MATHEMATICAL THINKING AND LEARNING, *6*(3), 261–284

The Roles of the Aesthetic in Mathematical Inquiry

Nathalie Sinclair
Department of Mathematics
Michigan State University

Mathematicians have long claimed that the aesthetic plays a fundamental role in the development and appreciation of mathematical knowledge. To date, however, it has been unclear how the aesthetic might contribute to the teaching and learning of school mathematics. This is due in part to the fact that mathematicians' aesthetic claims have been inadequately analyzed, making it difficult for mathematics educators to discern any potential pedagogical benefits. This article provides a pragmatic analysis of the roles of the aesthetic in mathematical inquiry. It then probes some of the beliefs and values that underlie mathematical aesthetic responses and reveals the important interplay between the aesthetic, cognitive, and affective processes involved in mathematical inquiry.

The affective domain has received increased attention over the past decade as mathematics education researchers have identified its central role in the learning of mathematics. Mathematicians however, who are primarily concerned with the doing of mathematics, have tended to emphasize the importance of another, related noncognitive domain: the aesthetic. They have long claimed that the aesthetic plays a fundamental role in the development and appreciation of mathematics (e.g., Hardy, 1940; Poincaré, 1908/1956). Yet their claims have received little attention outside the élite world of the professional mathematician and even less explanation or justification. This state of affairs might be inconsequential to the practices of the professional mathematician, but it severely constrains the ability of mathematics educators to analyze the possibilities of promoting aesthetic engagement in student learning.[1]

Requests for reprints should be sent to Nathalie Sinclair, Department of Mathematics, Michigan State University, East Lansing, MI 48864. E-mail: nathsinc@math.msu.edu

[1]In an editorial in *Educational Studies in Mathematics* (2002, volume 1, number 2, pages 1–7) this area of research is highlighted as one of a few significant, yet under researched, issues in mathematics education.

Those who have focused explicitly on the aesthetic in relation to mathematics learning have questioned the extent to which students can or should learn to make aesthetic judgments as a part of their mathematics education (Dreyfus & Eisenberg, 1986, 1996; Krutetskii, 1976; von Glasersfeld, 1985). Their doubt is based on a view of aesthetics as an objective mode of judgment used to distinguish "good" from "not-so-good" mathematical entities. However, other mathematicians (Hadamard, 1945; Penrose, 1974; Poincaré, 1908/1956), as well as mathematics educators (Brown, 1973; Higginson, 2000; Papert, 1978; Sinclair, 2002a), have drawn attention to some more process-oriented, personal, psychological, cognitive and even sociocultural roles that the aesthetic plays in the development of mathematical knowledge. At first blush, particularly because some of these scholars associate the aesthetic with mathematical interest, pleasure, and insight, and thus with important affective structures, these roles should be intimately related to the concerns and challenges of mathematics education. In fact, this position is supported by the researchers who have considered a broader notion of the aesthetic (e.g., Featherstone, 2000; Goldenberg, 1989; Sinclair, 2001). From this perspective, which I adopt, a student's aesthetic capacity is not simply equivalent to her ability to identify formal qualities such as economy, unexpectedness, or inevitability in mathematical entities. Rather, her aesthetic capacity relates to her sensibility in combining information and imagination when making purposeful decisions regarding meaning and pleasure. This is a use of the term *aesthetic*[2] drawn from interpretations such as Dewey's (1934).

The goals of this article are situated within a larger research project aimed at motivating student learning through manipulation of aesthetic potentials in the mathematics classroom. Here I draw heavily on prior analytic and empirical research of mathematical activity carried out using Toulmin's (1971) interdependency methodology[3] (for more details, see Sinclair, 2002b). That research was pragmatic in nature and aimed at mining connections between the distant but causally-linked worlds of the professional mathematician and the classroom learner.

[2]I distinguish aesthetics as a field of study from "the aesthetic" as a theme in human experience. A compelling account of the latter is found in Dewey (1934), whereas the former also includes the nature of perceptually interesting aspects of phenomena—including, but not limited to, artifacts. By using the singular form "the aesthetic," I do not intend to imply that aesthetic views are consensual across time and cultures—as I will make clear throughout the article.

[3]Toulmin (1971) used this methodological approach to study psychological development. It contrasts both with some researchers' strictly analytical approach and Piaget's strictly empirical one. The interdependency methodology acknowledges the need for a cross-fertilization—a dialectical succession—of conceptual insights and empirical knowledge when trying to grasp the true nature and complexity of constructs related to cognition and understanding. Thus, I relied on empirical discoveries to improve and refine my initial conceptual analysis, which in turn, led to improved explanatory categories and further empirical questions.

Although establishing these lateral connections—in this case, within a contemporary North American milieu—illuminates an important axis of the mathematical aesthetic, other studies are needed to delineate the sociocultural factors determining or influencing the aesthetic responses of these parties (the professional mathematician, the classroom student). In this work, I defer the sociocultural analysis in favor of a preliminary cartography of the contemporary mathematics environment. In other words, in this work, I am less interested in how the mathematical aesthetic comes to constitute itself historically than in how, at present, it deploys itself across the spectrum of mathematical endeavor.

This work also strives to reveal some of the values and emotions underlying aesthetic behaviors in mathematical inquiry, thereby forging links with the developing literature on the affective issues in mathematics learning.

Recognition of the beauty of mathematics (and claims about it being the purest form) is almost as old as the discipline itself. The Ancient Greeks, particularly the Pythagoreans, believed in an affinity between mathematics and beauty, as described by Aristotle "the mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful" (XIII, 3.107b). Many eminent mathematicians have since echoed his words. For instance, Russell (1917) wrote that mathematics possesses a "supreme beauty…capable of a stern perfection such as only the greatest art can show" (p. 57). Hardy's (1940) sentiments showed slightly more restraint in pointing out that not all mathematics has rights to aesthetic claims:

> The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics. (p. 85)

Despite the recurrent themes about elegance, harmony, and order, encountered in the discourse, we also find a diversity of opinion about the nature of the mathematical aesthetic. Russell emphasized an essentialist perspective by portraying the aesthetic as belonging to or existing in the mathematical object alone. His perspective closely resembles the traditional conception of aesthetics found in the domains of philosophy and art criticism (e.g., Bell, 1914/1992). A contrasting subjectivist position held, for example, by mathematician Gian-Carlo Rota (1997), saw the aesthetic existing in the perceiver of the mathematical object. A third possibility is the contextualist position, acknowledged by von Neumann (1956), which saw the aesthetic existing in a particular historical, social, or cultural context. In fact, D'Ambrosio (1997) and Eglash (1999) have reminded us that the high degree of consensus in aesthetic judgments stems in part from the domination of Western mathematics, which despite more recent research in the field of ethnomathematics, remains the cultural standard of rationality. Mathematics grows out of the specificities of our natural and cultural environments; it is an intellectual discipline with a history and, like other disciplines, it embodies myths. It is natural then, that mathematical developments in other cultures follow different tracks of intellectual inquiry, hold different visions of the self, and different sets of values. These different styles, forms, and modes of thought will result in different aesthetic values and judgments.[4]

In my analysis, I aim for an initial structuring of the diversity of aesthetic responses found in Western mathematics by gathering the various interpretations and experiences of the aesthetic—as presented by mathematicians themselves—under a more unified whole, focusing on their role in the process of mathematical creation. Thus, I have identified three groups of aesthetic responses, which play three distinct roles in mathematical inquiry. The most recognized and public of the three roles of the aesthetic is the *evaluative*; it concerns the aesthetic nature of mathematical entities and is involved in judgments about the beauty, elegance, and significance of entities such as proofs and theorems. The *generative* role of the aesthetic is a guiding one and involves nonpropositional, modes of reasoning used in the process of inquiry. I use the term *generative* because it is described as being responsible for generating new ideas and insights that could not be derived by logical steps alone (e.g., Poincaré, 1908/1956). Lastly, the *motivational* role refers to the aesthetic responses that attract mathematicians to certain problems and even to certain fields of mathematics. A number of mathematicians have readily acknowledged the importance of the evaluative role of the aesthetic, which operates on finished, public work. However, these mathematicians are somewhat less inclined to try to explain the more private, evolving facets of their work where the generative and motivational roles operate. Educators have tended to follow suit, considering the possibilities of student aesthetic response in the evaluative mode almost exclusively.

These three types of aesthetic responses capture the range of ways in which mathematicians have described the aesthetic dimension of their practices while suggesting the roles they might play in creating mathematics. As I will show, they are also useful for probing mathematicians' values and beliefs about mathematics and thus revealing aspects of the mathematical emotional orientation (Drodge & Reid, 2000), which in turn serves to connect the affective, cognitive, and aesthetic

[4]A too-brief review of the historical roots of early mathematical activity in India, China, and the Islamic region suggests a provocative mix of ubiquitous aesthetic values, as well as idiosyncratic ones (c.f. Joseph, 1992). On the one hand, time and time again within these different cultures, there is evidence of mathematicians seeking the more simple and revelatory solutions or proofs. On the other, there is evidence of distinct pervading aesthetic preferences, such as exactness in the medieval Islamic tradition, purity in the Ancient Greek tradition and balance in the Chinese mathematics. Although the preferences might differ, they all operate as criteria with which these mathematicians make judgments about their results.

dimensions of mathematics. As such, in illustrating each role of the aesthetic in mathematical inquiry, I also attempt to identify the emotions, attitudes, beliefs, and values—some of the elements of the affective domain (Goldin, 2000)—that are intertwined with aesthetic responses.

The three roles of the aesthetic in mathematical inquiry that I have identified have their theoretical basis in Dewey's (1938) theory of inquiry and are also consistent with Polanyi's (1958) analysis of personal knowledge in scientific research. The three roles occur primarily within the process of inquiry, rather than during other activities that mathematicians undertake, such as reviewing articles, presenting at conferences, or reading mathematical texts. Therefore, for the time being, this research may only be able to inform the research on the mathematical learning activities of students that are directly related to inquiry—such as investigation, problem posing and problem solving.

THE EVALUATIVE ROLE OF THE AESTHETIC

Hundreds of thousands of theorems are proved each year. Those that are ultimately proven may all be true, but they are not all worthy of making it into the growing, recognized body of mathematical knowledge—the mathematical canon. Given that truth cannot possibly act as a final arbitrator of worth, how do mathematicians select which theorems become a part of the body of mathematical knowledge—which get printed in journals, books, or presented at conferences, and which are deemed worthy of being further developed and fortified?

Tymoczko (1993) pointed out that the selection is not arbitrary and, therefore, must be based on aesthetic criteria. (I would add that the selection must also be based on factors such as career orientation, funding, and social pressure.) In fact, he argued that aesthetic criteria are necessary for grounding value judgments in mathematics (such as importance and relevance) for two reasons. First, as I have mentioned, selection is essential in a world of infinite true theorems; and second, mathematical reality cannot provide its own criteria; that is, a mathematical result cannot be judged important because it matches some supposed mathematical reality—mathematics is not self-organized. In fact, it is only in relation to actual mathematicians with actual interests and values that mathematical reality is divided up into the trivial and the important. The recent possibilities afforded by computer-based technology can help one appreciate the importance of the aesthetic dimension in mathematical inquiry: Although a computer might be able to create a proof or verify a proof, it cannot decide which of these conjectures are worthwhile and significant.

In contrast, mathematicians are constantly deciding what to prove, why to prove it, and whether it is a proof at all; they cannot avoid being guided by criteria of an aesthetic nature that transcend logic alone.[5] Many mathematicians have recognized this and even privilege the role of the aesthetic in judging the value of mathematical entities. If mathematicians appeal to the aesthetic when judging the value of other's work, they also do so when deciding how to express and communicate their own work. When solving a problem, a mathematician must still arrange and present it to the community, and aesthetic concerns—among others—can come into play at this point too. In the following section, I provide illustrative evidence of both functions of the evaluative role of the aesthetic.

The Aesthetic Dimension of Mathematical Value Judgments

Many have tried to formulate a list of criteria that can be used to determine the aesthetic value of mathematical entities such as proofs and theorems (Birkhoff, 1956; Hardy, 1940; King, 1992). These attempts implicitly assume that mathematicians all agree on their aesthetic judgments. Although mathematicians show remarkable convergence on their judgments, especially in contrast to artists or musicians, Wells' (1990) survey shows that the universality assumption is somewhat misguided (this survey, printed in *The Mathematical Intelligencer*, asks mathematicians to rate the beauty of 24 different mathematical proofs). Certainly, many mathematicians value efficiency, perspicuity, and subtlety, yet there are many other aesthetic qualities that can affect a mathematician's judgment of a result—qualities which may be at odds with efficiency or cleverness, and which may ignore generality and significance. Separately, Burton (1999a) has emphasized that some mathematicians prefer proofs and theorems that are connected to other problems and theorems, or to other domains of mathematics. Silver and Metzger (1989) also reported that some mathematicians prefer solutions that stay within the same domain as the problem.

The evaluative aesthetic is not only involved in judging the great theorems of the past, or existing mathematical entities, but is actively involved in mathematicians' decisions about expressing and communicating their own work. As Krull (1987) wrote: "mathematicians are not concerned merely with finding and proving theorems; they also want to arrange and assemble the theorems so that they appear not only correct but evident and compelling" (p. 49). The continued attempts to devise proofs for the irrationality of $\sqrt{2}$—the most recent one by Apostol (2000)—were illustrative of mathematicians' desired to solve

[5]I do not imply, as Poincaré (1913) did, that any nonlogical mode of reasoning is automatically aesthetic. Papert (1978) used the useful term *extralogical* to refer to the matrix of intuitive, aesthetic, and nonpropositional modes that can be contrasted with the logical. The extralogical clearly includes more than the aesthetic but, in using the term, he acknowledged the difficulty one has in teasing these modes of human reasoning apart.

problems in increasingly pleasing ways: no one doubts the truth of existing ones!

Several aesthetic qualities I identified in the previous section are operative at this expressive stage of the mathematician's inquiry as well. For instance, although some mathematicians may provide the genesis of a result, as well as logical and intuitive substantiation, others prefer to offer a pure or minimal presentation of only the logically formed results, only the elements needed to reveal the structure. Mathematician Philip Davis (1997) thought that the most pleasing proofs are ones that are transparent. He wrote:

> I wanted to append to the figure a few lines, so ingeniously placed that the whole matter would be exposed to the naked eye. I wanted to be able to say not *quod erat demonstrandum*, as did the ancient Greek mathematicians, but simply, 'Lo and behold! The matter is as plain as the nose on your face.' (p. 17)

The aesthetic seems to have a dual role. First, it mediates a shared set of values amongst mathematicians about which results are important enough to be retained and fortified. Although Hardy's criteria of depth and generality, which might be more easily agreed on, are pivotal, the more purely aesthetic criteria (unexpectedness, inevitability, economy) certainly play a role in determining value. For example, most mathematicians agree that the Riemann Hypothesis is a significant problem—perhaps because it is so intertwined with other results or perhaps because it is somewhat surprising—but its solution (if and when it comes) will not necessarily be considered beautiful. That judgment will depend on many things, including the knowledge and experience of the mathematician in question, such as whether it illuminates any of the connections mathematicians have identified or whether it renders them too obvious.[6]

Second, the aesthetic determines the personal decisions that a mathematician makes about which results are meaningful, that is, which meet the specific qualities of mathematical ideas that the mathematician values and seeks. The work of Le Lionnais (1948/1986) helped illustrate this. Although mathematicians tend to focus on solutions and proofs when discussing the aesthetic, Le Lionnais drew attention to the many other mathematical entities that deserve aesthetic consideration and to the range of possible responses they evoke. Those attracted to magic squares and proofs by recurrence may be yearning for the equilibrium, harmony, and order. In contrast, those attracted to imaginary numbers and *reductio ad absurdum* proofs may be yearning for lack of balance, disorder, and pathology.[7]

[6]In the past, mathematicians have called Euler's equation ($e^{i\pi} + 1 = 0$) one of the most beautiful in mathematics, but many now think it is too obvious to be called beautiful (Wells, 1990).

[7]Krull (1987) suggested a very similar contrast. He saw mathematicians with concrete inclinations as being attracted to "diversity, variegation, and the like" (p. 52). On the other hand, those with an abstract orientation prefer "simplicity, clarity, and great 'line'" (p. 52).

In contrast to Hardy, Le Lionnais allowed for degrees of appreciation according to personal preference. His treatment of the mathematical aesthetic highlights the emotional component of aesthetic responses. He also enlarged the sphere of mathematical entities that can have aesthetic appeal, including not only entities such as definitions and images that can be appreciated after-the-fact, but also the various methods used in mathematics that can be appreciated while doing mathematics. I will return to this process-oriented role of the aesthetic.

Students' Use of the Evaluative Aesthetic

Researchers have found that, in general, students of mathematics neither share nor recognize the aesthetic value of mathematical entities that professional mathematicians claim (Dreyfus & Eisenberg, 1986; Silver & Metzger, 1989). However, Brown (1973) provided a glimpse of yet other possible forms of appreciation that students might have, which mathematicians rarely address; moreover, he did not wrongly equate the lack of agreement between students' and mathematicians' aesthetic responses with students' lack of aesthetic sensibility.

Brown described what might be called a *naturalistic* conception of beauty manifest in the work of his graduate students. He recounts showing them Gauss' supposed encounter with the famous arithmetic sum: $1 + 2 + 3 + \ldots + 99 + 100$ which the young Gauss cleverly calculated as $101 \times 100/2$. Brown asked them to discuss their own approaches in terms of aesthetic appeal. Surprisingly, many of his students preferred the rather messy, difficult-to-remember formulations over Gauss' neat and simple one. Brown conjectured that the messy formulations better encapsulated the students' personal history with the problem as well as its genealogy, and that the students wanted to remember the struggle more than the neat end product. Brown's observation highlighted how the contrasting goals, partly culturally imposed, of the mathematician and the student lead to different aesthetic criteria. He rooted aesthetic response in some specific human desire or need, thereby moving into a more psychological domain of explanation, and highlighting the affective component of aesthetic response.

In contrast to Dreyfus and Eisenberg, who wanted to initiate students into an established system of mathematical aesthetics, Brown pointed to the possibility of instead nurturing students' development of aesthetic preferences according to the animating purposes of aesthetic evaluation. Accordingly, the starting point should not be to train students to adopt aesthetic judgments that are in agreement with experts,' but rather to provide them with opportunities in which they want to—and can—engage in personal and social negotiations of the worth of a particular idea.

Probing the Affective Domain

The aesthetic preferences previously articulated are by no means exhaustive. However, they do provide a sense of the various responses that mathematicians might have to mathematical entities, and the role aesthetic judgments have in establishing the personal meaning—whether it is memorable, or significant, or worth passing on—an entity might have for a mathematician. Such responses might also provoke further work: How many mathematicians will now try to find a proof for Fermat's Last Theorem that has more clarity or more simplicity? These preferences also allow some probing of mathematicians' underlying affective structures. In terms of the aesthetic dimension of mathematical value judgments, the emphasis placed on the aesthetic qualities of a result implies a belief that mathematics is not just about a search for truth, but also a search for beauty and elegance. Differing preferences might also indicate certain value systems. For instance, the more romantically inclined mathematician has a different emotional orientation toward mathematics than the classically inclined one, valuing the bizarre and the pathological instead of the ordered and the simplified. In terms of attitudes, when Tymoczko (1993) undertook an aesthetic reading of the Fundamental Theorem of Arithmetic, he was exemplifying an attitude of being willing to experience tension, difficulty, rhythm, and insight. He was also exemplifying an attitude of faith; he trusted that his work in reading the theorem would lead to satisfactory results; that is, that he will learn and appreciate.

Similarly, when mathematicians engage in aesthetic judgment, they are allowing themselves to experience and evaluate emotions that might be evoked such as surprise, wonder, or perhaps repulsion. They acknowledge that such responses belong to mathematics and complement the more formal, propositional modes of reasoning usually associated with mathematics. One of the respondents to Wells' (1990) survey illustrated the importance that emotions play in his judgment of mathematical theorems: "…I tried to remember the feelings I had when I first heard of it" (p. 39). This respondent's aesthetic response is more closely tied to his personal relationship to the theorem than with the theorem itself as a mathematical entity. It is not the passive, detached judgment of significance or goodness that Hardy or King might make; rather it is an active, lived experience geared toward meaning and pleasure.

In terms of the aesthetic dimension of expression and communication, when mathematicians are guided by aesthetic criteria as they arrange and present their results, they are manifesting a belief, once again, that mathematics does not just present true and correct results. Rather, mathematics can tell a good story, one that may evoke feelings such as insight or surprise in the reader by appealing to some of the narrative strategems found in good literature. This belief is also evident in the effort mathematicians spend on finding better proofs for results that are already known, or on discussing and sharing elegant and beautiful theorems or problems.

The links I have identified between the affective and aesthetic domains reveal some of the beliefs and values of mathematicians that are, along with their knowledge and experience, central to their successes at learning and doing mathematics—and thus of interest to mathematics educators. I now turn to the generative role of the aesthetic, which in terms of the logic of inquiry, precedes the evaluative.

THE GENERATIVE ROLE OF THE AESTHETIC

The generative role of the aesthetic may be the most difficult of the three roles to discuss explicitly, operating as it most often does at a tacit or even subconscious level, and intertwined as it frequently is with intuitive modes. The generative aesthetic operates in the actual process of inquiry, in the discovery and invention of solutions or ideas; it guides the actions and choices that mathematicians make as they try to make sense of objects and relations.

Background on the Generative Aesthetic

Poincaré (1908/1956) was one of the first mathematicians to draw attention to the aesthetic dimension of mathematical invention and creation. He sees the aesthetic playing a major role in the subconscious operations of a mathematician's mind, and argues that the distinguishing feature of the mathematical mind is not the logical but the aesthetic. According to Poincaré, two operations take place in mathematical invention: the construction of possible combinations of ideas and the selection of the fruitful ones. Thus, to invent is to choose useful combinations from the numerous ones available; these are precisely the most beautiful, those best able to "charm this special sensibility that all mathematicians know" (p. 2048). Poincaré believed that such combinations of ideas are harmoniously disposed so that the mind can effortlessly embrace their totality without realising their details. It is this harmony that at once satisfies the mind's aesthetic sensibilities and acts as an aid to the mind, sustaining and guiding. This may sound a bit far-fetched, but there seems to be some scientific basis for it. The neuroscientist Damasio (1994) pointed out that because humans are not parallel processors, they must somehow filter the multitudes of stimuli incoming from the environment: some kind of preselection is carried out, whether covertly or not.

Examples of the Generative Aesthetic

Some concrete examples might help illustrate Poincaré's claims. Silver and Metzger (1989) reported on a mathematician's attempts to solve a number theory problem (Prove that there are no prime numbers in the infinite sequence of integers 10001, 100010001, 1000100010001, …). In working through the problem, the subject hits on a certain prime factorization, namely 137×73, that he described as "wonderful with those patterns" (p. 67). Something about the symmetry of the factors appeals to the mathematician, and leads him to believe that they might lead down a generative path. Based on their observations, Silver and Metzger also argued that aesthetic monitoring is not strictly cognitive, but appears to have a strong affective component: "decisions or evaluations based on aesthetic considerations are often made because the problem solver 'feels' he or she should do so because he or she is satisfied or dissatisfied with a method or result" (p. 70).

Papert (1978) provided yet another example. He ask a group of nonmathematicians to consider the theorem that $\sqrt{2}$ is an irrational number, and presents them with the initial statement of the proof: the claim to be rejected that $\sqrt{2} = p/q$. He then asked the participants to generate transformations of this equation, giving them no indication of what direction to take, or what the goal may be. After having generated a half dozen equations, the participants hit on the equation $p^2 = 2q^2$, at which point Papert reported that they show unmistakable signs of excitement and pleasure at having generated this equation.

Although this is indeed the next step in the proof of the theorem, Papert (1978) claimed that the participants are not consciously aware of where this equation will eventually lead. Therefore, although pleasure is often experienced when one achieves a desired solution, Papert argued that the pleasure in this case is of a more aesthetic rather than functional nature. Furthermore, the reaction of the participants is more than affective because the participants scarcely consider the other equations, having somehow identified the equation $p^2 = 2q^2$ as the interesting one. Papert conjectured that eliminating the (ugly?) square root sign from the initial equation might have caused their pleasurable charge.

The first example illustrated how an aesthetic response to a certain configuration is generative in that it leads the mathematician down a certain path of inquiry, not for logical reasons but rather, because the mathematician feels that the appealing configuration should reveal some insight or some fact. The second example suggests the range of stimuli that can potentially trigger aesthetic responses; a quality such as symmetry might be expected to do so, but more subtle qualities such as the prettiness of an equation or the sudden emergence of a new quantity are also candidates. Both examples illustrate how mathematicians must believe in and trust their feelings to exploit the generative aesthetic. They must view mathematics as a domain of inquiry where phenomena such as feelings play an important role alongside hard work and logical reasoning.

Evoking the Generative Aesthetic

I could provide many more examples of aesthetic responses during inquiry that lead mathematicians to make certain decisions about generative paths or ideas. However, with educational concerns in mind, it is important to learn how such responses can be nurtured and evoked. Some might occur spontaneously, as Papert (1978) maintained, whereas others may take years of experience and acculturation, as Poincaré (1956) argued. However, there are also some special strategies that mathematicians use during the course of inquiry that seem to be oriented toward triggering the generative aesthetic. I have identified three such strategies: playing, establishing intimacy, and capitalizing on intuition.

The phase of playing is aesthetic insofar as the mathematician is framing an area of exploration, qualitatively trying to fit things together, and seeking patterns that connect or integrate. Rather than being engaged in a strictly goal-oriented behavior, ends and means get reversed so that a whole cluster of playthings are created. Featherstone (2000) called this mathematical play, drawing on Huizinga's (1950) theory of play, which consisted in the free, orderly, aesthetic exploration of a situation. In seeing play this way, Huizinga called attention to the possibility that the mathematician, freed from having to solve a specific problem using the analytical apparatus of her craft, can focus on looking for appealing structures, patterns and combinations of ideas.

Mathematicians also seem to develop a personal, intimate relationship with the objects they work with, as can be evidenced by the way they anthropomorphize them, or coin special names for them in an attempt to hold them, to own them. Naming these objects makes them easier to refer to and may even foreshadow its properties. Equally as important, it gives the mathematician some traction on the still-vague territory, some way of marking what she does understand. The mathematician Wiener (1956) did not underestimate these attempts to operate with vague ideas; he recognized the mathematician's

> power to operate with temporary emotional symbols and to organize out of them a semipermanent, recallable language. If one is not able to do this, one is likely to find that his ideas evaporate from the sheer difficulty of preserving them in an as yet unformulated shape. (p. 86)

The final category of the generative aesthetic relates to working with intuition. Many mathematicians talk about their best ideas as being based not on reasoning but on the particular kind of insight called *intuition* (Burton, 1999b). With intuition, the mathematician is able to perceive the properties of a structure that, at the time, may not be possible to deduce.

What are the types of things that look or feel right to the mathematician? Very generally, they are things that have some aesthetic import. For instance, Hofstadter (1992/1997) sensed the rightness of particular a relationship when he noticed that it produced parallel lines—had the lines been oblique, he would have skipped right over them. He also felt that a simple analogy in symbolic form, although meaningless to him geometrically, must be right—such a thing cannot just be an accident! That looking right is an elusive notion, one that stumps mathematicians who try to describe or explain it. Is there a perceptible harmony in terms of proportion or symmetry? Are there simply some inexpressible or tacit conceptions that have finally found a formulation?

The first option is interesting because it is somewhat self-fulfilling. The mathematician perceives and searches with some sense of order, and then is surprised by her own mind when she eventually found manifestations of her sense of order—

symmetry, balance, rhythm, order, simplicity. Mathematicians seem to like to perpetuate the notion of a magical intuition that leads to a sudden discovery. It lends credence to the mathematician's belief of glimpsing the truth or, for some, seeing the divine—such things can only be magic. The mathematician André Weil (1992) described the experience, also hinting at its other-worldliness, as "the state of lucid exaltation in which one thought succeeds another as if miraculously, and in which the unconscious seems to play a role" (p. 27). He goes on to explain why mathematicians seek this experience: "unlike sexual pleasure, this feeling may last for hours at a time, even for days. Once you have experienced it, you are eager to repeat [...]" (p. 27).

The second option emphasizes the aesthetic sensitivities that contribute to the mathematician's sense-making. In contemplating, experimenting with, playing with the elements of a situation, the mathematician is gaining a feel for patterns and potential patterns. Hofstadter (1992/1997) described the sudden insight, the aesthetic moment, as being when inner images and external impressions converge; it is "the concrete realisation of the abstract analogy—a lovely idea, irresistible to me" (p. 7). The mathematician may feel that she is bringing something beautiful but unfinished to its inevitable completion, to closure. In retrospect, she might appreciate the growth of her own grappling: She might be surprised (and thankful) that she pursued a certain path or she might realize how she wrongfully dismissed something as meaningless along the way. This appreciation alerts her to the mysteries of her own mathematical thinking process, mysteries that in many ways parallel in depth the mysteries she encounters in mathematics.

Probing the Affective Domain

Throughout this section, I have made explicit the intimate connection between the affective and aesthetic domains—to a much greater degree than in the evaluative section. Therefore, I will only highlight a few points here. First, because of the more subtle nature of the generative aesthetic, the reliance on emotions and feelings is much stronger. Mathematicians are alerted to their aesthetic responses through affective responses. A mathematician who ignores feelings of pleasure or repulsion or believes such feelings to be misleading may not be able to access aesthetically driven insights or hypotheses.

Second, the strategies that mathematicians use to evoke the generative aesthetic imply a belief that mathematics can be a pleasurable, intrinsically satisfying experience. They also imply that mathematicians value mathematics for the moments of intuition or insight it can provide. The first example I presented exemplifies the qualities mathematicians may value such as symmetry, order, closure, and the belief that such structures are indicative of truth. Drodge and Reid (2000) saw this as part of the mathematical emotional orientation—a belief that such structures are significant. These strategies also reveal a certain attitude that mathematicians have toward doing mathematics, accepting that problem solving will require some potentially frustrating, non-goal-oriented behavior. There will be times of play and many detours that must taken for progress to be made.

I turn now to the final role, one that has drawn much less attention but that, given its positioning in the process of doing mathematics, is the necessary precursor to mathematical inquiry.

THE MOTIVATIONAL ROLE OF THE AESTHETIC

Hadamard (1945), von Neumann (1956), and Penrose (1974) have all argued that the motivations for doing mathematics, as Penrose stated, "turn out to be ultimately aesthetic ones" (p. 266). Tymoczko (1993) argued that there is a logical imperative for the motivational role of the aesthetic. A mathematician has a great variety of fields to choose from, differing from one another widely in character, style, aims, and influence; and within each field, a great variety of problems and phenomena. Thus, mathematicians must select in terms of the research they pursue, the classes they teach, and the canon they pass on. Although there are some mathematical problems that are more famous, and even more fashionable, it would be difficult to argue that there is an objective perspective—a mathematical reality against which the value of mathematical products can be measured.[8] Contrast this with physics, for example, another discipline that makes strong aesthetic claims (see Farmelo, 2002), where questions and products can be measured up against physical reality: How well they explain the shape of the universe or the behavior of light.

One aesthetic criterion used by mathematicians is that of potential, as described by Hadamard (1945). He claimed that an aesthetic response to a problem can inform the mathematician of the fruitfulness of a future result, "without knowing any further, we *feel* that such a direction of investigation is worth following; we feel that the question in *itself* deserves interest" (p. 127). Penrose suggested another criterion, that of visual appeal, which motivated him to study the strange symmetries in irregular tilings. He emphasized the effect of the attraction: "when one is fascinated [by a problem], the internal aesthetics of the thing will drive it along" (p. 270). Visual appeal seems to be an increasingly available criterion for mathematicians; the computer-generated images that are now being produced have bewitched many—as Mumford et al. (2002) admitted in their recent and colorful book *Indra's Pearls*.

In analyzing scientific inquiry in general, including mathematics, Polanyi (1958) argued that the scientist's sense of intellectual beauty serves a crucial selec-

[8]Indeed, many would argue that a mathematical reality does not exist. But even if it did—independently from actual mathematicians—it does not announce which among its abundance of objects is trivial, which among the connections is important. As Tymoczko (1993) has also argued, mathematical reality would be too rich and too mute.

tive function: "intellectual passions have an affirmative content; in science they affirm the scientific interest and value of certain facts, as against any lack of interest and value in others" (p. 159). Moreover, Polanyi affirmed that the motivational aesthetic plays the psychological role that Penrose (1974) previously mentioned, "Intellectual passions do not merely affirm the existence of harmonies which foreshadow an indeterminate range of future discoveries, but can evoke intimations of specific discoveries and sustain their persistent pursuit through years of labour" (p. 143).

Although previously the mathematicians argued that aesthetic motivation is necessary for mathematical inquiry, they provided few examples of the types of aesthetic responses that might be motivational. In the following section, I provide categories of responses that are mentioned by mathematicians. These categories not only help elucidate the nature of the motivational aesthetic, I believe they might also help educators identify means through which the aesthetic could motivate student inquiry.

Categories of Aesthetic Motivation

Mathematicians can be attracted by the visual appeal of certain mathematical entities, by perceived aesthetic attributes such as simplicity and order, or by some sense of fit that applies to a whole structure. Penrose (1974) was aesthetically motivated by the visual complexity of nonperiodic tilings, but because so much of mathematics seems inaccessible to the senses, visual appeal is necessarily limited. Davis (1997) has provided a specific example. He described being hooked by the unexpected order emerging from an irregular triangle in Napoleon's theorem, and spending years of his life trying to figure out why it occurs. Apparent simplicity is another frequent occurrences of appeal and is exemplified by Gleason (in Albers, Alexanderson, & Reid, 1990), "I am gripped by explicit, easily stated things … I'm very fond of problems in which somehow an at least very simple sounding hypothesis is sufficient to really pinch something together and make something out of it" (p. 93).

A sense of surprise and paradox can also aesthetically motivate a mathematician. For example, the paradox of the hat problem recently intrigued and attracted many mathematicians across North America (Robinson, 2001). Surprise constantly arises in mathematics as mathematicians find things they have no reason to expect: a pattern emerging in a sequence of numbers; a point of coincidence found in a group of lines; a large change resulting from a small variation; a finite real thing proved by an infinite, possibly unreal object. Movshovits-Hadar (1988) revealed the motivational power of surprise by showing how the feeling of surprise stimulates curiosity that can, in small steps, lead toward intelligibility. In fact, Peirce (1908/1960) insisted that inquiry always begins with "some surprising phenomenon, some experience which either disappoints an expectation, or breaks in upon some habit of expectation of the *inquisiturus*" (p. 6.469).

Surprise makes the mathematician struggle with her expectations and with the limitations of her knowledge and, thus, with her intuitive, informal, and formal understandings. Gosper (in Albers et al., 1990) expressed surprise at the way continued fractions allow you to "see" what a real number is: "it's completely astounding [...] it looks like you are cheating God somehow" (p. 112). This sense of surprise has motivated him to do extensive work with continued fractions. Of course, to respond to surprise, one must have some kind of frame of reference that generates expectations, so that what surprises one mathematician may not surprise another.

The work on foundations of mathematics is a good example of a quintessentially nonsurprising problem. In fact, Krull (1987) suggested that those attracted to the study of foundations (investigating, e.g., the extent to which the set of all infinite decimals can be considered a logically faultless concept) are the least aesthetically oriented mathematicians because they are "concerned above all with the irrefutable certainty" (p. 50) of their results. On the other hand, Krull claimed that "the more aesthetically oriented mathematicians will have less interest in the study of foundations, with its painstaking and often necessarily complicated and unattractive investigations" (p. 50). Krull quite clearly situated himself in the latter camp but is perhaps too quick to judge the foundations mathematicians as nonaesthetically oriented. The inclination toward finding basic, underlying order is certainly also an aesthetic one—although different in kind from the inclination to surprise.

There is also social dimension to aesthetic motivation. Thurston (1995) agreed with Penrose (1974), Hadamard (1945), and von Neumann (1956) on the necessary aesthetic dimension to a mathematician's choice of field and problems, but he added another dimension rarely discussed: "social setting is also important. We are inspired by other people, we seek appreciation by other people, and we like to help other people solve their mathematical problems" (p. 34). This observation provides some indication of how mathematicians' aesthetic choices might be at least partially learned from their community as they interact with other mathematicians and seek their approval. In fact, there is certainly a long process of acculturation that begins with the mathematician learning that words like 'beautiful'—use more naturally for describing, say, rainbows and paintings—are even appropriate to use in mathematics. The findings of both Dreyfus and Eisenberg (1986) and Silver and Metzger (1989) suggested that this process only gains momentum in and after graduate school, when young mathematicians are having to join the community of professional mathematicians—and when aesthetic considerations are recognized (unlike at high school and undergraduate levels).

Of course, not all of the social interactions among mathematicians have an aesthetic dimension. The case of John Nash exemplifies a nonaesthetic social motivation. His biographer Sylvia Nasar (1998) described how he would only work on a problem once he had ascertained that great mathematicians thought it highly important—pestering them for affirmation. The promise of recognition, rather than the intrinsic appeal of the problem or situation, was the motivating factor.

The previous evidence presented suggests that aesthetic contributes to determining what will be personally interesting enough to propel weeks and maybe even years of hard work. In addition, Polanyi (1958) insisted that the various ways in which mathematicians become attracted to mathematical situations and problems do not only serve an affective motivational purpose. Rather, the attraction has a heuristic function by influencing the discernment of features in a situation, and thereby directing the thought patterns of the inquirer. According to Dewey's theory of inquiry (1938), the heuristic function of the aesthetic depends on the inquirer's qualitative apprehensions, and operates on vague suggestions of relations and distinctions rather than on firm propositional knowledge. These apprehensions give rise to and feed the foreground of qualitative thought, which is concerned with ideas, concepts, categories, and formal logic. They give what Dewey called a *qualitative unity* to objects or phenomena that are externally disparate and dissimilar. Such unity allows human beings to abstract discernible patterns in conjunction with the ideas and concepts at hand.

The theories of both Polanyi (1958) and Dewey (1938) suggested that the motivational aesthetic does not operate merely as an eye-catching device, neither does it provide merely the psychological support needed to struggle through a problem. Rather, it is central to the very process that enables the mathematician to deliberately produce qualitatively derived hypotheses: It initiates an action-guiding hypothesis. With the help of an example, I will attempt to illustrate this more cognitively significant motivational aesthetic.

An Example of Aesthetic Motivation in Mathematical Inquiry

Introspective analyses, which help shed light on the anatomy of mathematical discovery, are rare in the professional mathematics literature. Fortunately however, Hofstadter (1992/1997) provided a detailed description of his own geometric discovery that reveals the way in which the aesthetic draws him into a mathematical situation. The goal of my own analysis of his description is to show that the motivational aesthetic does not merely catch the mathematician's attention; rather, it serves the necessary role of framing the very way problems and initial conjectures are identified. The relevance to student motivation and interest is obvious here, particularly in the context of guided inquiry and problem posing.

The selective function of aesthetic motivation. Hofstadter (1992/1997) began by describing his moment of infatuation with geometry—"plane old Euclidean geometry"—when he tries to prove a "simple fact about circles." This proof concerns the special points of a triangle and the centers of various circles associated with them, such as the circumcenter, the orthocenter and the incenter. He is drawn into this world of triangles for several reasons. The first is a sense of surprise at how circles and triangles, which on the surface seem quite different, are so intimately connected. He wrote that he "came to love triangles, circles, and their unexpectedly profound interrelations," through the special points of a triangle (p. 1). The proof was more specifically about the Euler segment which, as Hofstadter discovered, connects four out of the five "most special of special points," and represents some kind of essential characteristic of its originating triangle. It is this fifth "neglected" special point, the "forgotten" incenter, that drew Hofstadter further in; his sense of balance and mathematical equity is betrayed by the fact that the incenter is "left out of the party." If the incenter was indeed a special point, then it also should have "its own coterie of special friends," just as the other special points did. He thus becomes intrigued by the triangle and its special points through his belief that there should be more symmetrical, inclusive relationships among the special points.

A third motivating factor is Hofstadter's (1992/1997) identification with the problem. He described his appreciation for the "metaphorical connection between my love for their [the triangles'] special points and a mathematical love that I had had from childhood: the love for *special points on the number line*, of which the quintessential examples are π and *e*" (p. 2). Hofstadter recalled his wondrous response to Euler's equation, which showed the "secret links" between some of those special points on the number line. And he is further enthused by the existence of an analogy between Euler's equation and the Euler segment—both of which relate four important mathematical objects in "a most astonishing way." Thus, Hofstadter became further "hooked" by the Euler segment because of this "emotional, somewhat irrational" connection between his previous interests and his current findings.

One final aesthetic dimension of Hofstadter's (1992/1997) immersion into triangle geometry stems from his keenness of analogical forms. I have already mentioned the analogy he perceived between the Euler segment and Euler's equation. Yet as he continued studying the Euler segment, he discovered two special circles: the 9-point circle and the Spieker circle. These two circles turn out to have some of "the most remarkable and complex mathematical analogies" (p. 5). Hofstadter claimed to ever have seen. As a scholar who has long been interested in analogy in human cognition, he is not only particularly attuned to things analogous, he also wants to and enjoys finding analogies—they are aesthetically pleasing to him. The wealth of analogies evident in his initiation into triangle geometry satisfies a strong aesthetic preference in him that motivates further inquiry.

This example underscores the temporal nature of a mathematician's aesthetic response. Anticipating educational concerns, it might be tempting to motivate students' mathematical activity by finding a hook using an aesthetic response such as surprise. However, Hofstadter (1992/1997) did not immediately see the plentiful analogies and the connection to his previous interests; that required a bit of playing around first. Very early on he perceived the surprising relationships between circles and triangles, which seemed to be enough to impel him to continue exploring. He was not looking for anything in particular—there was not a specific problem to solve—but was instead just looking around, noticing things, and playing. Thus he developed more layers of attraction. The motivational aesthetic operates and develops over time.

I previously argued that the motivational aesthetic is central to problem selection because the mathematician must select, on bases that cannot be entirely logical, areas of mathematics to pursue. Hofstadter's initial response to the unexpected quality he perceived between triangles and circles and his ensuing exploration indicate that the aesthetic quality can attract the mathematician's attention, and in so doing, direct it toward particular phenomena and relationships. This suggests that the motivational aesthetic can influence what the mathematician will find and create.

The heuristic function of aesthetic motivation. Recall that Hofstadter (1992/1997) was unhappy about the status of the incenter. His surprise at the fact that it was not as popular as the other triangle centers involved in the Euler segment prompted him to buy "all the relevant books" he could find. He finally hit on Coolidge's 1916 treatise that mentions a second, anonymous segment involving the incircle, the Nagel point and the centroid. Not only was this segment reminiscent of the Euler segment, Hofstadter saw that it was "deeply *analogous* to it." He called it the *Nagel segment*. His excitement only increased when he also discovered, reading a little further on, a set of analogies in the correspondances between the properties of the nine-point and Spieker circles (the 9-point circle involves the points of the Euler segment; the Spieker circle involves the incenter and the Nagel point). Hofstadter happily noted that the Spieker circle "restored the honor of the incenter" (p. 5) and made the Nagel point—"another special point that I already knew and loved"—more important as well. At this point, Hofstadter wondered why the Nagel segment is so unknown, especially if it was so deeply connected to the Euler segment: "Why does it not have a name? Why is it routinely not mentioned?" he asked. Then he said, "In mathematics, such a striking and intricate analogy can't just happen by *accident*!" This leads to his most important question: "why are the parallels so perfect and systematic?"

With this question in mind, he began to explore with *The Geometer's Sketchpad* by constructing a triangle and the two segments. His first heuristic, not surprisingly, was to look for more analogous properties of the segments. He found one almost right away, an analogy based on parallelism—"a lovely idea, irresistible to me"—but it turns out to be "a triviality," a consequence of the similar triangles formed by the two segments. His investigation continued, through a cycle of false leads and promising discoveries, making speculations based on potential analogies (he identified 10 of them on p. 34) and trying to find something "meaningful" or "significant" about the Nagel segment. He ended up finding another special segment related to the Nagel and Euler segments, naming the triangle composed of the three segments the "hemiolic crystal," thereby identifying even more special points, and creating a family of iterative equivalences between a triangle, its hemiolic crystal, and its auxiliary hemiolic crystal.

Hofstadter's (1992/1997) process of inquiry, at least the first part that I have described so far, illustrated the way in which the motivational aesthetic shapes inquiry. Hofstadter had a different opinion about the relative attention that should be accorded to these triangle centers, and it is this sense of imbalance that provides a qualitative unity to the mathematical situation. It has the capacity of bringing together many of the concepts involved under one metric. Yet the metric is not propositionally defined. There is no objective measure that describes how important one centre is over another—Hofstadter was not seeking one anyway—but there is some qualitative sense, which might include the number of relationships or analogies involved, of their relative importance. This unity allowed Hofstadter to abstract discernible patterns in the various points, segments, and circles involved with a triangle. It led him to notice and look for certain things; when reading through Coolidge's text, which is full of theorems and properties of triangle geometry, he found that the incenter is part of an unnamed segment and that it is related to the Nagel point, another forgotten object. The unity thus regulates the selection and weighing of the many facts he reads and observes. Had the imbalance of the incenter not troubled him, he would have attended to other ideas. His desire to seek more balance among the triangle's special points even shapes many of his interpretations as he tries to decide whether a new point or a new segment that he discovers is meaningful or not. It thus underlies all the details of his further explicit reasoning about the situation.

Intertwined with Hofstadter's (1992/1997) concern about the incenter is his investment in the meaningfulness of analogy. As he read, he was also selective toward analogy; he happened on the analogous properties of the 9-point and Spieker circles. He also noticed that analogies between the Euler and Nagel segments. Analogy is an intellectual passion for Hofstadter that has a selective function. He chose a situation that was full of analogies; he formulated a question about the reason for the relationship between the two segments based on his commitment to the idea that analogies are not accidental. It also has a heuristic function that becomes evident as he continues working, trying to find more analogies, speculating about properties based on analogy, and believing in discoveries because of analogy. Polanyi (1958) described well the self-fulfilling heuristic function of aesthetic value in inquiry: "The appreciation for scientific value merges here into a capacity for discovering it; even as the artist's sensibility merges into his creative powers" (p. 143).

Probing the Affective Domain

The motivational aesthetic, as illustrated in my analysis of Hofstadter's (1992/1997) discovery, is closely intertwined with effect. Hofstadter allowed many emotions to be evoked including tension, curiosity, bewilderment, as well as frustration and loss. The latter two emotions cannot be ignored for they certainly contribute to his ultimate excitement and satisfaction. In fact, Hofstadter's acceptance of these emotions revealed some of the attitudes that he had. He is inclined to try to resolve tension and curiosity; he accepts frustration as part of the process of resolution. Hofstadter's belief that he is able to resolve tensions and work through frustrations was also evident. Finally, he seemed to value the qualities of experience that result from resolving his initial feelings of tension and engaging in a process of inquiry. Hofstadter also valued mechanisms that provide structure to mathematical relationships, such as analogy.

Returning to the previous section where I expanded on several categories of the motivational aesthetic, two other affective connections can be made. First, the motivating factor of identification reveals a strong sense of "mathematical self-identity" (Goldin, 1999), a spectrum of related affect and cognition that allows the mathematician to establish herself in relation to mathematics. Second, the belief that mathematics is a social phenomenon, where value is negotiated through social interaction, is certainly at the root of the social dimension of aesthetic motivation emphasizes.

The importance that mathematicians place on surprise and paradox reveals the sense in which mathematicians look for and value that which will provide a new way of seeing and understanding. After all, surprises and paradoxes offer the greatest potential for that perceptual shift that can result in illumination or transformation.

CONCLUDING REMARKS

In some ways, it would seem absurd that students would find the same things appealing as mathematicians do—surely the mathematician's appreciation of structure, closure and order is a function of their mathematical knowledge, experiences and social acculturation. Yet even if the paradoxical hat problem is not appealing to a student, nor the order displayed in the Napoleon theorem configuration, nor even the symmetry of the Dirac equations, is there some way in which the qualities themselves—paradox, order, symmetry—can be accessible and valued by students in other mathematical contexts? For example, might students respond aesthetically to Zeno's paradox, or perhaps the way in which division by nine produces such surprising, patterned results, or that symmetry can be used to quickly add all the numbers between 1 and 100? That is, are there some common responses that engage both the professional mathematician and the student in inquiry? The prevalent experience of teachers that students are amazed and intrigued by such ideas suggests there are stimuli that commonly trigger aesthetic response. I have suggested that such responses can play an important role in motivating and sustaining inquiry.

Students may, in fact, share some aesthetic tendencies with mathematicians, but may not know how to use them in the context of mathematical inquiry. The emphasis that school mathematics places on propositional, logical reasoning might actually discourage students from recognizing and trusting the generative type of aesthetic responses that operate in inquiry. To nurture intuition, Burton (1999b) advocated providing students with prompts such as "ponderings, what ifs, it seems to be thats, and it feels as thoughs" (p. 30). She argued that such prompts explicitly invite feelings into the process of problem solving, and they can also release students from narrow foci to more global, qualitative framings. In contrast with Polyà's (1957) problem-solving heuristics, these processes specifically draw the inquirer back from the mechanics of the problem (the unknowns, the data, and the conditions) to the qualitative relations perceivable by the inquirer. Such processes might therefore be able to evoke more aesthetic modes of thinking.

I believe that further research into the aesthetic possibilities of mathematics education is both warranted and desirable. Of course, many challenges remain. One is to design and study mathematical situations that can evoke, nurture, and develop aesthetic engagement in students (see Sinclair, 2001, for such an attempt). Another is to examine the way in which the aesthetic operates in non-inquiry-based learning environments. Dewey (1934) argued that the aesthetic is a pervasive theme in experience that is intimately connected with cognitive growth. However, the nature of this connection, in the specific context of mathematics learning, needs further elucidation.

This article has attempted to make sense of the most tacit dimensions of human knowing and experience in mathematics. I hope it makes an initial step toward the recognition and celebration of the fundamentally aesthetic structure of mathematical inquiry, and, as a result, the possibilities for growth and learning that such inquiries can present for mathematicians and students of mathematics alike.

REFERENCES

Albers, D., Alexanderson, G., & Reid, C. (1990). *More mathematical people*. Orlando, FL: Harcourt Brace.

Apostol, T. (2000). Irrationality of the square root of two—A geometric proof. *American Mathematical Monthly, November*, 241–242.

Aristotle. *Metaphysics*. Retrieved October 22, 2000, from http://classics.mit.edu/Aristotle/metaphysics.1.i.html

Bell, C. (1992). The aesthetic hypothesis. In C. Harrison & P. Wood (Eds.), *Art in theory 1990–1990: An anthology of changing ideas* (pp. 113–116). Oxford, UK: Blackwell. (Original work published 1914)

Birkhoff, G. (1956). Mathematics of aesthetics. In J. Newman (Ed.), *The world of mathematics* (Vol. 4, pp. 2185–2197). New York: Simon & Schuster.

Borel, A. (1983). Mathematics: Art and science. *The Mathematical Intelligencer, 5*(4), 9–17.

Brown, S. (1973). Mathematics and humanistic themes: Sum considerations. *Educational Theory, 23*(3), 191–214.

Burton, L. (1999a). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics, 37*(2), 121–143.

Burton, L. (1999b). Why is intuition so important to mathematicians but missing from mathematics education? *For the Learning of Mathematics, 19*(3), 27–32.

Damasio, A. (1994). *Descartes' error: Emotion, reason, and the human brain*. New York: Avon Books.

D'Ambrosio, U. (1997). Where does ethnomathematics stand nowadays? *For the Learning of Mathematics, 17*(2), 13–17.

Davis, P. (1997). *Mathematical encounters of the 2nd kind*. Boston: Birkhäuser.

Dewey, J. (1934). *Art as experience*. New York: Perigree.

Dewey, J. (1938). *Logic: The theory of inquiry*. New York: Holt, Rinehart and Winston.

Dreyfus, T., & Eisenberg, T. (1986). On the aesthetics of mathematical thought. *For the Learning of Mathematics, 6*(1), 2–10.

Dreyfus, T., & Eisenberg, T. (1996). On different facets of mathematical thinking. In R. J. Sternberg & T. Ben-Zeev (Eds.), *The nature of mathematical thinking* (pp. 253–284). Mahwah, NJ: Lawrence Erlbaum Associates. Inc.

Drodge, E., & Reid, D. (2000). Embodied cognition and mathematical emotional orientation. *Mathematical Thinking and Learning, 2*, 249–267.

Eglash, R. (1999). *African fractals: Modern computing and indigenous design*. Piscatawy, NJ: Rutgers University Press.

Farmelo, G. (2002). *It must be beautiful: Great equations of modern science*. London: Granta Books.

Featherstone, H. (2000). "-Pat + Pat = 0": Intellectual play in elementary mathematics. *For the Learning of Mathematics, 20*(2), 14–23.

Goldenberg, P. (1989). Seeing beauty in mathematics: Using fractal geometry to build a spirit of mathematical inquiry. *Journal of Mathematical Behavior, 8*, 169–204.

Goldin, G. (1999). Affect, meta-affect, and mathematical belief structures. In E. Pehkonen & G. Törner (Eds.), *Proceedings of the Mathematical Beliefs and their Impact on the Teaching and Learning of Mathematics Conference*. Overwolfach, Germany: Gerhard-Mercator Universität.

Goldin, G. (2000). Affective pathways and representation in mathematical problem solving. *Mathematical Thinking and Learning, 2*, 209–219.

Hadamard, J. (1945). *The mathematician's mind: The psychology of invention in the mathematical field*. Princeton, NJ: Princeton University Press.

Hardy, G. H. (1940). *A mathematician's apology*. Cambridge, UK: Cambridge University Press.

Higginson, W. (2000). Amusing about aesthetics and mathematics. In J. McLoughlin (Ed.), *Proceedings of the 2000 annual meeting. Canadian Mathematics Education Study Group, Topic Group A*. St. John's, Newfoundland: Memorial University of Newfoundland.

Hofstadter, D. (1992). From Euler to Ulam: Discovery and dissection of a geometric gem. (Preprint version available from author. An abbreviated version of this paper was published in *Geometry turned on: Dynamic software in learning, teaching, and research*, pp. 3–14, by J. King & D. Schattschneider, Eds., 1997, Washington, DC: MAA)

Huizinga, J. (1950). *Homo ludens: A study of the play element in culture*. London: Routledge & Kegan Paul.

Joseph, G. (1992). *The crest of the peacock: Non-European roots of mathematics*. London: Penguin.

King, J. (1992). *The art of mathematics*. New York: Plenum.

Krull, W. (1987). The aesthetic viewpoint in mathematics. *The Mathematical Intelligencer, 9*(1), 48–52.

Krutetskii, V. (1976). *The psychology of mathematical abilities in schoolchildren* (J. Teller, Trans.). Chicago: University of Chicago Press. (Original work published 1968)

Le Lionnais, F. (1948/1986). La beauté en mathématiques. In F. Le Lionnais (Ed.), *Les grands courants de la pensée mathématique* (pp. 437–465). Paris: Editions Rivages.

Movshovits-Hadar, N. (1988). School mathematics theorems—An endless source of surprise. *For the Learning of Mathematics, 8*(3), 34–39.
Mumford, D., Series, C., & Wright, D. (2002). *Indra's pearls: The vision of Felix Klein*. Cambridge, England: Cambridge University Press.
Nasar, S. (1998). *A beautiful mind: A biography of John Forbes Nash, Jr.* New York: Simon & Schuster.
Papert, S. (1978).The mathematical unconscious. In J. Wechsler (Ed.), *On aesthetics and science*. (pp. 105–120). Boston: Birkhäuser.
Peirce, C. S. (1908/1960). A neglected argument for the reality of God. In C. Hartshorne & P. Weiss (Eds.), *Collected papers of Charles Sanders Peirce* (Vol. 6: Scientific metaphysics). Cambridge, MA: Harvard University Press.
Penrose, R. (1974). The role of aesthetic in pure and applied mathematical research. *The Institute of Mathematics and its Applications, 7/8*(10), 266–271.
Poincaré, H. (1913). *The world foundations of science* (G. B. Halsted, Trans.). New York: The Science Press.
Poincaré, H. (1956). Mathematical creation. In J. Newman (Ed.), *The world of mathematics* (pp. 2041–2050). New York: Simon & Schuster. (Original work published in 1908)
Polanyi, M. (1958). *Personal knowledge: Towards a post-critical philosophy.* New York: Harper & Row.
Polyá, G. (1957). *How to solve it.* Garden City, NY: Doubleday.
Robinson, S. (2001, April 10). Why mathematicians now care about their hat color. *The New York Times*, p. F5.
Rota, G. (1997). *Indiscrete thoughts.* Boston: Birkhauser.
Russell, B. (1917). *Mysticism and logic*. New York: Doubleday.
Sierpsinka, A. (2002). Reflections on education studies in mathematics. *Educational Studies in Mathematics, 50,* 251–257.
Silver, E., & Metzger, W. (1989). Aesthetic influences on expert mathematical problem solving. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving* (pp. 59–74). New York: Springer-Verlag.
Sinclair, N. (2001). The aesthetic *is* relevant. *For the Learning of Mathematics, 21*(1), 25–33.
Sinclair, N. (2002a). The kissing triangles: The aesthetics of mathematical discovery. *The International Journal of Computers for Mathematics Learning, 7*(1), 45–63.
Sinclair, N. (2002b). *Mindful of beauty: The roles of the aesthetic in the learning and doing of mathematics*. Unpublished doctoral dissertation. Queen's University, Canada.
Thurston, W. (1995). On proof and progress in mathematics. *For the Learning of Mathematics, 15*(1), 29–37.
Toulmin, S. (1971). 'The concept of stages' in psychological development. In T. Mischel (Ed.)*, Cognitive development and epistemology* (pp. 25–60). New York: Academic.
Tymoczko, T. (1993). Value judgements in mathematics: Can we treat mathematics as an art? In A. M. White (Ed.), *Essays in humanistic mathematics* (pp. 67–77). Washington, DC: MAA.
von Glasersfeld, E. (1985). *Radical constructivism: A way of knowing and learning*. London: Falmer Press.
von Neumann, J. (1956). The mathematician. In J. Newman (Ed.), *The world of mathematics* (pp. 2053–2065). New York: Simon & Schuster.
Weil, A. (1992). *The apprenticeship of a mathematician* (J. Gage, Trans.). Berlin: Birkhäuser. (Original work published in 1991)
Wiener, N. (1956). *I am a mathematician: The later life of a prodigy.* Garden City, NY: Doubleday & Company.
Wells, D. (1990). Are these the most beautiful? *The Mathematical Intelligencer, 12*(3), 37–41.