Int J Comput Math Learning (2007) 12:3–21
DOI 10.1007/s10758-006-9109-4

If this is our mathematics, what are our stories?

Lulu Healy · Nathalie Sinclair

Published online: 20 February 2007

Abstract This paper sets out to examine how narrative modes of thinking play a part in the claiming of mathematical territories as our own, in navigating mathematical landscapes and in conversing with the mathematical beings that inhabit them. We begin by exploring what constitutes the narrative mode, drawing principally on four characteristics identified by Bruner and considering how these characteristics manifest themselves in the activities of mathematicians. Using these characteristics, we then analyse a number of examples from our work with expressive technologies; we seek to identify the narrative in the interactions of the learners with different computational microworlds. By reflecting on the learners' stories, we highlight how particular features, common across the microworlds—motion, colour, sound and the like—provided the basis for both the physical and psychological grounding of the behaviour of the mathematically constrained computational objects. In this way, students constructed and used narratives that involved situating mathematical activities in familiar contexts, whilst simultaneously expressing these activities in ways which—at least potentially—transcend the particularities of the story told.

Keywords Narratives · Expressive technologies · Microworlds · Ideational and conceptual mathematics · Dynamic mathematisations

This paper was inspired by words spoken on behalf of the Gitxsan people of British Columbia, quoted in the title of Ted Chamberlin's book "If this is your Land, where

L. Healy (✉)
Programa de Estudos Pós-Graduados em Educação Matemática,
Pontifical Catholic University of São Paulo, Rua Marquês de Paranaguá,
111, Consolação, 01303-050 Sao Paulo, SP, Brazil
e-mail: lulu@pucsp.br

N. Sinclair
Faculty of Education, Simon Fraser University, 8888 University Drive,
Burnaby, BC, Canada V5A 1S6
e-mail: nathsinc@math.msu.edu

are your stories''. These words come from an episode in which a Gitxsan Elder is questioning the rights other groups have claimed to the land he sees as belonging to his people. If others want to claim the land, then surely they must have stories and myths that reflect their ties to it. The question is meant to assess whether the land really means something to them; if it does, they will have stories to prove it.

Over the past few decades, there has been an increasing emphasis on the relationship between stories and knowledge construction and a questioning of the kinds of stories students might have about their school mathematics. Many have lamented the pervasiveness of sad and fearful stories told by learners of mathematics, such as the one that follows—stories suggesting that rather than wanting to stake any claim, many experience mathematics as a forbidding and inaccessible land, a land that they would not call their own.

> And on the eighth day, God created mathematics. He took stainless steel, and he rolled it out thin, and he made it into a fence forty cubits high, and infinite cubits long. And on the fence, in fair capitals he did print rules, theorems axioms and pointed reminders. 'Invert and multiply.' 'The square on the hypotenuse is three decibels louder than one hand clapping.' 'Always do what's in the parentheses first.' And when he finished, he said 'On one side of the fence will reside those who are bad at math, and woe unto them, for they shall weep and gnash their teeth.'
> Math does make me think of a stainless steel wall – hard, cold, smooth, offering no handhold; all it does is glint back at me. Edge up to it, put your nose against it; it doesn't give anything back; you can't put a dent in it; it doesn't take your shape; it doesn't have any smell; all it does is make your nose cold. I like the shine of it – it does look smart, intelligent in an icy way. But I resent its cold impenetrability, its supercilious glare. (Buerk, 1982; p. 19 quoted in Brown, 1996, pp. 1291–1292)

Burton (1996) has suggested that such a stark vision of mathematical territory might be less likely to emerge if learners were encouraged to explore the meaning of their experiences in the mathematics classroom through narratives. For Burton, narrativising involves placing mathematics in its context and personalising it—creating a landscape which the learner can navigate. In contrast to this position, Solomon and O'Neill (1998) question whether a narrative approach is appropriate in the learning of mathematics, arguing that ''mathematics cannot be narrative for it is structured around logical and not temporal relations'' (p. 217). They contrast William Rowan Hamilton's personal letters written during his discovery of quaternions with the mathematical papers he later wrote in which his findings are communicated to the mathematical community. While the personal letters contain many narrative characteristics, including temporal relations, the mathematical papers contain none. Solomon and O'Neill thus contend that ''mathematics cannot be adequately conveyed in narrative terms'' (p. 219), and warn educators of the possible dangers of Burton's approach. Is their warning a legitimate one?

We might counter that, while Burton seems to concentrate on the construction of a personally meaningful mathematics, Solomon and O'Neill completely overlook the possible role of narrative in more personal acts of understanding in referring to the way mathematics can or cannot be communicated within the mathematical community. Did Hamilton's narrative letter-writing play no role at all in his own conceptual development? And, even when Hamilton is communicating in the more logical, de-temporalised forms of the mathematical community, are those personal narratives not useful or not used? Have they been completely subsumed by a new discourse? We propose that to be perceived as mathematically literate, learners need to construct mathematical meanings that make sense to them, but that are also coherent with those socially recognised (Balacheff, 1991). That is, there must be some meeting of the public and the personal—and it may well be that it is narrative that holds the key to this connection.

Moving away from the specific terrain of mathematics momentarily, we find that writers as diverse as Aristotle, Barthes and Bruner have all recognised the centrality of narrative in human cognition. For Bruner (1996) narrative is "a mode of thinking, […] a structure for organising our knowledge" (p. 119). According to him, narrative is one of the two fundamental styles of thinking enabling human beings to make their way in the world—the other style being the "paradigmatic" or logical/classificatory one that has typically been associated with mathematics. Bruner maintains that through narratives, we both organise and constitute our experience of the world; we tell stories, make up excuses, and impose plots that have a beginning, middle and end. This way of describing the narrative style might strike some as being quite a general style of thinking in that it could conceivably include the paradigmatic style too. This seems to be the view espoused by cognitive neuroscientists Young and Saver (2001), who write: "[n]arrative is the inescapable frame of human experience. While we can be trained to think in geometrical shapes, patterns of sounds, poetry, movement, syllogisms, what predominates or fundamentally constitutes our consciousness is the understanding of self and world in story" (p. 72). This raises the question as to how we harness our narrative tendencies in the construction of mathematical knowledge.

In this paper we intend to address the tension or dichotomy between narrative and paradigmatic modes of thinking we find in the mathematics education discourse. Are they really distinct? Do they interact? Does mathematics privilege one over the other? We also aim to investigate the conditions under which narrative modes of thinking can be productively evoked in the mathematics classroom—productively in the sense of leading to deeper and more satisfying understanding. We do so by considering examples of narratives in the mathematics classroom. We are not so much interested in the stories that imaginative teachers concoct in their mathematics classrooms as we are in the sometimes less polished, less didactic narratives that students formulate as they try to make sense of mathematical phenomena. We assume that narratives are central to meaning-making, but maintain that their specific creation by students as well as their function in student meaning-making, particularly in mathematics, has so far remained poorly understood.

1 The narrative mode

Before presenting examples that illustrate students' use of narrative in mathematics, we outline the characteristics of the narrative mode that enable us to identify these examples. Bruner describes narratives as particular types of discourses: "Narrative is a discourse, and the prime reason for a discourse is that there is a reason for it that distinguishes it from silence" (p. 121). For him, narratives involve the recounting of sequences of events: "The sequence carries the meaning: contrast the stock market collapsed, the government resigned with the government resigned, the stock market collapsed. But not every sequence is worth recounting'' (p. 121). Bruner claims that narratives are warranted or justified when the sequence of events they recount can address or explicate the unexpected, or resolve an auditor's doubt. He also draws attention to the dual nature that narratives take on: ''A story then has two sides to it: a sequence of events, and an implied evaluation of the events recounted'' (p. 121), stressing how, in recounting a series of events, the story-teller presents his or her interpretations of them.[1] The phrase 'sequence of events' is central to Labov's (1972) definition of narrative as well: ''We define narrative as one method of recapitulating past experience by matching a verbal sequence of clauses to the sequence of events'' (p. 359). A narrative can thus be seen to imbed experiences in time-dependent, contextual discursive flow. The time-dependent sequentiality of narrative distinguishes it from another mode of thinking that has gained attention in mathematics education, that of metaphor. Instead of organising events or phenomena into sequence, metaphor attempts to gain meaning by comparing events and phenomena. A second, though perhaps slightly less clear-cut, difference relates to consciousness. While, according to Lakoff and Nunes (2000), for example, the cognitive processes associated with metaphor occur predominately backstage, the narrative mode, with its emphasis on voicing our interpretations of our self and world brings cognition out of the realm of the unconscious and onto the centre stage.

1.1 The interplay between narrative and paradigmatic modes

The narrative mode, as Bruner (1986) construes it, ''strives to put its timeless miracles into the particulars of experience, and to locate the experience in time and place'' (p. 13). By contrast, the paradigmatic mode ''seeks to transcend the particular by higher and higher reaching for abstraction'' (p. 13). Paradigmatic thinking is an explicit form of reasoning about the world of facts whereas the narrative mode employs tacit knowledge implied in the telling (and often encourages readings between the lines) and while the paradigmatic favours the indicative mode of speech, the narrative mode is often expressed using the subjunctive verbal mood, or at least through linguistic markers that express possibility, wishes, emotion, judgements or statements that may be contrary to the facts in hand. Solomon and O'Neill (1998) provide excellent examples of each in their quotations from Hamilton's (1844–1850) work on quaternions; they provide entries from his personal letters, which reflect many instances of narrative thinking (see pp. 214–215), with excerpts of his writing in a research mathematical journal, which includes, the following statement: ''Let an expression of the form $Q = w + ix + jy\ w = kz$ be called a *quaternion*'' (p. 215, emphasis in original)—a statement that reflects paradigmatic thinking. Although it is the paradigmatic mode that has more immediate resonance to most conceptions of mathematical thinking, we propose that the narrative mode actually characterises many of the ways mathematicians have talked about their own experiences of mathematics. For example, Sfard (1994) quotes the mathematician ''ST,'' a set-theorist:

[1] While we are aware that some distinguish between *narrative* and *story* (see, for example, Abbott, 2002), in this paper we have adopted Bruner's practice of using the terms interchangeably.

> There is, first and foremost, an element of personification in mathematical concepts... for example yesterday, I thought about some coordinates... [I told myself] 'this coordinate moves here and... it commands this one to do this and that.' There are elements of animation. It's not geometric in the sense of geometric pictures, but you see some people moving and talking to each other. (p. 48)

For ST, coordinates become people and are placed into sequences of events where they move, talk to each other and issue commands. When he says, ''I told myself'' he signals his entrance into a discursive mode in which his past experiences are being recapitulated in a sequence of clauses. Devlin (2000) has also drawn attention to this narrative mode of reasoning in mathematics among mathematicians: ''mathematicians think about mathematical objects using the same mental faculties that the majority of people use to think about other people'' (p. 262). For example, for the calculating wizard Wim Klein, ''Numbers are friends to me.'' Taking 3,844 as an example, he says, ''For you it's just a three and an eight and a four and a four. But I say 'Hi, 62 squared!''' In saluting the number, Klein begins to recapitulate the set of past experiences he has had with 3,844—which includes finding its square root.

We can assume that both ST and Wim Klein also reason paradigmatically about coordinates and numbers; they work with de-temporalised, abstracted conceptions of coordinates and numbers too. It thus seems more accurate to say their mathematical thinking involves an interplay between Bruner's two modes. Given our particular educational interests, we might ask is this characteristic of mathematics? One thing is that, in the public sphere, we have more examples of paradigmatic modes of thinking than we do of narrative ones, the latter being essentially more personal, private and idiosyncratic. The mathematics of textbooks, journals, public talks, reflect the formal, paradigmatic modes of thinking used to communicate mathematics that produce what Schiralli and Sinclair (2003) have called ''conceptual mathematics'' (CM). CM strives to be de-temporalised, de-personalised and de-contextualised (see Balacheff, 1988), it consists of publicly accessible tools that exist outside in a public space of shared meanings. It strives to eliminate narratives, whose interpretive nature clashes with its objective, rigorous ideals.

These CM concepts are not necessarily the same as the mathematical ideas that individual mathematicians form of them, which will be influenced by many experiential and genetic factors. Schiralli and Sinclair call these individual, private conceptions that are used to produce mathematics ''ideational mathematics'' (IM). IM is the mathematics of individual people, which includes the ideas of mathematics actual people have when they are working on mathematical problems or discussing among peers. Narratives are admitted in IM, for they provide a discourse in which mathematicians can capture their own past experiences, which is crucial to IM. We should make clear that we are not equating IM with narrative thinking and CM with paradigmatic thinking, for they arise from different types of distinctions. What we do claim is that individual mathematicians think both in paradigmatic and narrative terms within their personal space of IM. Reconsidering the debate between Burton (1996) and Solomon and O'Neill (1998) in these terms, we see the approach espoused by Burton as emphasising IM, learners' individual contextualisations of mathematical challenges (p. 33), while for Solomon and O'Neill the mathematical is principally associated with formal texts, with traditional mathematical genres and hence with CM. Our position is that to understand the role of narrative in the construction of mathematical understandings, we need to understand how it functions in the meeting of the ideational (personal) with the conceptual (public).

1.2 Mathematical narratives

In this section, we clarify what narratives look like in mathematics; and how they might be characterised so as to be reliably identifiable. Returning to Bruner again, we find that narrative modes possess four characteristics that will help us offer an operationalisable definition:

- They have inherent sequentiality.
- They can be about real or imaginary events (a certain sense of factual indifference pervades).
- They forge connections between the exceptional and the ordinary.
- They have some kind of dramatic quality.

Can these characteristics be evoked in a mathematical setting, where, for example, factual indifference seems anathema, and sequentiality is constantly being chased away by de-temporalisation? We will consider these characteristics one by one.

The first will pose some difficulties for Solomon and O'Neill (1998), who insist on the time-independence of mathematics. However, in ST's description of coordinates, of *his mathematical thinking*, given above, we can clearly see sequentiality: this happens *and then* that happens; commands are issued and so a series of causal events is set up.

The second characteristic may, on first sight, pose even greater difficulties, given mathematicians' commitment to the true and the factual. And yet surely an interplay between the real and the imaginary is a critical facet of mathematical thinking, as well as mathematical objects—if one recalls their genesis. This interplay can be seen in the development of new numbers, such as the imaginary numbers and the quaternions, as well as in the development of new geometries. In each case, the new mathematics begins in an imaginary, and even fanciful, world and migrates to the "real" world as mathematicians create increasingly more connections to familiar, "real" objects and build consistency. The narrative mode permits the two worlds to co-exist, even if only temporarily.

The third characteristic, forging connections between the exceptional and the ordinary, may in fact present the least difficulties. The example Bruner provides involves a person vigorously waving a flag at the post-office, something that many would find extraordinary in that it does not fit into the usual scheme of things that happen at the post office. However, if one can find the right narrative that explains the presence of the flag-waver, if one can interpret the extraordinary event in more ordinary terms, the flag-waver becomes much less extraordinary. In the diaries of Hamilton, similar sorts of narratives around his discovery of quaternions occur. In trying to make sense of his own discovery, he forges connections between the ordinary world of geometry, algebra, space and time, with the exceptional world of his new entities:

> The quaternion was born, as a curious offspring of a quaternion of parents, say of geometry, algebra, metaphysics, and poetry... I have never been able to give a clearer statement of their nature and their aim than I have done in two lines of a sonnet addressed to Sir John Herschel:

And how the One of Time, of Space the Three
Might in the Chain of Symbols girdled be. (quoted in Graves, 1885, p. 525)

We find the fourth characteristic, that of dramatic quality, most difficult in that it involves a subjective judgement as to the quality of the narrative being told. Certainly, many narratives with dramatic quality exist in the mathematics lore: the discoverer of irrational numbers being put to death, for example. However, we are most interested in the narratives that are told by people trying to make mathematical sense and engage in mathematical thinking. Douglas Hofstadter (1997) provides a compelling example in describing his investigation of triangle centres. He begins by expressing dissatisfaction at the fact that the incentre is left out of the special relationship (the Euler line) that joins the most common special centres, namely, the circumcentre, the orthocentre and the centroid. ‘‘Why should the incentre be left out?’’ he laments. In developing concern for the left-out incentre, he has turned a rather dry phenomenon into a drama in which one character finds him/herself isolated and unloved. This motivates Hofstadter to try to find another party for the incentre to join, which he eventually does, through the tryst of the Nagel segment. The narrative's dramatic quality (for Hofstadter, at least) derives from the initial tension of isolation. The story Hofstadter tells (to himself, and then to us) plays a crucial part in his mathematical thinking in that it motivates his inquiry and thus provides him with a basis on which to seek mathematical relationships. We will return to the motivational role of narrative as we discuss examples of student-generated stories in the mathematical classroom.

As we have shown, the characteristics of narrative can be identified in various forms of mathematical activity undertaken by mathematicians, including mathematical discoveries and creations. However, can we also identify situations in which *students* are motivated to construct such narratives, to begin to make sense of some mathematical territory and hence claim it as their own? In the remainder of the paper, we consider the kinds of narratives we have observed students recount. Since both of our research interests have focussed on the use of expressive technologies, we take our examples from computer-based settings. While the focus may seem merely convenient at the outset, we have reason to believe that expressive technologies may provide unique opportunities for productive narrative in the mathematics activities of students.

2 Narrative and computer interaction

We have both been involved in the design and use of mathematical microworlds, computational environments composed of models of a domain of mathematical knowledge, represented by a formal system or a set of primitives along with phenomenological displays (be they physical, graphical and/or auditory) that depict the actions of these formal objects. If we trace the term ‘‘microworlds’’ to its beginning, we find the narrative of its history is replete with metaphors of the kind of ownership we would like learners to experience in relation to mathematics. The term was associated originally with the domain of Artificial Intelligence, and with simplified ‘‘fairyland’’ models of limited domains of the experienced world (Minsky & Papert, 1970, p. 70). Papert (1980) imagined accessible, evocative and engaging provinces of mathlands (p. 125), or mathematical cultures, in which learners would become immersed, and from which they would emerge as more mathematically fluent. His conception was of computational objects which would embed a mathematics that is not only formal but also "related to the self, the body, material and social objects, and activities" (Papert, 1992, p. xv) and hence permit a approach to mathematical sense-making that is both *body-syntonic*—it relates to learners' sense and knowledge about their own bodies—and *ego-syntonic* or coherent with learners' sense of themselves as people with intentions, goals, desires, likes and dislikes.

The narrative view of the microworld would suggest that, potentially at least, they are environments designed for story telling. So, using Bruner's characterisation of narratives, is it possible to identify stories that emerge as students negotiate mathematical activities using such expressive technologies? And perhaps more importantly are these stories productive mathematically—by grounding the activities of the formal proprieties and relations in interpretations connected to their embodied experiences and personal characteristics, do learners construct an IM that can be connected to the conceptual?

To explore these two questions, we present a number of examples from our observations of learners interacting with mathematical microworlds and we examine these examples in terms of Bruner's characteristics for narratives. They are organised into two sets. The first set, *exploring the landscape* focusses on activities involving explorations of the tools of different microworlds and the second, *building on the land*, includes interactions involving the construction by learners of new microworld tools.

It should be stressed that these examples come from a number of different and unrelated research projects, which occurred in different parts of the world. They are part of a larger collection that have tended to be put aside in our academic writings, as we concentrated on the more overtly mathematical, or rather as we emphasised in our analyses the paradigmatic details and archived these interactions as nice but somehow not worthy of the same attention. Yet, they impressed on us and continue to impress on us. They are like those magic moments when something important clicks into place. This paper is our first attempt to theorise about those moments and to recognise the importance of listening to and for the mathematical in the voices of our students, even when it is presented in a narrative genre.

2.1 Exploring the landscape

Our first example comes from a dynamic geometry setting and belongs to two girls, Meena and Haley (aged 12 and 13 years), who were interacting with a dynamic sketch that they had been asked to reproduce. The activity can be described an example of as what have been called "black-box" tasks—given a geometrical figure (a "black-box") defined on the basis of a number of unknown geometrical properties, the idea is that through extensive dragging activities, learners uncover the geometrical relations involved and use the construction tools to reproduce a figure with identical "behaviour" (Laborde, 1995). Unlike many black-box figures, the "box" in this task was not a conventional geometrical figure, but a (geometrically-defined) stick-person, shown in five different positions in Fig. 1.

As soon as the girls embarked on the task of uncovering the relations of the figure, they began to talk in terms of dancing.

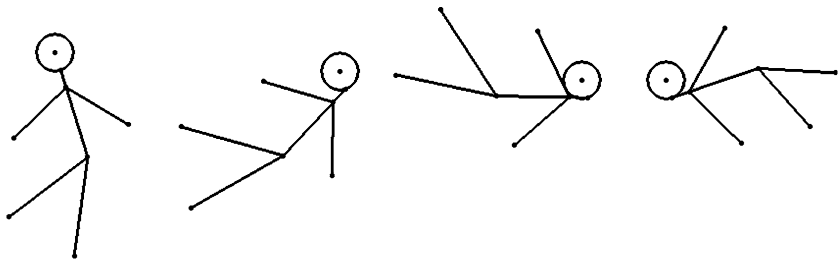


Fig. 1 The dancing stick-person

Meena: Well, *OK, see the arms can go and the legs, the legs can go like this, like this*.
Haley: *Groovy dancer* (begins to sing).

As Haley sang snatches of songs, Meena moved the points "in time" to the music. The figure's animation also animated the students and provoked them to describe its actions in very human terms as well as geometrical ones. The pair were particularly enamoured by the "hip" movements of the dancer, which they discovered traced a perfect circle as it somersaulted head over heals.

Haley: *Yeah!* (as Meena rotates the figure's "hips" over the "head" of the figure") *And one more time for the* ...(sings) *Are you ready? Here we go..do, do, do...*
Meena: *Here we go, and over he goes, like a somersault, head over heals...*
Haley: *...more like heals over head...*
Meena: *...Mmm, and in an exact and perfect circle*
Haley: *Make him dance slowly a minute. He keeps his body straight for the whole move, it doesn't ... bend, and this* (the "hip" point) *is always the same away from this, no this. Tricky!*

The figure's movements were hence concretisable both in terms of known geometrical objects and by connection with movements of real human bodies.

Does this example represent a narrative? We think that it does—and, more precisely, that the characterising features of narratives help explain the role of story of the dancers in mathematical-meaning making. We can identify the presence of all four of Bruner's characteristics, although two in particular stand out: the factual difference between the real or imaginary and the dramatic quality.

The central actor in this story is the stick-person, but this actor was not seen as real—quite the contrary, the figure clearly represents an extremely limited model of a living being: the hips of a real person hardly rotate following the trajectory of a perfect circle. Indeed, part of the dramatic quality of the story was the impossibility, in human terms, of the dance steps performed by the computational agent. And it was the virtual nature of this agent that made this extraordinary behaviour more understandable. Its ordinariness can be related to its mathematical nature and to a mathematical object familiar to both girls, the circle.

In interacting with the dynamic geometry software, and in creating a sequence whereby the stick-figure is brought to life and passes through a series of dance movements (accompanied by Haley's soundtrack), it made sense to Haley and Meena to tell a story in which circles were treated as a set of points associated with a dancing figure, with the point representing the hip always the same distance from the point at which the neck joins the body. The narrative is well connected to a view of circle as the locus of a set of points equidistant from a centre. So while situated in the context of dancing, the learners express an abstraction whose validity—at least potentially—transcends the context. There is also some evidence of movement from a narrative to a more paradigmatic mode of thinking. When they are discussing dancing there is less of a need to be precise, it is not necessary to clarify whether it is head over heals or heals over head, for example, but once the connection to the circle is made by Meena, Haley also switches to a more indicative form of language use in order to identify an invariant property in the dance move. Although still couched in terms of the dance, she turns her attention to fact rather than fiction.

Motion is clearly a central factor in the narrative created by Haley and Meena, helping to allow the stick-person model to be an evocative, body-syntonic, object-to-think-with. We might hypothesise that its movements activate the finely developed meaning-making mechanisms we possess for perceiving things in motion, enabling, in this case, the girls to link the mathematical behaviour of the computational agent with a very human activity, by mapping the tracing of a circle onto the gyrating of hip. Of course, in this example the anthropomorphisation of the computational object does not come as a great surprise, given that the points, circumferences and segments were represented in the form of a person. Is this example, then, an isolated case or do similar interactions occur in the face of motion on screen even when the objects have no obvious connection to real-world agents?

Our next example considers this question. It again involves a dynamic geometry setting, but this time, instead of portraying stick people, or anything resembling human figures, the sketch was composed of four dynagraphs (originally described by Goldenberg, Lewis, & O'Keefe, 1992). Dynagraphs are representations of graphs in which both the *x*- and the *y*-axis are horizontally-configured so that dragging an input along the *x*-axis causes an output to move along the *y*-axis (see Fig. 2). As the input is dragged toward the right in dynagraph *f* the output moves as well, going right when the input goes right and left when the input goes left. As can be seen in Fig. 2, the output for *f* is twice as far from the hash marks as the input, suggesting a $y = 2x$ relationship.

Typically, each dynagraph is explored, one by one, with the audience (teachers or students) being asked to focus on the behaviour of the outputs and to reason about the kind of function that each dynagraph expresses. When dynagraph *h* is first manipulated, an inevitable loud and sustained burst of laughter occurs: as the input is dragged, the output moves along in stuttering steps made possible by the step function relationship between the input and output, thus contrasting with the continuous fashion in which outputs *f* and *g* move along the axis. When asked to explain

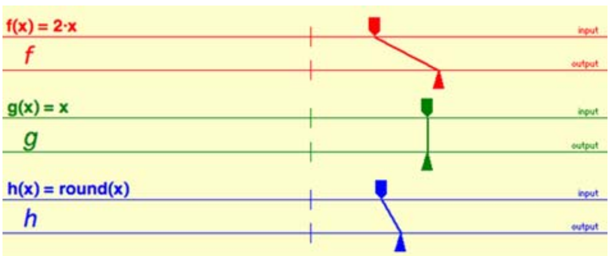


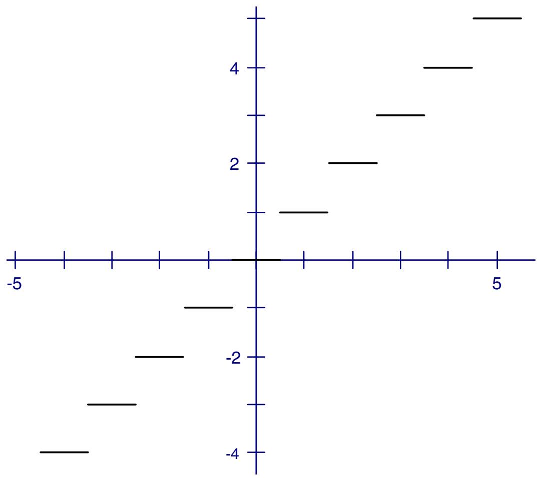
Fig. 2 Three examples of dynagraphs

why they are laughing, the teachers respond with explanations such as "it looks like a cool dude walking across the screen" or "I've just never seen a function with so much personality before." The first explanation contains tacit knowledge about how a cool dude sort of hops as he walks, lingering on discrete locations on the ground just as the dynagraph stays on one integer for a short while before hopping to the next. The second explanation, though not employing the subjunctive verbal mood, offers a statement that is clearly contrary to fact: the function has personality!

These narratives are short, and inferred from certain reactions along with explicit articulations, but we can still find traces of Bruner's narrative characteristics. They depend on an inherent, perceived sequentiality as well as a perception of dramatic quality. Past experiences are recapitulated to make sense of an initially extraordinary event, a jerking response to a smooth mouse movement, and, once again, to connect to the mathematical an imagined human activity is evoked. In this case, the anthropomorphisation does not rely on stick-figure shapes; instead, the motion of the objects on screen, with its characteristically human rhythm, evokes a narrative in which some "cool" person is "grooving" across the screen, and the behaviour, initially extraordinary in functional terms, becomes more ordinary, or at least accessible to interpretation. Consider the way in which this motion-based representation differs from the step function shown in Fig. 3. The canonically-represented step function may elicit metaphors, such as stairs, but it does not as persuasively and dependably evoke characteristics of narrative proposed by Bruner (nor does it make teachers laugh).

In these two examples, we argue that learners construct their own stories to account for the behaviours of computational objects, whose actions on the screen were defined by others. The narratives may not necessarily represent a "whole story" but at the very least they involve a fleshing out of the characters involved, attributing purpose to these characters and defining relationships between them in ways that gives meaning to the activity. While the systems behind the behaviour are formal, rigorous and certainly of a paradigmatic nature, the narratives recounted to interpret the screen effects produced by the computational objects are anything but—the formally defined objects come to life as purposeful agents who dance and

Fig. 3 The canonical, static representation of a step function



jive. Perhaps this is only to be expected, perfectly reasonable, given the nature of the feedback provided. After all, in these examples, the learners were not actually interacting with the formalisms behind the computational agents, but interpreting movement. What happens when learners are not asked only to explore, but to construct computational tools? Are their activities better characterised as paradigmatic or does the narrative mode continue to play a significant part? We now turn to consider the interactions that emerge as student work on this kind of construction activity.

2.2 Building on the land

A quality distinguishing microworlds from many other computational learning environments is that users are expected to build upon the given restricted environment to create a world of their own. The next examples are associated with this extensibility and concern learners' interactions during attempts to construct new microworld tools. More specifically, the examples occurred during interactions with a Logo microworld designed for the exploration of geometric transformations and the construction of a tool for reflecting a (turtle) point in a (turtle) line. This challenge was embedded in the task presented in Fig. 4.

In this microworld, learners can generally communicate with any of the different turtles on screen to change their position (they can ask them to go forward or backward a certain number of steps or turn through a specific angle to the right or left, for instance) or to discover information about the relationships between turtles (the distance between two turtles, for example, or the angle that one might turn to point towards another). The idea was that learners would begin by building a procedure in which a blue turtle and a red turtle with the same initial position would produce two halves of a quadrilateral having its diagonal as one line of symmetry (leaving the blue and red turtles in positions symmetrical to each other in relation to the given line). In the second part of the task, the aim was to place another turtle so that it was also the image of the blue turtle under reflection—this essentially meant sending a new turtle to the location of the red turtle (already the image of the blue), but without communicating with the red turtle to determine any necessary distances or turns. The mathematical aim of the task hence is the construction of a general procedure that can be used to produce the image of a turtle under reflection irrespective of the relative positions on screen of the turtle point and turtle line that

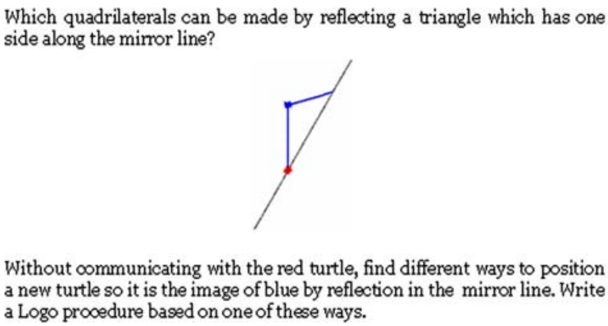


Fig. 4 The construction task

represent the input variables. However, what was interesting was how the task constraint of banning communication with one of the turtles impacted the students' interactions.

The explanations developed to explain this imposed condition by the 13-year-old girls who interacted with the task varied. In one case, the two turtles became lovers who had rowed and Jodie described the purpose behind the positioning of the new image turtle as follows:

> *They are not speaking but blue wants to make up...he's going to ask his friend to go and talk to her* (she points to the red turtle)...

For Jodie, the unusual appearance of an uncommunicative turtle and the requirement that a third turtle become involved became more commonplace when she associated it with recapitulations about lovers' quarrels and the sequences of events that evolve to resolve them. They also brought a certain dramatic quality into play. The idea of meeting that permeates this narrative emerged frequently during microworld interactions, not least perhaps because of the tool `meet`, a tool that allowed the construction of a new turtle at the point at which the paths of two others meet (a turtle version of an intersection point). As this tool is executed, the user sees two turtles crawling in unison from the location of the original turtles until they meet. The motion of the turtles, apparently, ''running into each other arms'', along with social experiences associated with the word ''meet'' help explain why the students tended to evoke friendship and partnership to interpret the action played out on screen. At the same time, this feedback also permits several of the mathematical relationships characterising reflection to be visually experienced. Hence, as well as treating the computational objects as feeling beings, participating in the dance of life, they could also be seen as mathematical beings. When it came to actually defining new behaviours for these beings in the formal language that they understand, even descriptions suggesting paradigmatic thought—static, factual and explicit, as the first sentence below clearly shows—often contained traces of the narratives that had accompanied their genesis, as the following description illustrates:

> *Every turtle has its own reflection turtle with the same distance away from the mirror and the same angle, except for lefts and rights. This one* [she is talking about the turtle which traced the axis for reflection] *has no distance away and no angle, but it still has its own reflection turtle... So its like going RT zero and forward zero, and his partner is on top, completely covering...*

This description suggests a meaning for reflection that can be closely related to the notion of function: each turtle has its own image—even when both are the same—and about mathematical relationships, the distances and angles, by which elements in the input set are related to their pairs in the output set. And yet, when Aimee refers to the independent and dependent variable pair as partners, a glimpse of her narrative, in which each turtle and its image were also romantically related, can also be seen.

Love stories were not the only narratives to emerge. When one pair, Laurel and Candy, came across the task constraint disallowing communication with the red turtle, they tried to think of reasons for the constraint, which both bemused and amused them. As they sought explanations, they entered the subjunctive mood of possibility:

Candy: *Perhaps he just fainted?...or could be deaf maybe?*

And as the girls' embarked on the given task, their objective became to rescue the red turtle.

Laurel: *Oh, and oh, so we need to send a blue to the rescue!*

The girls began again by considering the problem of positioning a turtle on the mirror.

Candy: *OK, SOS red turtle...what should we do?*
Laurel: *I don't know, we need to know, the point on the mirror, but we can't meet red.*
Candy (sings) *Cos red could be dead, or asleep in his bed...*(says) *we could meet mirror* (the black turtle that had traced the mirror line), *he can still hear can't he? Let's just try meet mirror.*

Making use of the idea developed in previous microworld activities, that symmetrical turtle paths are made by turning through angles with equal value but in opposite directions and traversing through equal distances, along with the visual feedback to each of the commands they typed, the pair succeeded in placing the new turtle exactly on top of the red. Operation Rescue Red was not forgotten as Candy exclaimed "*We've done it! We're geniuses. Now, mouth to mouth resuscitation.*" Yet, the pair knew they had not quite finished. They knew that they needed to develop a procedure that could be used regardless of the initial position of the blue turtle and they went on to test their command set with other examples—Operation Rescue Red: Take 2 and 3, encapsulating their results in the procedure which they named "freda" (Fig. 5).

Although this procedure is not completely general (it works for all turtle positions in which the turtle is pointing towards the mirror line), it encapsulates both the distance and angle properties that characterise the isometry reflection, and represents a functional relationship between independent variable (the original blue turtle) and dependent variable (the reflective image of the blue turtle in the line traced by the mirror turtle).

The story that accompanied its construction was also based on the idea of one turtle dependent on another for its rescue—the narrative around rescue implied the dependence relationship. Within it, we can identify all of Bruner's four characteristics. There is sequentiality: first, a sequentiality constrained by the demands of the task, in which interactions with the red turtle preceded the positioning of a new turtle in an identical location; and second, a sequentiality in terms of the procedure necessary for this positioning. There is a pervading sense of factual indifference: it

```
to freda
blue, meet "mirror
remember towards "mirror {:m1}
run :m1
run :m1
remember distance "blue {:m2}
bk :m2
end
```

Fig. 5 Laurel and Candy's reflection procedure did not matter to the story-tellers that the red turtle was not really dead, nor asleep in its bed and indeed both these explanations reflect attempts to forge connection between the exceptional (a turtle that does not respond to commands) and the ordinary. Finally, there is its dramatic quality. This quality is not only present in the story line, but also in the suspense that accompanied the pair's interactions, particularly the sense of bated breath when they tested whether their rescue strategy worked for different positions of the input turtles.

Ostensibly, the narrative constructed appears to be related to meeting the objectives of the task and less to understanding what is going on mathematically. But, as with the stories presented in the previous section, the mathematical is also present: in addition to drawing students into the task itself, their narrative is based on the notion of dependency that also characterises the mathematical meaning the task was intended to highlight.

3 Learning mathematics through narrative: reflections on our story

By using narrative, or, more particularly, the four features Bruner suggests to characterise stories, as our organising principles, we have attempted to show how learners, as well as mathematicians, make claim for mathematical territories by populating the landscape with fictional beings engaged in purposeful activity. While we are convinced that the stories we have presented are truly representative of Bruner's narrative mode, we also propose that the paradigmatic mode asserts itself in each and every one of them. The stories provide a means to locate mathematical concepts in the particulars of experience, but not to necessarily imprison them there.

We believe the stories contain the seeds from which we can begin to construct an understanding of the role of narrative in mathematical-meaning making. With this aim in mind, we end by reconsidering them from three different angles, offering our own story on the connections between the narrative mode of thinking, expressive technologies and the construction of mathematical meanings. First, we focus on the microworld features we see as central in affording story-telling; second, we seek to uncover relationships between tool and task design and the narratives that emerge during their use; and finally we offer our reflections on the place of narrative in school mathematics.

3.1 Motion, time and agency: story-affording features of microworlds?

Permeating the microworld-mediated stories we have presented are three interrelated themes: motion, time and agency. As the stories unfolded, we came to see these as closely related to the four characteristics Bruner posits as central to stories worth telling. Time is clearly a factor in the inherent sequencing present in any good story and we suggest that motion is behind the dramatic quality and the connections made between exceptional and routine occurrences, which are also central to story telling. Perhaps, however, it is the agency ascribed to microworld objects that holds the key to understanding why such objects might be particularly effective, not only in permitting good stories to emerge, but in forging productive connections between the narrative and the paradigmatic and in creating mathematical meanings that make sense beyond the specifics of context in which they arise. To examine their story-affording features, we consider each of these themes in more detail.

The physical movements of screen objects seemed to invite the learners to construct narratives, thus imbuing the objects with a purpose and intent that involved situating the agency of these supposedly paradigmatic actors within a real-world activity. The presence of motion, indeed, the primacy of motion in many microworlds, often has the effect of reversing two of the ''de's'' cited by Balacheff (1988) above: motion returns mathematics to the realm of time, since motion can only occur over time.

This brings us to the second theme, the time-dependent nature of activities with expressive technologies. While amazingly fast, computers run in real time and programs are based on sequences of carefully ordered commands. The necessary sequencing of actions with expressive technology has resonance with the sequencing that characterises narratives as described above. Hence as our students strove to build and understand computational constructs characterised by paradigmatic intentions and to transcend the particularities of the examples they worked with, perhaps it was the need to organise their interactions with the computational objects in terms of a meaningful sequence of events that provoked a narrative perspective to emerge.

And it is the human-computer interaction which accounts for the third theme, that of agency. Computational objects can be viewed as dynamic interactional agents and microworld interaction akin to interpreting and constructing behaviours of agents who act and interact in real time. Making sense of this behaviour through story-making is perhaps what permits initial entry into a mathematical territory, particularly for those whose previous experiences have suggested it to be a barren land with little to offer and with customs very different to those they follow. This is the motivational aspect.

But story also has an important role to play beyond facilitating the completion of the border formalities. Once exploring the land, learners know that the computational agents are made up of a different substance: they are not of flesh and blood, neither are they necessarily governed by the same bodily or cultural constraints as us; rather, they are merely virtual beings—ideal for story construction as factual indifference (in one sense at least) is present from the start. Nonetheless, despite their virtual nature, the behaviour of these agents is not unconstrained. It is mathematically constrained. Very human terms may couch the stories told about them—they become dancers, artists, friends and lovers. Yet, and for us this is a critical point, as we hope our examples have shown, it is not just any context that serves. The storylines we have presented, and the mathematical ideas, the IM expressed by the learners during their interaction with the computational agents, are all connectable with aspects CM, mathematics as the shared intellectual construct represented in the public sphere—the mathematics they are supposed to learn.

3.2 The shaping of storyline by microworld tasks and tools

It is the shaping of intended learning that we use as a second angle onto the stories. The microworld activities we have described occurred in the context of carefully constructed tasks, designed with particular learning objectives in mind. Can we identify specific ways in which the design of the tasks and tools impacts upon the emergence of particular story lines and particular mathematical meanings? Reflecting back on our examples, we notice that the tasks are far from neutral in the stories we have presented. As designers, our conscious intention was to craft sequences of activities that would bring learners into contact with particular aspects of CM. We were perhaps less conscious of how our own IM impacted on the narratives inherent in this sequencing of mathematical ideas, relationships and properties. In retrospect, we can see that all the tasks reported in this paper have in common the presentation of something unusual—a step function appearing suddenly in the midst of continuous ones, a turtle that cannot be communicated with, a stick-person who moves mathematically. Built into the tasks, then, was at least one incentive for story-making—something exceptional that needs explaining. The computational objects became the characters who, because of their dynamic capabilities, were able to act out the extraordinary event and the storyline provided mathematically orientated interpretations for it.

If tasks cannot be seen as neutral, the same is true for the microworld tools. The behaviours programmed into the computational agents and even the names of the tools which control these behaviours may well have a hand in the particular interpretations that arise—we suspect, for example, that if instead of "meet" the MTG tool had been called something like "Intersection turtle", and if movement had not been highlighted during its execution then rather different narratives would have emerged. Whatever, once in progress, the narratives that do emerge act back on the tools, giving meanings to them beyond those originally intended by the microworld designer, their function being extended and modified as they are used to solve the tasks in hand.

In terms of the design of learning situations involving expressive technology, it seems that we not only created resources for productive narratives, we also built into microworld tools and tasks traces of our own narratives, our own IM. It is not that we expect learners to retell our stories, indeed one of the wonderful things about stories is that each time they get told they can include new twists and turns. What we as designers who are also teachers are aiming for are learning activities that draw learners into constructing their own twists and turns that help them to interpret mathematical phenomena, while at the same time appropriating a means to express themselves mathematically.

3.3 Returning to narratives in school mathematics

This brings us back to Solomon and O'Neill (1998) and their concerns that the narrative emphasis on authorship, creativity, and personal expression will favour the genre of recounting and will direct learners' attention away from the way mathematical texts are constructed and the reason for their unique construction (p. 211). Solomon and O'Neill see the narrative approach to mathematics as consisting of first engaging in some mathematics and second recounting how one engaged with it or explaining "what you did". We do not think this has much to do with storytelling—or to Bruner's mode of narrative thinking—at all. It may elicit accounts based around a particular sequencing of events, but necessitates no forging of connections between the exceptional and the ordinary, no dramatic quality and certainly no sense of factual indifference. If the "narrative" merely recounts a sequence of events, without evaluating or interpreting it, then it cannot be counted as a story.

In relation to the second objection, we would agree that it is important for learners to have the opportunity to focus on the particularities of mathematical texts and mathematical discourses and to understand that expressions involving formal symbol-systems represent a fundamental part of the practice of mathematics. What we have difficulty in accepting is that respecting the validity of learners' stories as a vehicle for mathematical understanding necessarily diverts attention away from the structure of a mathematical text. Indeed, our stories illustrate rather the contrary: to communicate with the computational participants in the mathematical activities, attention must be given to the structure of the language these agents respond to.

We hence differ on two points. First, Solomon and O'Neill seem to equate mathematics exclusively with CM, whereas we also value IM, perhaps especially as our own ideas of mathematics are constantly being rethought in the light of the technological tools we use to express them. Second, we take a rather different perspective on of the potential role of the narrative in the mathematics classroom. In our view, the question is not so much whether or not a so-called narrative approach is adopted—learners, we believe, will continue to construct stories as a result of their participation (or non-participation) in mathematical activities. Rather, our question is how we can support them in constructing productive narratives: narratives that imbue a sense of ownership of the mathematics they are studying and that make possible a sharing of their IM. In this paper, we have attempted to present the germs of an answer. We have argued that productive narratives, in terms of mathematical sense-making, are those in which learners are able to connect mathematical objects, and their paradigmatic relationships and properties, with things they already know—and care—about: stories in which the mathematical is given meaning through its grounding in experienced phenomena. In our view, expressive technologies afford productive narratives precisely because they permit such grounding. The feedback they provide in terms of motion, colour, sound and the like, provides the basis for physical grounding, for embodied mathematisations, in which physical movements of computational objects on computer screens can be associated with sensory-motor experiences of bodies in the real world. They also seem to enable a more psychologically based grounding, in which the mathematical agency of computational objects becomes associated with familiar feelings, objects and desires. The mathematically constrained agents who populate expressive technologies hence appear to have the potential to become for learners mathematical characters who bring life to their IM, while also permitting them to express their ideas in forms that can be connected to the conventions of CM. We still need to understand better the conditions under which this potential might be best exploited and the specific mechanisms through which the interplay between the two worlds can occur. We believe that this is a research agenda worth pursuing.

References

Abbott, H. P. (2002). *The Cambridge introduction to narrative*. Cambridge: Cambridge University Press.

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics (D. Pimm, Trans.). In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216–235). London: Hodder and Stoughton.

Balacheff, N. (1991). Treatment of refutations: Aspects of the complexity of a constructivist approach to mathematics learning. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 89–110). Dordrecht: Kluwer Academic Publishers.

Brown, S. (1996). Towards humanistic mathematics education. In A. Bishop (Ed.), *First international handbook in mathematics education* (pp. 1289–1331). Dordrecht: Kluwer Academic Publishers.

Bruner, J. S. (1986). *Actual minds, possible worlds*. Cambridge, MA: Harvard University Press.

Bruner, J. S. (1996). *The culture of education*. Cambridge, MA: Harvard University Press.

Burton, L. (1996). Mathematics, and its learning, as narrative – A literacy for the twenty-first century. In D. Baker, J. Clay, & C. Fox (Eds.), *Changing ways of knowing: In english, mathematics and science* (pp. 29–40). London: Falmer Press.

Devlin, K. (2000). *The math gene: How mathematical thinking evolved and why numbers are like gossip*. New York: Basic Books.

Goldenberg, P., Lewis, P., & O'Keefe, J. (1992). Dynamic representation and the development of a process understanding of function. In G. Harel & E. Dubinsky (Eds.), *The function concept: Aspects of epistemology and pedagogy* (pp. 235–260). Washington, DC: Mathematical Association of America.

Graves, R. (1885). *Life of Sir William Rowan Hamilton* (Vol. 2). Dublin: Hodges, Figgis and Co.

Hamilton, W. (1844–1850). On quaternions; or on a new system of imaginaries in algebra. *Philosophical Magazine*, (Vols. 25–36) (Reprinted in H. Halberstam & R. Ingram (Vol. 3, Eds.). (1967). *The mathematical papers of Sir William Rowan Hamilton* (pp. 227–297)). Cambridge: Cambridge University Press.

Hofstadter, D. R. (1997). Discovery and dissection of a geometric gem. In J. R. King, & D. Schattschneider (Eds.), *Geometry turned on!: Dynamic software in learning, teaching, and research* (pp. 3–14). Washington, DC: The Mathematical Association of America.

Laborde, C. (1995). Designing tasks for learning geometry in a computer based environment. In L. Burton, & L. B. Jaworski (Eds.), *Technology in mathematics teaching – a bridge between teaching and learning* (pp. 35–68). London: Chartwell-Bratt.

Labov, W. (1972). *The transformation of experience in narrative syntax*. Philadelphia: University of Pennsylvania Press.

Lakoff, G., & Nunes, R. (2000). *Where mathematics comes from. How the embodied mind brings mathematics into being*. New York: Basic Books.

Minsky, M., & Papert, S. (1970). Draft of a proposal to ARPA for research on artificial intelligence at MIT, 1970–1971.

Papert, S. (1980). *Mindstorms: Children, computers and powerful ideas*. London: Harvester Press.

Papert, S. (1992). Foreword. In C. Hoyles, & R. Noss (Eds.), *Learning mathematics and logo* (pp. ix–xvi). Cambridge, MA: MIT Press.

Schiralli, M. & Sinclair, N. (2003). A constructive response to 'Where mathematics comes from'. *Educational Studies in Mathematics, 52*(1), 79–91.

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics, 14*(1), 44–55.

Solomon, Y., & O'Neill, J. (1998). Mathematics and narrative. *Language and Education, 12*(3), 210–221.

Young, R., & Saver, J. (2001). The neurology of narrative. *Substance 94/95, 30*(1–2), 72–84.