRAGON

I draw a sequence of three shapes on the blackboard starting with a simple segment (see Figure 1). In fact, this sequence of shapes shows the first iterations of the dragon curve, a non-intersecting fractal curve which some claim take its name from an ancient Chinese creation myth. These students aren’t quite sure what this has to do with a Chinese dragon they saw a moment ago, but they are curious.

“Can you see a pattern between the three shapes, a way of moving from one step to the next? What would be the next step in the pattern?”

I ask the students to work on those questions. Some prefer to work in groups and some prefer to work individually. They play around with different possible patterns. Many of the students focus on the number of segments that make up each shape (1, 2, 4) and attempt to extend the pattern based on this characteristic. Others experiment with the structure of the shapes, some seeing stair-like ones and others pyramid-like ones. Those who think that they have found possible solutions draw them on the blackboard for class discussion. After fifteen minutes, we gather together again in order to test whether the solutions on the blackboard “work.” It appears that not one of them do, so I suggest that they try different strategies for finding a pattern, instead of focusing solely on the number of segments in each shape. I hint that perhaps the work we had done in the previous class might be helpful.

![Figure 1. First generation curves.](image-url)

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1 See Scott (1997) for the story behind this myth.

2 While the initial questions framing this investigation were novel to the students, there is continuity with the classroom’s prior activity. We had just spent the several classes on questions emerging from the challenge of folding a paper strip into fractional parts, with a special focus on “thirding” and on discovering iterative end-to-end folding patterns (a rhythmic right-to-left, left-to-right folding) to approximate the non-exact (thirds, fifths, etc.) paper strip divisions. In fact, the continued use of the paper strip as a theme of investigation guided the selection and development of this activity.
Even before technology is introduced, the microworld activity context establishes group work and classroom discussion as “normal” modes of operation from the start. This helps establish a common set of experiences for the students—a set of problems, investigations, solutions, and vocabulary—on which they can draw in their later work with Sketchpad.

Not long after, Jasmine exclaims that she has figured it out: “I can make the pattern by using this strip of paper!” She shows the small group gathered around her how to do this step by step by (1) pointing out that a strip looks just like the first line segment when viewed “on-edge;” (2) folding the strip in half, and then unfolding it to form a right angle as shown in Figure 2, which corresponds to the second shape on the blackboard; and (3) refolding the strip, folding it in half once again, and unfolding it to form three right angles. The third step produces a shape that also corresponds to the shape on the blackboard.

She then folds her strip up one more time, unfolds it into right angles and produces the next iteration of the dragon curve. This new curve, with its seven right angles, provides Jasmine with quite a challenging shape to draw on the blackboard.

While all this paper-strip discovery was going on, another group of three girls had been working on trying to find symmetries in the shapes. They explain to me that the second shape contains “a reflection symmetry” and that if you take the initial segment and reflect is across a diagonal line, you

![Figure 2. Curves from a folded paper strip.](image-url)
would get the vertical part of the second shape. Of course, they tell me, you have to keep the horizontal part there too to make the whole shape. Then, not yet sure of this hypothesis, they try to explain the process using their arms. Starting with one arm horizontally, each girl rotates her other arm into a vertical position. Now they seem more sure of how it works. So I ask them whether they can use this same process to take the L shape to the next shape drawn on the blackboard. Now they struggle through apparent confusion between their claimed “reflection” and the rotations that appear to work for the first two shapes. So, they move to the computer where, using Sketchpad, we try to model their arm movements. Since they have to provide Sketchpad with explicit instructions to effect a transformation, the girls are forced to think about what exactly is rotating, where it is rotating around and by how much it is rotating. Having given Sketchpad these instructions, they try the rotation and are delighted to see the first segment, with its rotated image, form the second shape. Now, the girls are poised to move on to the third shape. They know they have to pick a center of rotation, an object to rotate, and an angle of rotation. They tentatively repeat the instructions of the previous rotation and Sketchpad displays for us a shape that matches the third shape on the blackboard. Instead of continuing on, the girls want to go back to their notebooks and draw their shapes.

This episode illustrates the use of Sketchpad as a tool for reference. When the three girls first describe a pivoting motion as “reflection,” turning to Sketchpad clarifies the distinction between rotation and reflection, and identifies the constituent components that well-define both operations’ relation of pre-image to image (a center point and angle for rotation; a linear mirror for reflection).

Katya is a student who takes quickly to the paper strip; she announces that her third shape is different from the one on the blackboard. Her group is surprised, and perplexed. She explains:

“There are different ways of folding, some of us alternate folding from right-to-left then left-to-right and the others always fold from the right.” I leave the students wondering how this small discrepancy can make such a difference, and let them continue investigating the phenomenon.

By now, having exhausted their pencil-and-paper searches and having caught on to the paper-strip furore, other groups of students become interested in the various patterns emerging as they folded their strips to create different dragons. One group starts keeping track of the relationship between the number of folds they make with their paper strip and the number of segments created; another group becomes interested in the number of “boxes” that appear at each iteration (see Figure 3) a few students notice that different methods of folding (right-over-left or left-over-right and right-over or right-under) are producing different curves and therefore different numbers of boxes. For instance, making three folds from the right
produces a curve with no boxes (like the shape on the blackboard), whereas alternating the folds produces a curve with one box (see Figure 4).

By the end of the class, we have a series of conjectures to test for the following class and students are starting to complain about the limits of the lowly paper-strip. They are having trouble keeping track of the folds they are making and want to be able to make more folds so that they can see what kind of patterns emerge with higher iteration dragon curves.

The introduction of the microworld through off-line activity allowed students to connect to their previous work. (It also allowed them to realize apparent limitations of their paper-folding technology, and later appreciate the versatility of the dynamic geometry technology.) The paper models
served as concrete, initial representations of the dragon curve, which we could later compare to different virtual representations. Students who felt more comfortable with the paper medium frequently returned to it when initiating new investigations and solving new problems.

It’s time to show them the SuperDuperDragon (one of several Sketchpad figures that author Jackiw and I had created in our own explorations and in anticipation of student-generated questions). This sketch not only dynamically shows eight iterations, either step-by-step or all at once, of the dragon curve (see Figure 5 for a sketch of the dragon curve after the fifth step), but can shrink or expand the curve, turn it about, and unfold it continuously. The students had been creating their small dragons in discrete, careful folds; this sketch gives them the experience of a smooth transition from a straight segment to a complex, yet recognizable figure. The students ask to see different steps so that they can compare Sketchpad’s curves to their own; they ask to move the dragon around, to drag its tail to make it very small—looking just like the

Figure 5. SuperDuperDragon sketch showing a 5th iteration dragon and the angle controller.
“Chinese dragon” they had seen at the beginning of the class. Then they ask to make it very big, which fills the screen with a lattice shape. Since the fold angles of the dragon curve are controlled by a parameter, I show them that you can fold the dragon from the initial segment by decreasing the included angle from 180° to 90° (as shown in Figure 5; here 180° signifies no fold, hence a straight line), and then you can unfold it by reversing from 90° to 180°. (Flip the pages of this article for a sense of this visual contraction, from a straight segment to a 90° dragon curve.) The students then try to count all the boxes that form from the segments coming together.

This introduction of Sketchpad was planned in an emergent sense—that is, with the anticipation that students would outgrow their paper folding and would be prepared to extend their investigations into a more complex world of dragon curves. This use of Sketchpad acts as a cognitive amplifier by extending our paper-folding investigations past the notorious limit of seven folds imposed by physical paper. In addition, by positioning their dragons within a context where mathematical constraints are observed simultaneously rather than sequentially, the students are able to draw connections between disjoint aspects of their experience with physical manipulatives.

The students then ask whether the sketch can make the kinds of dragons that they were creating with their non-standard folds. Ah! Just what I was waiting for! Another version of the SuperDuperDragon sketch lets you choose which folds should be right-over-left and which to be left-over-right. As shown in Figure 6, it is possible to toggle the fold direction between right and left for any of the four folds: the dragon curve shown is RLRR folded. (By contrast, Figure 5’s sketch involves only right-over-left folds.) Now I can hardly keep up with their requests. They want to try all the combinations, they want me to skip some folds in between (a decidedly non-paper-strip procedure!), they want to try to make the dragon as boxy as possible. In fact, we discover that alternating types of folds produces the most boxy, and hence, most compact dragon.
During the first few class meetings on paper-foldings and on dragons, the nature and purpose of our enabling dynamic geometry technology appears neither fixed by a particular interpretation nor governed by a single approach or framework. Rather we find several complementary uses of technology emerging more or less independently within the whole-class context, reflecting different conceptions and evaluations of its significance by various classroom actors. Specifically, beyond its traditional uses (for visualizing and exploring constructed dependency dynamically, and for measuring properties or calculating geometric properties and their ratios) the technology’s position within a shared public space allows it to act in various ways. First, our strips of folded paper are inherently not public artifacts: they are small, and if we’re enthusiastic, often crumpled beyond recognition. On
the technology blackboard, they become large enough to be visible to all in a form that reveals their essential properties. Secondly, the SuperDuperDragon sketch acts as a point of common return: students repeatedly ask (individually and in class-wide discussions) to bring up the SuperDuperDragon sketch and its related variations, and—by associating previous discussions with the microworlds around which they had unfolded—orient themselves to new questions through apparent harmonies or contradictions found in relationships to their previous work. Thus, in addition to providing a forum for acting, the technology captures our action for subsequent and deliberate review, analysis, and modification.

As we see in the next narrative, students develop precisely this sense of relationship in the SuperDuperDragon sketch by isolating and experimenting with individual folding choices (as shown in Figure 6), regardless of their position within the overall folding sequence, while simultaneously watching the result on a partially-unfolded dragon. This inspection leads them to the startling realization that the direction of the final fold in an arbitrary sequence has absolutely no effect on the shape of the unfolded dragon—a finding they supported with transformational arguments (each fold acts as a mirror applied to subsequent folds, but the last fold in a sequence has no subsequent folds). Separately, by varying the rotational angle of dragons resulting from various folding sequences between 90° and an open angle (a process students call “lying the dragon on its back”—again, see the flip-book illustration), students become convinced of a dragon property that had eluded them in paper-folding: that while corners of a right-angled dragon may touch each other, they will never cross each others’ edges. (This is not even a question one thinks to ask, with paper!) This discovery in turn plays a critical role in refining the symbolic production grammars they later construct for predicting a dragon’s ultimate shape from its (symbolically-expressed) folding pattern.

Obviously, these plural conceptions do not emerge from a void. In that students had only very limited exposure to (and certainly no “hands-on” use of) dynamic geometry tools prior to this unit, the teacher is clearly instrumental in figuring students’ first perceptions of the software’s relevance and purpose. But the teacher’s whole-class role is one much more of providing as-needed direction rather than instruction. By deferring hands-on and activity-dominated laboratory interactions with the software, and more subtly by first introducing the software as a classroom affordance (like a blackboard) for pursuing active questions, the teacher encourages students to engage with the technology critically but non-judgementally, to remain open to its possibilities.

Contrast this plural approach to the more conventional hands-on technology setting governed by a scripted activity. If we introduce dynamic geometry through the (canonical) activity in which
students construct a triangle’s centroid, it is not unreasonable to assume that students’ primary conception of the tool is first and foremost as an electronic centroid-constructing device. Of course, they may eventually sense that this primary activity is embedded in a technology matrix of greater potential, or they may evolve this singular conception to a more general sense of the opportunities or versatility of the tool with time. But they do so under preconceptions strongly structured by their first contact with the tool, for two reasons. First, the activity is extrinsically given: it has a sequence and logic that emerges not from the classroom but that is imposed by some external mathematical authority. Second, implicitly or explicitly (depending on its phrasing), the centroid activity immediately forces students into a position of strong ego investment in fixing their relationship to the technology. Once we are told to construct a centroid using a particular set of tools, we are set to a task in which eventually we shall either succeed or fail—and our conception of our relation to the tools becomes intertwined with our relation to our selves. By contrast, encountering the tools first in an environment less structured by outcomes or goals allows us to explore and contextualize them, individually and consensually, in ways that differ usefully from learner to learner and from class to class. Our claim is less that these diverse technology conceptions and technology-based modes of interaction are necessarily correct (or even abstractly “better” than specific interpretations which one might cultivate through various scripted activities), than that they are appropriate to these students—precisely because they reflect technology as appropriated by these students.

NARRATIVE II: SYMBOLIC MANOEUVRES

The previous class ended with some unanswered questions about how different folding patterns produce different dragons. I suggest that we might try to come up with a way of describing the dragons that would make them easier to compare. During the previous class, a group of students had produced several fourth-iteration dragons and we now try to figure out who’s were the same and whose were different. Carmen observes that there are many different ways of looking at the folded paper strip, starting either with its head or tail and looking either at one edge or the other. Roman suggests that we try to standardize this by marking one end as the head of the dragon and the other as the tail, and by agreeing to always look at our dragons from tail on the left to head on the right. We all fold our own paper-strips using the right-over-left sequence 4 times, and follow Roman’s suggestion. Having noticed that the dragon could be traveled from the tail to the head by a series of either right turns or left turns, I suggest that we could describe our dragons by simply writing down its sequence of left (L) and right (R) turns. The students think this is a good idea and start to describe, using L and R sequences the dragon curves that we are holding in our hands. But this proves quite dizzying, trying to figure out which ones are left
turns and which ones are right turns: students have their heads going in all directions. So instead, we develop another system to use: the students notice that if we unfold the dragons slightly on a horizontal plane—or “lay the dragon on its back”—the right and left turns start looking like mountains and valleys, or M’s and V’s. (Note: there is a point during the “lying the dragon on its back,” where one can see the horizontal sequences of M’s and V’s; this is almost easier with paper since fold lines are clear even when the paper is nearing its limiting straight segment: see Figure 7.) This proves to be a much more manageable system.

Since the SuperDuperDragon microworld allows the dragon curve to be unfolded like this, we move to Sketchpad to analyse the first few iterations, one after the other, of the regular dragon (always folding from the right). Since we can drag the tail of the dragon, we also flip it on its “front” to see how the sequence changes, observing that all the mountains became valleys and vice versa. Having constructed five sequences of M’s and V’s this way, the students write them down in their notebooks and we start to investigate the properties of the sequences themselves. The students move into small groups or work individually to do this.

We see a move toward symbolization: there arises a need to categorize the dragon curves in a systematic way, and to generalize the shape of the myriad particular dragons being folded or drawn. This move also leads to abstraction in two senses: the students can refer to specific dragons by using an abstracted

Figure 7. Seeing the M’s and V’s.
notation (their sequences of M’s and V’s); and they begin manipulating the symbols themselves, without specific reference to the visual or concrete representations of the dragon curve. This manipulation of symbols leads to algebraic expressions describing relationships within and between sequences. The unfolding of the dragon allows them to invent their own symbols to describe the curve. In fact, although they eventually decide that looking at the sequence of creases (and thus, M’s and V’s) would be most interesting, there were students who had proposed that we consider “troughs” and “plateaux” instead and who even pursued this idea in their individual project work.

As new observations or conjectures are made, again, the students write them up on the blackboard. Tarquin notices right away that the middle term is always the same—an M or a V depending on whether the dragon is on its back or its front. Charles and Roman are intrigued by the growth of M’s and V’s from one generation to the next and start formulating an algebraic description of this. They find that the number of V’s is equal to $2^f$, where $f$ is the number of folds. Simone is also interested in growth, but growth in creases; she comes up with an iterative relationship for the number of creases: that is, the number of creases on the $n$-th fold $c_n = 2c_{n-1} + 1$.

As the students work with their own strips, they start to appreciate the importance of keeping track of the way that you fold, i.e., right-over-left/ left-over-right and right-under-left/left-under-right. In the corner of the classroom, a group of girls is verging on finding an important symmetry: they notice that the letters alternated around the middle one, but can’t really explain why this might happen. At this point, I take them over to the computer again and together we construct an animated version of the paper folding. We first have the strip lying straight, then fold it over in half twice; then we unfold it make the right angle. Finally, we lay the dragon almost on its back so that we can still see its creases, or its V’s and M’s (as shown in Figure 7). While it is lying down, we label the vertices M and V and fold it back up again. Then, as we unfold our model, we can see the vertices carrying opposite letters—but made from the same crease—separating from each other. We try the process again. This time, with more folds, Jasmine is beside herself when she realizes that all the letters are opposite, all the way down the spine of the dragon.

Here Sketchpad is again used as a reference, providing needed insight and clarification for a larger group of students. The teacher’s use of Sketchpad in this episode allows students to observe and follow construction techniques. They have not yet engaged directly with Sketchpad, nor explicitly learned anything about how to use Sketchpad, but they are developing a sense of the types of actions that are possible.

The idea of symmetry is making its way around the class. Roman and Charles begin to see the symmetry caused by the method of rotation that we
had briefly seen in the previous class. Roman explains to the class that you could build the symbolic sequence similar to the way that you can build the curve with rotations. He writes the following on the blackboard:

We check that Roman’s method generates the same sequences as the ones that we had at the beginning of the class. The students seem satisfied.

After Roman’s description, Pierre announces that he has another way to generate the dragon that is different than the other ones (see Figure 8). He writes his method on the blackboard:

Pierre’s picto-symbolic description resembles the $L$-system approach where the starting point is the initial segment and the rule is to add to the initial segment a rotated copy of it (represented by his curved arrow). It is based on a generalized conception of the components of the dragon. He defines the generation of the dragon as a function of the initial line segment, i.e. $x_{n+1} = x_n + f(x_n)$, where $+$ is a geometric concatenation operator and $f(.)$ a right angle rotation. The move to symbolism is natural in the sense of being appropriate and justifiable in the context of its introduction, rather than artificially injected after a sufficient amount of “concrete” investigation. And finally, instead of being an
endpoint, or ultimate goal, the symbolization and abstraction leads to further concretizing, as we shall see in the next narrative.

We are at the end of the class, so I briefly recapitulate the proposals that we have on the blackboard and ask the students which ones we have verified as being true or false, and which ones are still pending. Karly notes one pending question. In fact, it’s the question that we started the class with: we know that different ways of folding will produce different looking dragon curves, but will they produce different sequences? She asks whether the proposals that we verified on the blackboard apply to the non-standard dragons as well.

**THEME: WHOLE-CLASS EFFECT ON STUDENT AFFECT AND CREATIVITY**

The whole-class microworld design appears to motivate students by increasing affective aspects such as students’ confidence level in their ability to understand a concept or solve a problem, students’ interest in the activity, and students’ sense of satisfaction, independence, and self-direction. These dimensions of student affect overlap with those identified by Stipek, Salmon, Givvin, Kazemi, Saxe, and MacGyvers (1998) as contributing to nurturing positive motivation, and in turn, increased achievement.

In the whole-class dragon microworld, students clearly rely on each other for help, be it technical or conceptual: Roman and Pierre emerge as experts in symbolization; Katya leads paper-folding efforts; the group of three girls head the rotation explorations. Later in the narrative, we see that these local experts continue to evolve, and contribute to the efforts and confidence level of the whole classroom. This sharing of expertise toward common goals saves each individual student from potential anxieties of having to master all ideas and skills.

Boaler (1997) has shown that students’ sense of independence and self-direction can be encouraged in open problem environments. However, in the
dragon microworld, the students as a whole also begin to act in an independent way, that is, according to the their own initiatives and questions as opposed to the teacher’s pre-set agenda. Although individual students have provoked certain directions of inquiry, the receptivity of the microworld contributes to students’ sense that their own collective questions and interests are guiding the trajectory of the activity. We can observe the students taking ownership when they suggest and adopt their own names or conventions when discussing properties of the dragons. The students’ project work (described later) also shows the students’ sense of independence, one that is nurtured by the whole-class microworld in which ideas and questions for these projects were born.

In terms of enjoyment and interest, while students working individually with sketches can certainly become interested in or excited by them, the public, shared nature of the SuperDuperDragon Sketch, for example, engenders a different type of excitement. This excitement—over the possibilities of the virtual dragon—is perhaps only initially felt by a small group of students, but propagates through the classroom through student interaction—through expressions of wonder, urgent requests to manipulate the sketch, questions about the dragon. This “in the air” excitement touches those students who might not individually perceive the attraction or interest of the situation, but who by listening to the comments and questions of their classmates are provoked to develop their own interests.

Our classroom observations have suggested to us ways in which student affect is related to the whole-class design of the microworld. We caution that without more informative data—notably interviews or questionnaires with students—we are not in a position to verify that student affect is directly correlated with whole-class design. However, we now empirically informed categories of student affect that can be used in further research.

We now briefly consider the whole-class effect on student creativity. Instead of focusing on the creativity of individual students, we take the classroom of students as a whole as our unit of analysis. This fits well with Csikszentmihalyi’s (1996) view of creativity that draws attention to the role that the field—the members of the community of a particular domain—plays in the motivation and legitimization of creative ideas or acts. In the whole-class microworld, the students as a whole generate a number of differing strategies and solutions; share them with members of the classroom; and develop criteria through discussion for deciding which strategies are worth pursuing, and which solutions are acceptable and appropriate. An individual student’s creativity is typically judged by the degree of flexibility, fluency and originality shown in her strategies and solutions (Sternberg, 1988). Yet here we can see the classroom as a whole showing flexibility and fluency in generating a wide-ranging set of strategies and
solutions. We also see how the students’ original solutions are motivated by each others’ creative ideas and acts. As these solutions and strategies are shared among students, they are compared and discussed in terms of “sociomathematical norms” (Yackel & Cobb, 1996) such as relevance, appropriateness, beauty or interest. The classroom as a whole thus legitimates these solutions, functioning as a knowledge-building community not unlike that of research mathematics.

**NARRATIVE III: GENERATING DRAGONS**

Since quite a few of the students had shown interest in Pierre’s rotation method of generating the dragon, I suggest we use Sketchpad to see whether we can build the dragon curve using Pierre’s method. (Note that we are extending the work begun by the three girls who had first modelled the rotations of their arms.) The students provide instructions for the Sketch that is projected onto a screen, and I carry them out. They call first for a line segment, and then they tell me to select the line segment and rotate it around its “tail” by 90 degrees. This produces the first step L shape. They then ask me to rotate the L shape, but this time around its vertex. So I select the L shape, rotate, and this time we get a T shape, clearly not what they had intended! Stéfan proposes that we rotate it “the other way” so I ask him what angle would produce “the other way.” Roman suggests 270 degrees. So we try that and this time we get yet another T, oriented on its side. Then Jasmine suggests that we rotate the L around its tail. This produces the desired effect.

Jasmine then describes how we can go on to the next step, always rotating the previous step around its tail and we create several iterations of the dragon curve. We then “lay the dragon down” to read the sequence of M’s and V’s so that we can compare it to the sequences that we had produced in the previous class. Roman explains how this rotation sequence agrees with the rule he had suggested before. Since all our sequences seem to be agreeing, I ask the students whether they can produce an example of a sequence that couldn’t be a dragon. Jeremy goes to the blackboard and writes down this illegal sequence: VVMMMV. Carmen explains how it violates the symmetry rules dictated by the Roman’s method and the rotation principle. I ask the students whether there are any differences between the two methods. Sabine suggests that their sizes are different. In the rotation method, the length between creases stays the same, whereas in the paper-folding method, the lengths of the segments are halved after each new fold. Karly had worked this out in the previous class and this gives us an opportunity to share her insights with the entire class: the length between creases in the paper-folding method is $1/2^n$ the original size of the strip, where $n$ is the number of folds.

In this episode, we see Sketchpad playing the role of an enhanced blackboard. The students are providing the program’s inputs through the teacher’s mediation.
As students express the actions they want, the teacher reconciles them with Sketchpad commands and constructions. Since it is the technology itself that requires a formal vocabulary in order to effect one’s intent, students show little resistance to adopting one. (Contrast this situation to the case in which teachers try to enforce formal vocabulary—Hewitt (2001) discusses this phenomenon.) A whole-class use of Sketchpad eventually leads to a shared vocabulary that the students come to apply to their offline arguments, as well as to their eventual direct manipulation of the tool.

The students have found two ways of describing the dragon sequences: one by a method of symbolic generation, and one by identifying necessary symbolic symmetries. But, we still didn’t know whether they were necessary conditions and symbolic generation and symbolic symmetry were related. I give the students six sequences and ask them to decide whether they are “legal” dragons or not, a dragon being declared legal if we can make it with our paper-strips. They work on these individually for about five minutes and then we discuss them as a class. Katya claims that the first sequence (MMVMMVV) is a legal dragon because it satisfies the symbolic symmetry rule (the letters on either side of the middle M are “opposite”). Jonathon is concerned that there is only one V surrounded by so many M’s; he doesn’t think this could happen in a legal dragon. He draws that part of the sequence on the blackboard to verify his conjecture. The shape doesn’t overlap or look wrong, but nobody can remember seeing it as part of their dragons. Jeremy remarks that it also had the right number of creases. Having no other ideas of how to test the sequence, we leave it noting that it might be a legal dragon.

The next example (VMVVM) only has five terms in it and the students are quick to point out that it isn’t legal. We talk about how this argument is different than the previous one: it is easier to show that a sequence is not a dragon than to show that it is. At this point Michelle decides to tell us that the sixth sequence (VMVVMVV) is definitely not legal either because it violates the symmetry rule. Suddenly, Tarquin shouts out that he has built the dragon curve corresponding to the first sequence by folding right-over-left, left-under-right, and then right-over-left: a proof by existence! This marks the first time that we are able to integrate non-standard dragons into our sequence explorations.

We write our four dragon curve methods on the board: the symmetry/creases/3 rule, the symbolic generation, the paper-folding, and the rotation. I tell the students that there is yet another way of making the dragon curve. I described this recursive method briefly on the blackboard,
showing how the initial segment is replaced by a right-angled triangle (whose hypotenuse is the initial segment), and how each new segment undergoes the same replacement. I move to Sketchpad where I can show them more precisely how this works. Now though, we have to construct that point which is the vertex of the right-angled triangle. To do this, we rotate the end point of the first segment around its midpoint. Pierre doesn’t believe that this construction produces a right-angled triangle. So, we measure the angle and verify that it is indeed a 90° angle.

Then I explain that we were going to iterate the construction on each new segment. This is a new word for them, but one that Sketchpad will have an easy time illustrating. We use a program “script” to record the first replacement. After a false attempt, the students reason that the replacement of one segment has to be opposite the other segment in orientation, in order to produce our well-known symmetry property. We then play the script step-by-step on two new endpoints (dragon head and tail) for two iterations (see Figure 9).

The students are fascinated by the way the curve is being built, so quickly with intermediate steps that don’t look like dragons at all. Students shout out for increased iterations.

At this point, with our five methods on the board, I ask the students if appeal more than others. Roman suggests that they are not all the same: some make more dragons than others (the symmetry/creases/3 rule and the paper-folding can generate more dragons—i.e., the non-standard ones—than the others), some only describe and some construct. Adrian notes that the paper-folding method has lots of variations that we haven’t completely described. Several students add that the fractal method is the most interesting and attractive. Several other students counter that the symmetry/crease/3 rule is the most informative. This is, after all, the one that they have developed entirely on their own.

Figure 9. Recursive dragon script.

\begin{figure}
\centering
\begin{verbatim}
Given:
1. Point A
2. Point B

Steps:
1. Let [l] = Segment between Point B and Point A (hidden).
4. Let [k] = Segment between Point [A'] and Point B.
5. Let [l] = Segment between Point A and Point [A'].
6. Recurse on A and [A'].
7. Recurse on B and [A'].
\end{verbatim}
\caption{Recursive dragon script.}
\end{figure}
By this point, a clear technology trajectory appears in the whole-class approach to dynamic geometry. After its unassuming introduction as an electronic blackboard or display device, the software’s first appropriation by students was for calculation purposes—for generating numerical answers (such as the degree to which an iterative folding sequence approximated a 1:3 fold; or, as the situation expanded to include dragons, to verify that curves in the canonical dragon’s display could be modelled by angles of 90°). This conception of computer as calculator at once reflects both broader social assumptions about the relationship of computational technologies to mathematics (computer as “number-cruncher”) and the relatively constrained forms of interaction afforded by the domain-specific parameterized models (authored sketches) which we had conceived—and in some cases, actually constructed outside of class—as teachers mapping out the situation’s problem space.

From this initial conception, however, students quickly begin to interact with the teacher as she develops computational models to probe classroom discussion points. Pierre’s request that the class use Sketchpad to explore rotation—that is, that the teacher show them how a rotation can be constructed geometrically—and the widespread desire to extend the visualization of an n-folded dragon to include the process of contraction by which it folds from an unfolded strip depict fundamental changes in students’ conceptions of their relation to the technology medium. From model manipulators, they have become model describers and developers. Although they are not yet engaged in constructing models themselves, the frequent shifting groupwork of the rich mathematical situation naturally situates different students working at different times in more immediate circumstances with the teacher and the computer, and thus local technology expertise begins to evolve among the students themselves. By the time the class constructs a dragon using Pierre’s rotation method, classroom culture has evolved to a point at which students clearly see themselves as authors (and the teacher, at least in relation to technology, as an almost superfluous mediator). The students’ move—in the next narrative section—to sophisticated hands-on and self-directed use of the software in pursuit of individual projects occurs naturally in this setting, and legitimates their self-conception as technology authors.

This whole-class trajectory—from passive or scripted interaction with parameterized models, through a participatory design context in which students contribute both motivational questions and aesthetic or design input to a model-building process, to hands-on and self-directed individual student authoring—has immediately practical
implications. The whole class embeds technology in a collective problem-posing and -exploring context that replaces the need for formal software training, for devoting classtime to student skill building.

The issue of training is significant. Case-study research of student/technology interactions tends either to focus on students’ very preliminary and “white slate” conceptions of a new piece of software or to commence at a moment when software expertise is sufficiently established to ground a more domain-specific investigation. Thus issues of access and sustainable impact are less frequently addressed (Roschelle & Jackiw, 2000, p. 779). But the not-insubstantial classroom overhead of introducing “rich” or multi-functional technology is a frequent concern and obstacle confronting teachers managing classrooms outside the research setting. The prospects of dedicating class time at the start of a semester to technology training—to learning (or teaching) the mechanics of various hardware and software devices, of studying how to drive as distinct from driving to particular destinations—becomes more difficult to justify as teachers find the complexity and scope of these technologies ever increasing, while class hours per semester remain quite fixed. Even when technology instruction is coupled with more overtly mathematical material, the result is often wanting: graphing calculators, for instance, have introduced an era which transforms many algebra texts into impenetrable compendia of keystroke sequences, obscuring recognizable mathematics behind yet another wall of opaque symbolism. (The same claim can be made of some of the published dynamic geometry curricular materials.)

Against this backdrop, teachers confront a Hobson’s choice: waste precious class time following tightly scripted instruction materials in which the subject is software mechanics as opposed to mathematics, or throw students “off the deep end” and then manage ensuing chaos as best one can. The former is often impossible given class schedules and external directives for constant progress through a curriculum sequence; the latter often complicated by teachers’ assumptions that they must first achieve (mythical) mastery of the technology in order to fend off the swarm of questions they’ll confront on the first day of its deployment. In practice, many classrooms falter here just on the cusp of meaningful technology adoption. Mathematics classrooms seem particularly prone to this effect: recent surveys of computer technology adoption across the United States find that teachers of secondary mathematics are among the least likely to use computers in class (Becker et al., 1999). In this light, the whole-class approach appears a promising exit from the prevailing quagmire of stalled technology diffusion. Introducing technology as a communication affordance rather than a focal subject on its own permits multiple conceptions of its relevance and usefulness to grow and intertwine in the classroom. Keeping it before the whole class while modelling these various interpretations—rather than immediately transferring students to computer lab settings—both maximizes the effectiveness of limited in-class computer resources and more importantly
enables “control” of the software to shift from a single trained teacher\(^5\) to the learning community incrementally rather than abruptly. Both motions foster creative and personal appropriations of the technology by students: of the fourteen pairs of students who submitted projects at the end of our dragon unit, all but three chose to base their projects at least in part on self-directed dynamic geometry investigations.

**NARRATIVE IV: SOLO PERFORMANCES**

Project time! We have learned a lot about dragon curves but there are many questions that came up during our explorations that we left unanswered. I mention some of these questions and invite the students to conduct their own investigations either on one of these, or on one of their own. Most of them choose to work in pairs; they are first required to formulate a line of investigation before proceeding and to indicate what materials they would like to use. Three computers are made available to those who want to use Sketchpad to help them in their investigations.

One student creates a “Canadian dragon” and investigates its properties, comparing it with the properties we had explored in the “Chinese dragon.” One pair of students want to build a high iteration dragon by combining low iteration dragons, and specifying how many people—if they each make one little dragon—you would need to build a 10-iteration dragon; another pair decides to describe the aesthetic aspects of the different methods and stages of dragon construction.

The students that choose to work with the computer each follow a slightly different path. Some know exactly what they want to look for (how the numbers of boxes changes with the number of folds; how many different dragons you can make with 5 folds if you allow right-over-left, left-over-right and right-under-left folds; how the right-over-left/ left-over-right folds are related to the right-over/right-under folds) and use the SuperDuperDragon sketch variants that we had already developed in class to carry out their investigations. The students that use the SuperDuperDragon sketch gather their data during classroom time and then write up their analyses. While using the sketch, they are able to play with different properties that we had touched upon in

\(^5\) Lest an ease-of-deployment argument based in part on a “single, trained teacher” appear disingenuous coming from Jackiw (who has more than a decade’s intimate acquaintance with Sketchpad), we note that the teacher in question, Sinclair, had encountered dynamic geometry for the first time less than eight months before the first class described above. This was her first semester teaching with Sketchpad.
class: the two primary (re)discoveries they make are (a) the fact that the last fold, whether it’s right-over-left or left-over-right, doesn’t affect the shape of the dragon, and (b) that the SuperDuperDragon sketch allows you to create curves with “skipped” folds, something that was pretty much impossible to do with the paper strip.

Other pairs of students want to experiment a little with Sketchpad before posing a particular question; they also want to be able to construct their own dragons using either the rotation or the recursive method. For example, one pair wants to change the look of the dragon by turning the right angles into curves. We do this together in class but then they decide to turn the right angles into 75° ones when they continue their work at home. Another pair wants to make a different fractal based on a recursion similar to that of the dragon curve. Instead of building right-angled triangles on each segment, they build trapezoids, where the slanting sides are constructed by rotating half the segment by 45°. This pair produces four different fractals and investigates which ones resemble the dragon curve the most and which ones produce the most attractive images. Another pair, interested in pursuing the way the dragons could be rotated to fit together, decide to tessellate a sketch area with different coloured copies of the dragon. Instead of just copying and pasting the dragons into place, they attempt to use rotations around both the head and the tail of their dragons in order to fill the screen. Finally, another pair is interested in the self-similarity of the dragon curve; they want to see whether parts of the dragon can be dilated to fit over the whole dragon. At first they want to use a photocopier to zoom printed copies of the dragon, but then they decide to create a dragon using the rotation method that they can enlarge by dragging its tail.

Finally, across or alongside these other dimensions of technology’s use, students turn to Sketchpad as a source of inspiration and as a medium for improvisation. Sometimes this quality emerges serendipitously—as when students using a rotation to generate higher-order curves accidentally rotate an existing curve around its head rather than its tail, and discover the four-fold rotational symmetry of a closed dragon. At other times, it emerges as the result of more systematic research, as when two students later build new fractals based on a fractal definition of the canonical dragon. In either event, both for students and for teachers (inside the classroom and in their preparatory work), dynamic geometry’s constrained construction space—and facility for unlimited undo—encourages us to explore without strong preconceptions of our final destination.

We spend a large portion of the following class presenting these projects. The different lines of investigation pursued by the students allow us to fill in some of the gaps that we had left during our group work. A few pairs of students explain the relationship between the different ways of folding the paper-strip, as modelled in the SuperDuperSketch and the effect of these
foldings on M, V sequences. We are finally able to see how the number of boxes grows at each iteration, and we are struck by the ubiquity of exponential relationships. Many students spend their time constructing their own sketches and proudly present them to the class: fractal images, tessellations, and skewed dragons.

**THEME: WHOLE-CLASS LEARNING**

Prior to the students’ individual project investigations, we see the cultural culmination of a classroom’s work over four classes building a body of knowledge about dragon curves. Creative ideas and acts emerge along the way, sometimes from individuals, sometimes from group discussions, but are always explicitly evaluated by the entire class against the context of their collective experience with the particular dragon-curve domain. For example, Michelle’s insight (Narrative IV) about the overlapping square grew from work on boxes with her small group, experience with paper-folding and observations on Sketchpad. It could not have passed as an explanation for other students had they not also shared these experiences. Thus the whole-class “balanced discussion” (Bartolini Bussi, 1998) is oriented to the socialization and collective evaluation of strategies already set up by the students in individual problem-solving.

In contrast to traditional teacher-led classrooms, student interests, questions and solutions are mixed with those of the teacher. The rule of symmetry/crease/3 was co-constructed over two classes by the students, and became a staple of class comparison and reflection in many further discussions. The recursive construction, though introduced by the teacher, emerged as a similar focal point of future reference. Local expertise continued to develop within the classroom: we would ask Roman for information on the symbolic methods, Jasmine for information on rotations, Karly for paper-folding, and Simone about crease/length/symbol growth, and Tarquin for help with Sketchpad constructions. In this context, the whole class becomes a collection of pods that move together, interacting amongst themselves at a classroom level. The pods form as students work independently from the whole. Having done previous work with the same class, we were keen to note that the pods’ make-up changes with different situations and that different students emerge as local experts in different contexts.

The interactionist perspective of learning developed by Bauersfeld proposes that the relationship between individual students’ mathematical activity and classroom mathematical practices is reflexive (Cobb, 1989). Reflexivity, in this sense, implies that neither an individual student’s mathematical activity nor the whole classroom’s activity can be adequately accounted for without considering the other. We have chosen to
focus on the whole classroom unit in part to address the lack of such analyses (Cobb & Yackel, 1998), especially in inquiry based instructional settings using microworlds. This has allowed us to appreciate the range of contributions that students can make in a microworld designed through a whole-class approach.

This reflexive relation also implies that participation in classroom practices both allows and constrains individual activity and learning. How was student performance and “capability to perform” as individuals in their project investigations mediated by the classroom mathematical practices and activities? The students’ individual investigations lead us to four observations:

• the individuals’ investigations were all rooted in classroom conjectures and discussions, yet there were distinct differences in the interests and pursuits of different students—the whole classroom activity acted as a springboard but not a straitjacket;
• the disposition toward conjecturing and problem-posing was maintained during the individual project work, as was student-initiated communication between pairs;
• concepts developed in the whole-class setting of exponents, transformations, similarity, and recursion were put into action at the individual level.
• The individual project work initiated some of its own new mathematical concept development, for example: several pairs of students had to confront the combinatoric issue of generating all possible four-fold dragons; the tessellating pair had to develop more sophisticated geometric vocabularies for describing their tessellations; the fractal pair had to elaborate their understanding of fractals given their brief introduction to the idea through the recursive dragon.

CONCLUSIONS

In this article, we follow a class through a rich mathematical situation, paying particular attention to the role of students’ interactions with dynamic geometry-based microworlds. In this pursuit, we have taken a whole-class approach—both as a pedagogical model and as a perspective for critical reflection—that emphasizes students’ collective interactions with the whole learning environment (with each other, the classroom, teacher, ideas, and tools) over their individual relations either to teacher or mathematical content.

A mathematical situation characterized by this approach evolves less according to a master lesson plan or sequential activity model than according to the trajectory of a socially-constructed center of gravity. In the whole-class approach, communication becomes the central vehicle of mathematization, and discussing variations—negotiating as a group criteria of importance, relevance, beauty, interest, and value—becomes the distinguishing attribute of ensemble playing.
Group work—either as an entire class, or in collaborative subgroups that change membership according to interest, expertise, and necessity—emphasizes expression (calling out construction ideas and approaches, deciding and agreeing upon them) over unreflective action (wiggling dynamic geometry points until they “seem right”).

Dynamic geometry technology’s position within such a context no longer appears to reflect straightforward curricular objectives of its microworld- or educational-software designers, but instead joins sense of purpose, the individual’s role in a wider collaborative context, and mathematics itself, within a matrix of meanings open to classwide social improvisation, negotiation, and evolution. It plays an integral role in the whole-class microworld, inseparable from the other components that make up the classroom environment. In fact, we can identify four roles that Sketchpad plays within the shared, public space of our case-study—roles that extend beyond those identified as contributing to individual student mathematics learning (that is, as a cognitive amplifier; as a tool for exploration and feedback, for student expression and action, for connecting different representations; and, as a window on mathematical thinking (Clements, Battista, & Sarama, 2001; Noss & Hoyles, 1996):

1. **As a Blackboard and Stage**

In its central position before the class (projected from a single computer), Sketchpad becomes a forum for performance and the sharing of ideas (and, prosaically, as an attention-focusing device). At the same time, Sketchpad differs from a traditional blackboard in the convenience by which its figures may be altered or expanded, changed or redrawn. A traditional blackboard invests its images with a degree of authority—they are not only public, but relatively fixed. When a teacher draws an image on the blackboard, this image often acts as the “big picture” framing illustration toward which subsequent verbal discussion relates. (With chalk, we cannot readily redraw the entire image in response to each question we field or variation we discuss.) From a student’s perspective, the situation is worse. Once placed on a blackboard, a student’s work is subject to public scrutiny and assessment. But the only way we can change or “fix” a student’s illustration (to clarify a misperception or address a missing case) is by erasing and redrawing—an essential act of invalidation, of “throwing away” that student’s public contribution. By contrast, because the dynamic geometry process of modifying illustrations is frequently one of continuous visual transformation, such modifications often visually encapsulate exactly our rationale for changing the picture in the first place. Thus the dynamically manipulable
figures of Sketchpad not only permit the teacher to modify her own illustrations to address questions or explore variations, but allow both teacher and students to extend and modify student illustrations as well, building on—and enriching—their contributions rather than throwing them away.

Though the virtual blackboard contributes less than a conventional one to the rigidity of visual ideas and the anxiety of their articulation, it still acts as a place in which public, shared conventions can be established and valorized by the whole class. Even if not fixed in appearance or form, by virtue of having been explored in a communal discourse, our virtual illustrations become points of reference for future discussions. When at their most effective, they offer us new cognitive tools or vocabularies for class-wide discussion of related ideas we subsequently encounter.

2. As a Reference

In that Sketchpad offers the students a recognized if not altogether familiar mathematical vocabulary for discussing shapes and their manipulation, it frequently defines, clarifies, or expands their access to various geometric operations. In this sense it serves as a reference, a geometric dictionary, available as a shared, public resource. (Like a dictionary, however, its authority in the public space is tempered by its neutral role as a shared resource rather than a guiding text; it is authoritative without being authoritarian.) Unlike a traditional reference text, however, Sketchpad also provides a vehicle for enacting the concepts or operations it defines, frequently in terms of the specific problem or context in which their definition becomes appropriate or necessary. Thus the traditional division between “looking up” a definition and applying or using it may be less dichotomized in the dynamic environment.

3. As a Recorder

The previous roles—blackboard and reference—are extended by software’s ability to act as a recorder of both ideas and work. By saving sketches produced in class either from scratch or as variations on previous sketches, the class builds an archive of returnable points of collaboratively-authored reference for clarification and expansion.

4. As a Muse

In an individual setting, Sketchpad can be a medium for improvisation as well as a source of inspiration and of serendipitous insights. In a public, shared setting, the muse effect is multiplied as one student’s idea tips off another’s, as one student’s actions reveals a solution or a new question for another student, as communal efforts—including those of the teacher—enrich improvisation and problem-solving.
We argue that this position of Sketchpad within the whole-class microworld elevates its potential impact on students’ processes of discovery and meaning-making. From a utilitarian perspective, the whole-class approach overcomes several traditional classroom barriers to effective technology integration. The rich variety of technology-assisted explorations we describe occurred predominantly in a setting containing a single computer: the whole-class model is not particularly jeopardized by limited computer resources or limited availability of computer resources, and constructionist modeling experiences are still available to students (individually or in groups) despite lack of computer labs and one-on-one access to machines. Nor is class-time devoted to—or from many teachers’ perspective, wasted upon—technology introduction, explanation, and training. The whole-class approach simultaneously provides ample opportunity for students to develop models of, and facility with, technology use, providing an experiential trajectory in which students move at their own rate from passive to active relations to technology tools; and the same approach removes any requirement that each student achieve the same level of facility as every other in order to avoid incremental exclusion from classroom activity or from stalling such activity altogether.

From the perspective of meaning construction, dynamic geometry within a whole-class microworld becomes multivalent, open to diverse personal epistemologies that include motivational and executional aspects. For some students, it serves primarily as an expressive medium for documentation and presentation—an enhanced blackboard or notebook. For some, it provides the locus of aesthetic discoveries that captivate and compel them to pursue new ideas. For many, it becomes a fundamental language, a tool—like a notation or programming language—not only for describing mathematical insights, but for creating them, reasoning with them, articulating and adapting them. The whole-class discourse allows these differing roles to emerge simultaneously, and for many students, they overlap and shift with time, developing into vernaculars suitable for rich forms of mathematical self-expression.

REFERENCES


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