The Geometer's Sketchpad: Non-Euclidean Geometry & The Poincaré Disk

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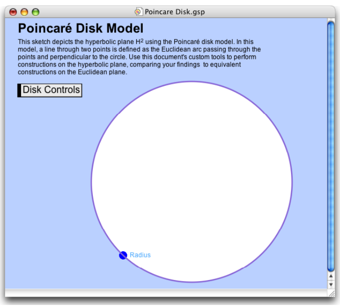
Overview. The study of hyperbolic geometry—and non-euclidean geometries in general—dates to the 19th century's failed attempts to prove that Euclid's fifth postulate (the *parallel* postulate) could be derived from the other four postulates. Lobachevsky, Bolyai, and Gauss all independently conceived a geometry in which the 5th postulate is "broken" by allowing *many* lines—rather than just one—to be defined as parallel to a given line through a point not on that line. The resulting *hyperbolic geometry* can be made particularly vivid by Henri Poincaré's remarkable disk model, which allows that geometry to be visualized—and, in Sketchpad, manipulated—within the Euclidean plane. In this session, we'll examine the implications of breaking the 5th postulate by constructing and exploring hyperbolic geometry, using Poincaré's disk model of the hyperbolic plane.

Contents

the Poincaré disk model & disk tools

In Poincaré's model, the hyperbolic plane is seen as the interior of a Euclidean circle. (The hyperbolic plane does not include this *fundamental circle* itself, only its interior.) Since you'll be fitting all of infinite hyperbolic space into a (relatively tiny!) single circle, expect to encounter distortion! Hyperbolic lines appear in the model as Euclidean arcs, running from one edge of the disk to another.



1. Choose **Sample Sketches & Tools** from the **Help** menu when connected to the internet. Then navigate to:

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Advanced Topics/
    Poincaré Disk Model of Hyperbolic Geometry
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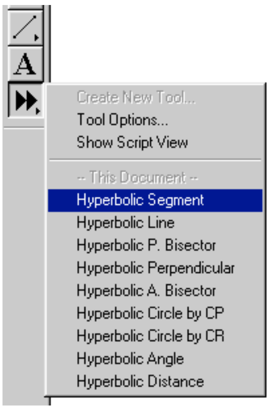
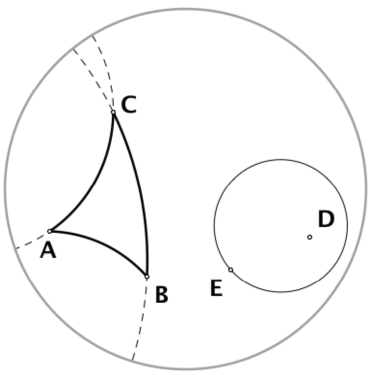


2. To work with Poincaré's model in Sketchpad (which knows nothing, by itself, of hyperbolic geometry), use the **custom tools** attached to this document. These are found in the menu attached to the bottom tool of the Toolbox. To use a Poincaré Disk tool, choose it from this menu, then click one or more times inside your sketch's Disk.

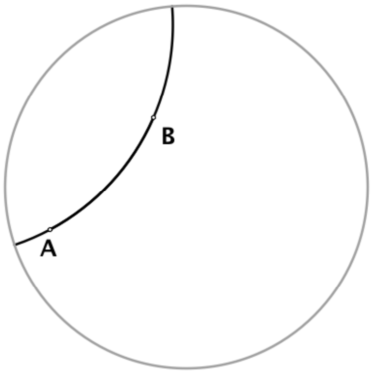
For example, to create a Hyperbolic Segment, choose the **Hyperbolic Segment** custom tool, and then click twice—or click and drag—in your disk to create the segment's two endpoints. (This is very similar to using Sketchpad's built-in segment tool, to construct a Euclidean segment.)



3. Experiment with each of the Poincaré disk tools to get a feel for how they work. If you have trouble understanding how a specific tool works, ask a neighbor. (Or choose **Show Script View** from the Custom Tools menu after activating the tool. A detailed description of the tool will appear, with comments about how to use it.)

To start fresh, close the document (don't save changes!) and reopen it a second time. Once you've explored the tools, try answering the questions on the next page.

lines and angles



Construct a hyperbolic line through *A* and *B*. Then drag *A* or *B*.

A1. What happens to the line *AB* when *A* or *B* leaves the Poincaré Disk?

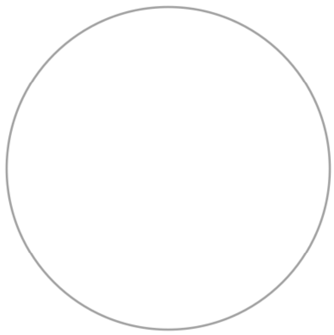
A2. What do you notice about how each "end" of line *AB* meets the fundamental circle surrounding the disk?

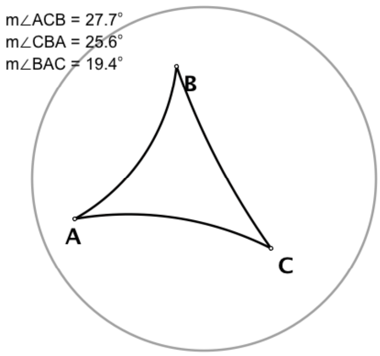
A3. What other special cases do you notice?

Remember: The curved appearance of lines in the Disk model are an effect of the model, not of hyperbolic geometry. In the hyperbolic plane, these lines are still "straight"—they only appear curved in this Euclidean model of that non-Euclidean place.

B. Recall that the 5th postulate of Euclidean geometry claims that through any point not on a given line, there is exactly one other line that is parallel to—i.e. that does not cross—the given line. Construct a demonstration that this postulate does not hold in the hyperbolic geometry of the Poincaré Disk.

(If handing this in, draw your construction in the Disk at right and write a statement describing how your construction contradicts the 5th postulate.)





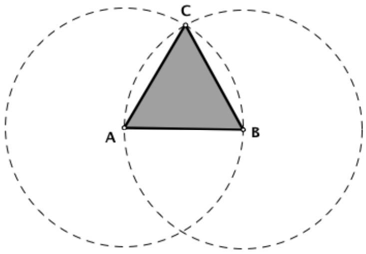
Construct a hyperbolic triangle ABC, and then use the Hyperbolic Angle tool to measure its angles. (Click three times to determine an angle.) Use **Measure | Calculate** to sum the three angles.

C1. As you drag A, B, and C, what do you notice about the angle sum?

C2. What is its largest apparent value? When does this happen? Can you find another triangle that produces this value?

C3. What is its smallest apparent value? When does this happen?

length, shape, congruence, and similarity on the Poincaré disk



Euclid's first proposition describes the construction of an equilateral triangle as shown to the left. Two circles *AB* and *BA* are constructed with equal radii (*AB* = *BA*). A point of their intersection *C* is clearly on both circles, meaning that *AC* = *AB* = *CB*. Equilateral means all sides equal, so QED.

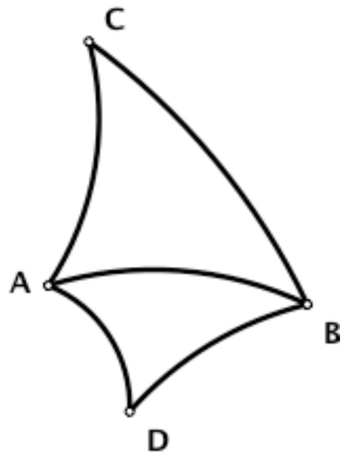
Now interpret this construction hyperbolically. Use the hyperbolic tools to perform the same construction and to produce a corresponding hyperbolic triangle on the Poincaré disk.

Once you've constructed your triangle, answer these questions *without* using hyperbolic measurement tools:

D1. Is your hyperbolic triangle equilateral? What is the largest possible length of any side? The smallest?

D2. Is your hyperbolic triangle equiangular? What is the largest possible interior angle at any vertex? The smallest?

D3. The second point of intersection of your two circles—let's say *D*—forms a second triangle whose base is *AB*. Construct this. What is the relationship between Δ*ABC* and Δ*ABD*?



If you were uncertain of any of your answers, explore and confirm them now using the Hyperbolic Length and Hyperbolic Angle tools to measure your triangles.

Start with a new Poincaré Disk. Can you construct…

E1. …a hyperbolic square?

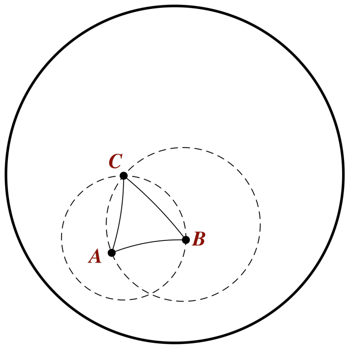
E2. …a hyperbolic triangle congruent to another arbitrary hyperbolic triangle?

E3. …a hyperbolic triangle similar but not congruent to another hyperbolic triangle?

regular tessellations on the Poincaré disk

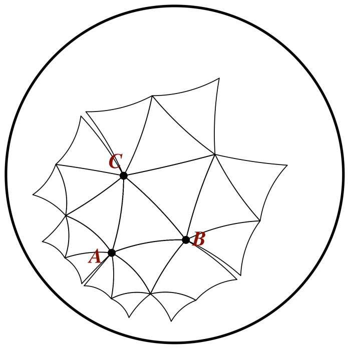
You've established that, in hyperbolic geometry, the angles of a regular polygon vary from some upper limit down to almost 0°, as a function of the polygon's size. In this activity, you'll exploit this property to find a size—and therefore, an angle—at which a regular triangle tessellates the hyperbolic plane.

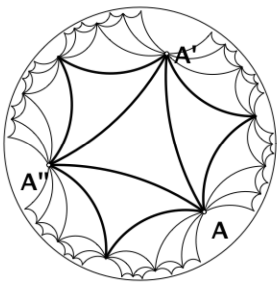
1. Construct a hyperbolic equilateral triangle. You can use the construction of Euclid's first proposition, since this does not depend on parallel lines. (Hide the construction circles when done.)



2. Now select the two defining vertices of your equilateral triangle—*A* and *B*—and choose **Transform | Iterate**.

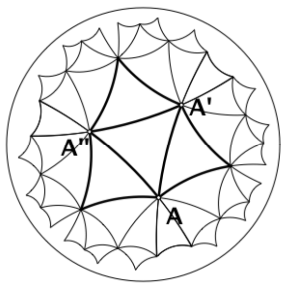
Map *AB* multiply to *BA*, *AC*, and *CB*, to recursively build similar equilateral triangles on the three edges of your original triangle (and, by recursion, on each of those, etc.).





4. Increase the depth of iteration once or twice by pressing + when the iterated triangles are selected. (Pressing – decreases the iteration depth.)

5. Finally, drag *A* to resize your family of triangles—bigger or smaller—until your reflected triangle tessellates precisely (or almost precisely).



F1. How many triangles meet at a vertex in your tessellation?

F2. Look for other tessellations by dragging *A*. What is the maximum number of triangles that can meet at a vertex? The minimum? Why?

F3. Imagine a hyperbolic tessellation $\{n, k\}$ of regular n-gons meeting k times at each vertex. What is the internal angle of each n-gon in this configuration?

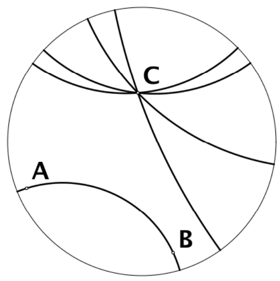
F4. For what integer values $k>2$ and $n>2$ is the hyperbolic tessellation $\{n, k\}$ possible?

teacher notes

The Poincaré disk offers an accessible and self-contained experience in non-Euclidean geometry for students at many levels. Where explorations in *spherical* geometry provide a non-Euclidean counter-point to traditional Euclidean investigations, the sphere remains a surface with which we are intuitively comfortable. The hyperbolic surface modeled by the Poincaré disk, on the other hand, is decidedly non-intuitive on first encounter, and so provides a perhaps bolder contrast to the Euclidean plane.

These activities introduce the model and tools for working on it (page 1), and then establish some of these non-intuitive hyperbolic properties (page 2 & 3), focusing on ones having counter-parts in Euclidean geometry that should be familiar to high school geometry students. Finally (page 4) these results are put to novel use, leading to the startling conclusion that any regular polygon can tessellate the hyperbolic plane in an infinite variety of tessellations. The first three pages should be accessible to students with very little experience with Sketchpad; the fourth page requires somewhat more Sketchpad experience as well as acquaintance with the idea of tessellation. This entire sequence may be enhanced by use of the **Full Poincaré.gsp** sketch as a whole-class overhead. That sketch contains some historical background, motivating detail, and clarifying examples relating to this sequence. If you don't have a copy, contact the author (njackiw@keypress.com). For additional background, there are many excellent introductions to hyperbolic geometry on the web. Visit the Math Forum (www.mathform.org) for an accessible starting point, or Math World (www.mathworld.com) for a more rigorous treatment.

specific activity notes



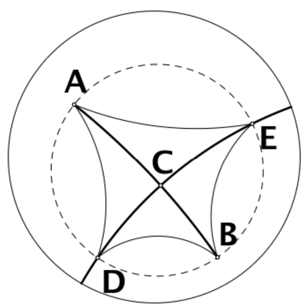
A. Emphasize that *only the interior* of the disk is considered "hyperbolic" in this model.

B. The picture at right demonstrates four distinct hyperbolic lines passing through *C*, all of which are *parallel* to the line *AB* (in the sense that they never intersect it).

C. Students will find here that the larger a triangle is in hyperbolic terms, the smaller its angles, and vice versa. The sum of the angles of a hyperbolic triangle will always fall between 0° and 180°, but not reach either limit. (At 0°, the triangle *could not* exist; at 180°, the triangle would be Euclidean, rather than hyperbolic.) It's interesting to note that in *spherical* geometry, the angle sum would be *larger* than 180°. Thus Euclidean geometry—with its perfect 180° triangles—is the border *between* hyperbolic (smaller angle sums) and spherical geometry (larger angle sums). A nice extension here is to revisit the Euclidean

proof that the angle sum is 180°, to point out how by requiring the identification of a unique parallel this proof depends on the 5th postulate.

D. Both triangles here are equilateral, equiangular, *and congruent*—regardless of how their lengths appear to be distorted by the disk model! Remember, they have been *constructed* as equilateral and congruent—so they are congruent by definition. (Also note that though lengths appear distorted by the disk, angles don't suffer similar distortion.)



E. Since you can't build a square by compass-and-straightedge without invoking the 5th postulate's unique parallel, you can't have a hyperbolic square (or, for that matter, any quadrilateral of four 90° angles). However, you *can* construct a regular quadrilateral, with equal side lengths and equal angles (though they'll be smaller than 90°) The figure at right shows one such construction. Use the **Hyperbolic Segment** tool to start with a segment *AB*. Then construct its perpendicular bisector (**Hyperbolic P. Bisector**), and their intersection *C*, which is the midpoint of *AB*. Then construct a circle with center *C* passing through point *A* (**Hyperbolic Circle by CP**). The resulting quadrilateral *AEBD*, with perpendicular equal diagonals, is equilateral and equiangular.

F. The tessellation activity requires more time and a greater degree either of previous Sketchpad experience or teacher support, since it assumes familiarity with Euclidean rotation (to generate the vertices of your central triangle) and reflection (as a technique for tessellating the plane starting from a given tile).

Before starting this activity, it may be worth reviewing Euclidean tessellation, by reflecting a (Euclidean) equilateral triangle over its own edges multiple times. The result is a tessellation of triangles (or 3-gons) meeting six times at each vertex. This would be called the Euclidean {3, 6} tessellation. What other Euclidean tessellations of regular polygons can students find? (There are only two others. Why?)

When you move to the hyperbolic plane, to perform hyperbolic reflections (step 2), you'll need a **Hyperbolic Point Reflection** tool, which is *not* part of the hyperbolic tools include in the Sketchpad sample file named **Poincare.gsp**. Such a tool is provided in the **Full Poincare.gsp** document instead, and you might want to have students switch to using that document before trying the tessellation activity.

(Where does that tool come from? Recall that in Euclidean geometry, you can construct the reflected image of a point *P* over a line *m* by extending the perpendicular to *m* through *P*, and then finding the point equal distance from *m* on the far side from *P*. Since this doesn't require parallels of any form, the same construction could be used in hyperbolic geometry, so the idea of point reflection is well-defined in the Poincaré disk. At a more advanced level, if you research how Poincaré's disk model is actually synthesized geometrically, you'll discover that the reflection, in the disk, of point *P* through hyperbolic line *m,* is the same as the Euclidean inversion of point *P* through the Euclidean circle of whose circumference *m* is part!)

As students explore their tessellating triangles, they should realize they can have four triangles meeting at a vertex, or any number more than five, if they have enough reflections ({3, 4}, {3, 5}, {3, 6}…). Though it may appear as if they can also have {3, 3} if their triangles are sufficiently small, this is visual approximation error. If three congruent triangles fit around a vertex, their external angles are each 120° (since 120° x 3 = 360°), so the triangles in {3, 3} have internal angles of 60° each. That would mean their angle sum is 180°—and that's a Euclidean triangle, not a hyperbolic triangle! This vividly demonstrates that a very small locality of the hyperbolic plane closely resembles the Euclidean plane. As hyperbolic triangles become larger, their deviation from Euclidean equivalents becomes more noticeable.

F3. To fit k times around a vertex (that is, around 360°), a regular polygon would have to have angles each of which was $360°/k$.

F4. Since a regular n-gon has n equal angles, each of which in a tessellation is $360°/k$, the sum of the angles of the n-gon is $n(360°/k)$. A Euclidean convex n-gon has an interior angle sum of $180°(n-2)$, so our hyperbolic n-gon must have a smaller sum. Therefore $n(360°/k) < 180°(n-2)$. Thus (expand and divide), a hyperbolic tessellation $\{n, k\}$ is possible only if $(2/k) + (2/n) < 1$.