Research Statement

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1 Introduction

My primary field of research is algebraic geometry. I am especially interested in questions regarding Fano varieties and deformation theory. Pervasive in much of my research is the exploitation of combinatorial structure in order to attack geometric problems. This combinatorial structure often arises due to a group action, in particular, the action of an algebraic torus.

The remainder of this introduction gives a brief overview of the three thematic areas into which my research falls. Subsequent sections describe current and proposed research in these areas in more detail.

- **Torus Actions and Combinatorics**: A toric variety is an algebraic variety endowed with the action of an algebraic torus having a dense open orbit. Such varieties are used in many areas of mathematics such as geometric modelling, mirror symmetry, algebraic statistics, and coding theory. In addition, they provide a useful class of examples in algebraic geometry. One attractive feature is that these varieties may be described completely in combinatorial terms. Altmann and Hausen began generalizing this combinatorial approach to varieties with lower-dimensional torus actions, that is, $T$-varieties. A portion of my research has consisted of developing this approach further, ranging from topics such as torus equivariant vector bundles and polarized $T$-varieties, to applications of $T$-varieties in coding theory.

- **Moduli Spaces and Deformation Theory**: Objects in algebraic geometry often arise in families; in many situations there exists a geometric object called a moduli space whose points correspond to these objects. The local study of such moduli spaces is deformation theory. My research addresses a number of moduli and deformation problems, including the deformation theory of toric varieties, and moduli of linear spaces of singular matrices.

- **Fano Varieties and Mirror Symmetry**: Fano varieties, that is, projective varieties with ample anticanonical bundle, appear in many areas of algebraic geometry such as the minimal model program, homogeneous spaces, and mirror symmetry. It is an important open problem to classify all deformation families of smooth Fano varieties of a fixed dimension; this was completed by del Pezzo in dimension two, and Iskovskikh and Mori-Mukai in dimension three. Some of my current and planned research projects approach this problem in higher dimensions from the perspective of mirror symmetry, toric geometry, and combinatorics.

These three thematic areas cover a broad swath of algebraic geometry and touch on many fundamental problems. In my research, a wide range of methods are applied, including tools from toric geometry, combinatorics, mirror symmetry, and computer algebra.

2 Torus Actions and Combinatorics

Recall that a toric variety is a normal variety endowed with a faithful action by an algebraic torus having a dense orbit. Such varieties are easily approachable: a toric variety may be encoded by a
collection of rational polyhedral cones called a fan. Many aspects of the geometry of a toric variety can be read off directly from this combinatorial data, including its cohomology, information on line bundles, and behaviour of singularities.

A broader class of varieties is that of $T$-varieties: a $T$-variety is a normal variety endowed with a faithful action by an algebraic torus $T$. Its complexity is the codimension of a general orbit of $T$ in $X$; complexity zero $T$-varieties are hence the same thing as toric varieties. $T$-varieties of higher complexity arise naturally as total spaces of toric vector bundles, universal torsors over Mori Dream Spaces, and homogeneous spaces $G/P$. Combinatorics still plays an important role in this situation. Given some $T$-variety $X$ of complexity $k$, the basic idea is to split the information describing $X$ into a $k$-dimensional variety $Y$ along with some combinatorial data. The variety $Y$ is a sort of quotient of $X$ by $T$, and the additional combinatorial data encodes the information lost by passing from $X$ to $Y$, see e.g. [AIP+12].

My research projects described below use combinatorics to gain insight into the structure and geometry of $T$-varieties.

2.1 Torus Equivariant Vector Bundles

Equivariant vector bundles on toric varieties have been described in combinatorial terms by Klyachko [Kly89]. Let $X$ be a toric variety corresponding to a fan $\Sigma$. Then the category of torus equivariant vector bundles on $X$ is equivalent to the category of finite dimensional vector spaces $E$ endowed with filtrations $E^\rho(i)$ for each ray $\rho$ of $\Sigma$, subject to a compatibility condition among the filtrations. Along with R. Birkner (Berlin) and L. Petersen (Frankfurt), I have written a computer package ToricVectorBundles [BIP10] for the computer algebra system Macaulay2 which implements this description of toric vector bundles. In particular, the package facilitates efficient computation of the graded cohomology groups of toric vector bundles.

In joint work with H. Süß (Edinburgh), I have generalized Klyachko’s equivalence of categories to torus equivariant vector bundles on $T$-varieties of higher complexity [IS13]. Let $X$ be a factorial $T$-variety over an algebraically closed field, $W \subset X$ the open subset with finite stabilizers, and $Z = [W/T]$ the stack quotient.

**Theorem 2.1** ([IS13]). There is a natural equivalence of categories between $T$-equivariant vector bundles on $X$, and the category of vector bundles $E$ on $Z$ endowed with filtrations $E^D(i)$ for every prime divisor $D \subset X$ not meeting $W$, subject to a compatibility condition among the filtrations.

This equivalence of categories can also be applied to give an explicit formula for the space of global vector fields on rational complexity-one $T$-varieties. Furthermore, it can be used to show that low rank equivariant vector bundles on projective space split as a sum of line bundles:

**Theorem 2.2** ([IS13]). Let $1 \leq m \leq n$ and consider the action of $T = (\mathbb{K}^*)^m$ on $\mathbb{P}^n$ by multiplication of the first $m$ homogeneous coordinates. Then any $T$-equivariant vector bundle $V$ on $\mathbb{P}^n$ splits as a direct sum of line bundles if $\text{rank } V < \min\{n, m + 3\}$.

In [Pay08], Payne utilized Klyachko’s description of toric vector bundles to construct the moduli space of equivariant vector bundles on a toric variety. In future research, I aim to extend this construction to more general $T$-varieties.
2.2 Other Related Research

There is a well-known correspondence between lattice polytopes, and projective polarized toric varieties. H. Süss and I generalize this correspondence to the setting of projective polarized complexity-one $T$-varieties [IS11]. Indeed, we show that such $T$-varieties correspond to so-called divisorial polytopes: convex piecewise-linear maps $\Delta \rightarrow \text{Div}_Q Y$ from a lattice polytope $\Delta$ to the group $\text{Div}_Q Y$ of rational divisors on a curve $Y$, satisfying a certain integrality and positivity condition.

Consider any variety $X$ over $\mathbb{F}_q$. By evaluating sections of an $\mathbb{F}_q$-rational line bundle at some finite number of $\mathbb{F}_q$-rational points of $X$, one may construct an error correcting code, see e.g. [Gop81]. In [IS10], H. Süss and I study this setup when $X$ is a complexity-one $T$-variety. Using the combinatorics of $T$-varieties, we give nontrivial upper and lower bounds for the dimension and minimum distance of such codes. Furthermore, I have used this construction to find an example of a code over $\mathbb{F}_7$ with parameters better than were previously known for any linear code [Gra07].

In work with R. Vollmert (Berlin), I show how the combinatorial description of a $T$-variety changes when restricting or enlarging the torus which acts [IV13]. Besides giving useful insight into the underlying combinatorics, this can be applied to describe the Cox rings of certain rational $T$-varieties.

Together with H. Süss [IS11] and L. Kastner (Berlin) [IK13], I describe algorithms which reconstruct the coordinate ring of an affine $T$-variety from combinatorial data. These algorithms provide important tools for translating combinatorics and geometry into algebra.

3 Moduli Spaces and Deformation Theory

A set of guiding problems in algebraic geometry are classification problems, for example, the classification of all smooth projective curves, or classification of all smooth Fano varieties of dimension $d$. In many situations, there exists a scheme called a moduli space whose closed points correspond to the objects one wants to classify, and which satisfies some nice functorial properties.

Although explicit descriptions of moduli spaces are often difficult to attain, one might hope that, when studying geometric objects with lots of structure, this structure is reflected in the moduli space as well. A first step in understanding a moduli space is often to understand its formal local structure, where one can bring the tools of deformation theory to bear. Even when the functor at hand doesn’t admit a moduli scheme, one may often construct a formal local substitute called a versal deformation.

3.1 Deformation Theory of Toric Varieties

Toric varieties provide an important and accessible class of examples in algebraic geometry. It is thus natural to try to understand their deformation spaces. In [It09], I study the deformation theory of two-dimensional cyclic quotient singularities from the toric perspective. In particular, I give a necessary and sufficient criterion for when two one-parameter torus equivariant deformations may be combined into a family of surfaces with two-dimensional base.

The remainder of my research on toric deformation theory focuses on the non-affine case. In [It11], I describe combinatorially the first-order deformations of a smooth, complete toric variety...
X: the degree \(-u\) graded piece of \(T^1_X\) is isomorphic to

\[
T^1_X(-u) = \bigoplus_{\rho \in \Sigma(1) \atop (\rho, u) = 1} H^0(\Gamma_\rho(u), \mathbb{C})/\mathbb{C},
\]

where \(\Sigma\) is the fan of \(X\) and \(\Gamma_\rho(u)\) is a certain graph depending on the \(\Sigma\), \(\rho\), and \(u\) in a combinatorial fashion. Furthermore, if \(X\) is a surface, I show that \(T^2_X = 0\).

R. Vollmert and I investigate higher-order deformations of varieties with torus action in [IV12]. Let \(T\) be a torus, and \(X\) be a rational complexity-one \(T\)-variety. We describe a method for constructing deformations \(\pi : X \to S\) of \(X\) over an affine space \(S\) based on some input combinatorial data. The torus \(T\) acts on the total space \(X\), and this action preserves fibers of the map \(\pi\). In a related project with A. Hochenegger (Cologne) [HI13], I describe an explicit map comparing the Picard groups of the fibers for such a family.

The above construction may be used to describe homogeneous deformations of a toric variety \(X\). Indeed, if \(\chi^u\) is a character of the big torus acting on \(X\), let \(T\) be the kernel of \(\chi^u\). Then \(X\) may be viewed as a complexity-one rational \(T\)-variety, and \(T\)-equivariant deformations may be constructed combinatorially as described above.

**Theorem 3.1** ([IV12]). Let \(X\) be a smooth, complete toric variety, and \(u\) any multidegree. Then the homogeneous piece \(T^1_X(-u)\) of the space of first-order deformations is spanned by restricting to first order \(T\)-equivariant deformations constructed as above.

If \(T^1_X\) is concentrated in a single degree, then the above approach may be used to construct the versal deformation of \(X\). If \(T^1_X\) is not concentrated in a single multidegree, it is unknown what the versal deformation of \(X\) is. In future research, I aim to prove the following:

**Conjecture 1.** The versal base space for a smooth complete toric variety is smooth.

Although the obstruction space \(T^2_X\) of a smooth complete toric variety \(X\) in general does not vanish, ongoing research hints that the cup product map \(T^1_X \times T^1_X \to T^2_X\) does. This is a first sign that all deformations may in fact be unobstructed. If true, this would be similar to the situation of complex Calabi-Yau manifolds, which have unobstructed deformations despite a non-trivial obstruction space.

### 3.2 Fano Schemes of Determinants and Permanents

Let \(X \subset \mathbb{P}^n\) be a projective scheme. The **Fano scheme** of \(k\)-planes on \(X\) is the fine moduli space \(F_k(X)\) parametrizing \(k\)-dimensional linear subspaces of \(\mathbb{P}^n\) which are contained in \(X\). In an ongoing project [CI13], M. Chan (Harvard) and I study Fano schemes \(F_k(X)\) for a special class of varieties \(X\), namely, the varieties \(D_{m,n}\) and \(P_{m,n}\) cut out by the determinants or permanents, respectively, of the maximal square submatrices of a general \(m \times n\) matrix. This choice of varieties is motivated by Valiant’s conjecture [Val79], which postulates a fundamental difference in the complexity of computing determinants and permanents.

Chan and I determine many geometric properties of \(F_k(D_{m,n})\) and \(F_k(P_{m,n})\), including characterizing for which values of \(k, m, n\) these schemes are irreducible or smooth. More interestingly, we are able to characterize connectedness:

\(^1\)Recall that a **permanent** is just an unsigned determinant.
Theorem 3.2. For $0 \leq s \leq m - 1$, define $\kappa(s) = mn - (m - s)(s + n - m + 1) - 1$. Assume that the Fano scheme $F_k(D_{m,n})$ is non-empty. Then $F_k(D_{m,n})$ is disconnected if and only if either $\kappa(0) \geq k > m^2 - 2m$ or there exists $s \in \mathbb{N}$, $0 < s < m - 1$, such that $\kappa(s) \geq k > \kappa(s) - \min\{m - s - 1, n - m + s\}$.

A similar result holds for $F_k(P_{m,n})$. In the special case of $F_1(D_{m,m})$, we show that this scheme is a reduced local complete intersection; this fails for higher values of $k$. In future work, we aim to gain more explicit descriptions of $F_k(D_{m,n})$ and $F_k(P_{m,n})$ for low values of $m, n$. Likewise, we wish to show that also $F_1(P_{m,m})$ is a reduced local complete intersection.

3.3 Computational Tools

I have authored the Macaulay2 package VersalDeformations [It12b] which facilitates formal deformation-theoretic calculations, including the calculation of versal deformations and local (multi-graded) Hilbert schemes. In future work, I plan to expand the package’s functionality to include deformations over rings of mixed characteristic. This will give an algorithmic tool to construct explicit liftings of characteristic $p$ schemes to characteristic 0. In particular, I hope to construct examples of 0-dimensional schemes which do not lift to characteristic 0; currently, no such schemes are known to exist.

4 Fano Varieties and Mirror Symmetry

Recall that a complex variety $V$ is called a Fano variety if its anticanonical divisor $-K_V$ is ample. A fundamental result concerning smooth Fano varieties is that in any fixed dimension, there are only finitely many deformation families [KMM92]. This, in theory, makes it possible to classify all families of smooth Fanos in a given dimension. This classification is known in dimensions three and below (see e.g. [MM82]), but is wide open for higher dimensions. While the sheer number of families of Fano varieties makes a complete classification in higher dimension most likely unfeasible, it would still be highly desirable to have a rough idea of what Fano varieties exist. My research outlined below approaches this problem from a variety of angles, using techniques from combinatorics, toric and log geometry, and mirror symmetry.

4.1 Fano Varieties with Torus Action

Just as there are only finitely many deformation families of smooth Fano varieties in any given dimension, there are also only finitely many toric Gorenstein Fano varieties in any fixed dimension. Such varieties correspond to reflexive polytopes, for which a classification algorithm exists [KS97].

To see a larger class of Fano varieties, H. Süß and I propose to study Fano varieties with mild singularities admitting a complexity-one torus action. Since polarized complexity-one $T$-varieties can be understood combinatorially in terms of divisorial polytopes, see Section [222] this special class of Fano varieties is approachable via combinatorics. We conjecture that

Conjecture 2. In any fixed dimension, there are only finitely many equivariant isomorphism classes of smooth complexity-one Fano $T$-varieties which are not toric.

A proof of this conjecture would likely come with a classification algorithm, again giving a new method for producing new higher-dimensional Fano varieties.
4.2 Extremal Laurent Polynomials and Toric Degenerations

Recall that an extremal Laurent Polynomial for a Fano variety $V$ is a Laurent polynomial $f : (\mathbb{C}^*)^n \to \mathbb{A}^1$ whose constant term series $\sum_{k=0}^{\infty} (f^k)_{0t^k}$ is a solution of the regularized quantum $D$-module of $V$, cf. [ILP13, Section 1.2]. A new approach to classifying smooth Fano varieties suggests instead classifying extremal Laurent polynomials, and understanding a sort of duality between these polynomials and smooth Fanos $\text{ToricDeg}$ [Prz09]. Following ideas of Batyrev, this duality should be related to toric degenerations:

**Conjecture 3** (cf. [Prz09, Question 20]). Let $V$ be a smooth Fano variety and $\Delta$ a normal reflexive polytope. The following are equivalent:

1. There exists an extremal Laurent polynomial $f$ for $V$ with Newton polytope $\Delta$ and positive integral coefficients.
2. $-K_V$ is very ample and $V$ has an embedded degeneration to the normal toric variety with moment polytope $\Delta^*$.

J. Lewis (Vienna), V. Przyjalkowski (Moscow), and I have strengthened a result of Przyjalkowski to show:

**Theorem 4.1** ([ILP13]). Every smooth Fano threefold $V$ with Picard rank one has an extremal Laurent polynomial $f$ such that $V$ degenerates to the toric variety with moment polytope dual to the Newton polytope of $f$.

We also show in [ILP13] that any Fano complete intersection $V$ in a weighted projective space similarly has an extremal Laurent polynomial satisfying the same properties. Furthermore, together with J. Christophersen (Oslo) [CI12], I completely classify embedded degenerations of Fano threefolds of degree at most 12 to toric Gorenstein Fano varieties. Together with calculations of [CCG+11], this implies that for any embedded toric degeneration of a Fano threefold of degree at most twelve, there exists a corresponding extremal Laurent polynomial.

In [Ilt12a], I consider special birational transformations of Laurent polynomials called *mutations* which preserve the property of being an extremal Laurent polynomial.

**Theorem 4.2** ([Ilt12a]). Let $f'$ be a mutation of an extremal Laurent polynomial $f$, and denote the toric varieties with moment polytopes dual to the respective Newton polyhedra by $X_{f'}$ and $X_f$. Then there is a flat projective family over $\mathbb{P}^1$ with $X_f$ and $X_{f'}$ as special fibers.

This fits together nicely with Conjecture 3 since, if true, it would imply that if $f$ and $f'$ are extremal Laurent polynomials for the same Fano variety, then $X_f$ and $X_{f'}$ appear as fibers in a flat projective family over some irreducible variety.

In upcoming research with G. Musiker (Minnesota), I plan to relate mutations of Laurent polynomials in two variables to perfect matchings of bipartite graphs on punctured Riemann surfaces. This work is inspired by considerations in supersymmetric gauge theory [HS12] and would give enumerative meaning to the constant terms of powers of extremal Laurent polynomials in two variables.

In planned research with H. Ruddat (Mainz), I aim to find sufficient criteria for the existence of a smoothing of a toric Fano variety, leading to a better understanding of Conjecture 3. A major obstacle in finding such a smoothing lies in patching local deformations together. We propose to control these patchings using logarithmic geometry and the machinery of Gross and Siebert [GST11].
References


