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Equivalent-expenditure functions and expenditure-dependent equivalence scales

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Abstract

This paper presents and investigates a new class of equivalent-expenditure functions that is a generalization of the one that corresponds to exact (independent-of-base) equivalence scales. It provides less restrictive household demands, especially for children's goods, and has associated equivalence scales that may depend on expenditure. We show that, under certain conditions, equivalent-expenditure functions and the associated expenditure-dependent equivalence scales can be uniquely estimated from demand data. We estimate them using Canadian data and find that the resulting scales are both plausible and expenditure dependent. The estimated equivalence scales for households with children decline significantly as expenditure increases.

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1. Introduction

Although economic data report the expenditures made by households, individual levels of an index of well-being (utility) must be known for social evaluation and

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the measurement of inequality and poverty. When making comparisons of economic well-being across households, it is important to take account of 'economies of scale' in the consumption of commodities such as housing, and special needs of some people such as those with disabilities.

Equivalence scales make use of a reference household type, usually a single adult, and are able to provide interhousehold comparisons of well-being. If, for example, a family of four has an expenditure level of \$42 000 and the scale for its type takes on the value 2.1 at the prices it faces, the household is equivalent, for welfare purposes, to four (reference) single adults who spend $\$42\,000/2.1 = \$20\,000$ each. In this example, \$20 000 is the equivalent expenditure for the household.

Equivalent expenditure for a household is the expenditure level which would make a reference single adult as well off as the members of the household, and it can be written as a function of prices, expenditure and household characteristics. The corresponding equivalence scale is actual expenditure divided by equivalent expenditure.

Equivalence scales can, in general, depend on prices, expenditure and household characteristics. Blackorby and Donaldson (1991, 1993) and Lewbel (1989) have investigated the case in which scales are independent of expenditure or, equivalently, utility and they say that such scales satisfy equivalence-scale exactness (ESE).¹ In that case, equivalent expenditure is proportional to household expenditure and the corresponding equivalent-expenditure function has an expenditure elasticity of one.

In this paper, we investigate equivalent-expenditure functions with expenditure elasticities that are independent of household expenditure but may depend on prices as well as household characteristics.² This generalization of ESE, which we call generalized equivalence-scale exactness (GESE), ameliorates some of the difficulties associated with ESE: (1) the equivalence scale can depend on expenditure so that its value can be different for rich and poor; (2) the restriction on preferences implied by ESE is weakened; and (3) the demand functions for children's goods may have expenditure elasticities that are different from one.

There are two arguments that suggest that equivalence scales should depend on total expenditure. First, because economies of household formation are associated with sharable commodities such as housing whose expenditure share decreases as total expenditure rises, it is reasonable to expect expenditure-dependent equivalence scales for multi-person households to increase with expenditure. Second, because the consumption of many luxuries, such as eating in good restaurants or

¹Lewbel (1989) calls this condition 'independence of base'. See also Blundell and Lewbel (1991), Browning (1988, 1992), Deaton and Muellbauer (1980), Donaldson (1992), Grunau (1988), Muellbauer (1974), and Pollak and Wales (1979).

²In Donaldson and Pendakur (1999), we investigate equivalent-expenditure functions that are affine in expenditure as well as those that satisfy GESE.

attending the theatre, are more enjoyable when done in groups, we may expect equivalence scales for households with more than one member to decrease with expenditure. A priori, one cannot say which of these effects will dominate; we estimate models in which either is possible.

Both ESE and GESE structure interhousehold comparisons of well-being. Suppose that the members of two households of different types are equally well off. ESE implies that a one-percent increase in expenditure by one of the households must be matched by a one-percent increase in expenditure by the other in order to preserve equality of well-being. GESE allows the matching percentage increase to be different from one percent and to depend on household types and prices but not on expenditure levels.

ESE imposes a restriction on the way non-reference and reference household preferences are related. This restriction makes it possible to identify the equivalent-expenditure function and associated equivalence scale from demand behaviour alone if ESE holds and the reference expenditure function is not PIGLOG³ (Blackorby and Donaldson, 1993).

We show that this result can be extended to the case of GESE. If GESE is a maintained hypothesis and the reference expenditure function is not PIGLOG, the equivalent-expenditure function can be identified from demand behaviour (Theorems 2 and 3).⁴ GESE implies that expenditure functions for different household types are related in a particular way, a restriction which can be tested. The relationship also makes it possible to identify the equivalent-expenditure function. GESE is stronger than the restriction on behaviour, however, and the whole of GESE cannot be tested by using demand data alone.

Confidence in an estimated equivalent-expenditure function therefore depends, in part, on a commitment to the way in which GESE structures interhousehold comparisons of well-being. Such a commitment might be based on intuition or on independent research such as the survey conducted by Koulovatianos et al. (2001), which finds that people believe that equivalence scales decline with expenditure.

Econometric research on equivalent-expenditure functions has focussed on those that support ESE equivalence scales. Researchers have used consumer-demand techniques to assess the scales' sizes, price sensitivity and exactness. Many papers (for a survey, see Browning, 1992; Blundell and Lewbel, 1991; Dickens et al., 1993 and Pashardes, 1995) use parametric methods to estimate exact equivalence scales, and three papers (Blundell et al., 1998; Gozalo, 1997; Pendakur, 1999) use semiparametric methods to estimate exact scales. Most of these papers allow for price-dependence and find that equivalence scales do depend on prices. Indeed, Blundell and Lewbel (1991) argue that price responses are the only observable

³We say that the expenditure function E^r is PIGLOG if and only if it can be written as $\ln E^r(u, p) = C(p)f(u) + \ln D(p)$.

⁴There is a body of recent research which suggests that expenditure-share equations are not PIGLOG (see, for example, Banks et al., 1997).

features of equivalence scales. Several papers test the exactness of equivalence scales. Parametric papers (Blundell and Lewbel, 1991; Dickens et al., 1993; Pashardes, 1995) and semiparametric papers (Blundell et al., 1998; Gozalo, 1997; Pendakur, 1999) alike reject the ESE hypothesis. Pendakur (1999) suggests that exactness may hold for some comparisons but not others and finds that equivalence scales are not exact for comparisons of childless households to households with children, but may be exact for comparisons within these groups. He further notes that this may be due to the tight restrictions that exactness puts on the demand for children's goods. Our results are consistent with this hypothesis.

In the empirical section of this paper, we use parametric estimation techniques and Canadian price and expenditure data to estimate equivalent-expenditure functions and equivalence scales. We assume that expenditure-share equations are quadratic in the logarithm of expenditure and show how GESE restricts the demand equations and how equivalent-expenditure functions can be calculated if GESE is maintained. We test GESE against an unrestricted quadratic alternative, estimate GESE-restricted equivalent-expenditure functions and test down to ESE. Equivalence scales given GESE are quite different from those given ESE, with the former larger than the latter and decreasing in expenditure.

Our estimates suggest that equivalence scales for households with children decrease significantly with expenditure. For example, the GESE-restricted equivalence scale for dual parents with one child is 1.93 at low expenditure and 1.62 at high expenditure. The scale for dual parents with two children ranges from 2.14 at low expenditure to 1.81 at high expenditure. On the other hand, the GESE-restricted equivalence scale for households without children is insignificantly downward sloping.

The dependence of equivalence scales on expenditure permits greater flexibility in the demand for children's goods, allowing the share spent on those goods to depend on expenditure. We show that this dependence is quantitatively important by examining the demand for children's clothing directly.

Not surprisingly, we find that the ESE restrictions are rejected against a GESE alternative. However, we find that the restrictions on demand functions implied by GESE are rejected against an unrestricted quadratic alternative. This rejection does not imply that GESE restricts demand behaviour in an important way, however. We assess the economic importance of the GESE restrictions by examining the cost-of-living indices—which are estimable without restrictions on unobservable features of utility functions—given by our GESE model compared to those given by an unrestricted quadratic model. We find that the restrictions on behaviour imposed by GESE do not distort cost-of-living indices by much.⁵

Section 2 presents the model of the household that we employ along with its relationship to equivalent-expenditure functions and equivalence scales and

⁵Blundell and Lewbel (1991) call the cost-of-living-index ratios 'relative equivalence scales'.

Section 3 introduces generalized equivalence-scale exactness (GESE). Section 4 provides our theorems on theoretical identification. Section 5 discusses the empirical model and the data, Section 6 presents our empirical results, and Section 7 concludes the paper.

2. Equivalent-expenditure functions and equivalence scales

Suppose that \mathcal{Z} is a set of vectors of possible household characteristics. Elements in \mathcal{Z} may describe household characteristics exhaustively, including the number and ages of household members, special needs and so on, or it may include only a subset of all possible characteristics. To use equivalent-expenditure functions in a normative setting, it is necessary to assume that any two households with the same characteristics have identical preferences and that each household member enjoys the same level of utility.⁶

We restrict attention to the consumption of m private goods only and use the indirect utility function $V: \mathcal{R}_{++}^{m+1} \times \mathcal{Z} \rightarrow \mathcal{R}$, $m \geq 2$, to represent household preferences and measure the levels of well-being of household members. For a household with characteristics $z \in \mathcal{Z}$ facing prices $p \in \mathcal{R}_{++}^m$ with expenditure $x \in \mathcal{R}_{++}$, the well-being of each household member is

$$u = V(p, x, z). \quad (2.1)$$

We assume that V is continuous, increasing in x and homogeneous of degree zero in (p, x) for each $z \in \mathcal{Z}$. The expenditure function corresponding to V is E where $E(u, p, z)$ is the minimum expenditure needed by a household with characteristics z facing prices p to attain utility level u for each of its members. E is continuous in (u, p) for each z , increasing in u and homogeneous of degree one in p .

To define equivalent-expenditure functions and equivalence scales, a reference household type is needed and, although other choices are possible, we use a childless single adult as the reference and denote his or her characteristics as z^r . For a household with characteristics z , equivalent expenditure x^e is that expenditure which, if enjoyed by a reference single adult facing the same prices, would result in a utility level equal to that of each household member. Thus, equivalent expenditure is implicitly defined by

$$V(p, x, z) = V(p, x^e, z^r) = V^r(p, x^e) \quad (2.2)$$

where $V^r = V(\cdot, \cdot, z^r)$ is the indirect utility function for the reference household. We assume that, for every (p, x, z) in the domain of V , (2.2) can be solved for $x^e \in \mathcal{R}_{++}$, and we write

⁶See Blackorby and Donaldson (1993) for a short discussion of more complex formulations.

$$x^e = X(p, x, z). \tag{2.3}$$

Because the indirect utility function V is homogeneous of degree zero in (p, x) , X is homogeneous of degree one in (p, x) . In addition, X is increasing in x and $X(p, x, z^r) = x$ for all (p, x) .

Because

$$V(p, x, z) = u \Leftrightarrow E(u, p, z) = x, \tag{2.4}$$

$$X(p, x, z) = E^r V(p, x, z), p, \tag{2.5}$$

where $E^r = E(\cdot, \cdot, z^r)$ is the expenditure function of the reference household.

For welfare purposes, equivalent-expenditure functions permit the conversion of an economy with many household types into an economy of identical single individuals. If, for example, a household has four members, then it is equivalent, in terms of utilities, to four single adults, each of whom spends x^e .⁷

Two indirect utility functions, \hat{V} and \tilde{V} , represent the same preferences for each household type if and only if there exists a function ψ , increasing in its first argument, such that

$$\hat{V}(p, x, z) = \psi(\tilde{V}(p, x, z), z) \tag{2.6}$$

for all $(p, x, z) \in \mathcal{R}_{++}^{m+1} \times \mathcal{Z}$. If, in addition, \hat{V} and \tilde{V} , make the same inter-household comparisons of utility levels, then there exists an increasing function ϕ such that

$$\hat{V}(p, x, z) = \phi(\tilde{V}(p, x, z)) \tag{2.7}$$

for all (p, x, z) . We show, in Theorem 1 in Appendix A, that \hat{V} and \tilde{V} represent the same reference preferences (Eq. (2.6) holds for $z = z^r$) and the associated equivalent-expenditure functions \hat{X} and \tilde{X} are identical if and only if Eq. (2.7) is satisfied. That is, reference preferences and the equivalent-expenditure function together determine the overall indirect utility function V up to an increasing transform which is independent of household characteristics. It follows that equivalent-expenditure functions make interhousehold comparisons of utility.⁸

An equivalence scale S is defined by

$$u = V(p, x, z) = V^r(p, x/s) \tag{2.8}$$

⁷The definition of the equivalent-expenditure function ensures that utility levels are invariant to the choice of reference household but the same is not true of equivalent expenditures themselves. In the context of inequality measurement, this and other difficulties are investigated in Ebert and Moyes (2001).

⁸Only comparisons of utility levels are needed; for discussions, see Blackorby and Donaldson (1993), Blackorby et al. (1984) and Sen (1977).

where s is the value of the scale. If household expenditure is divided by s , equivalent expenditure results. Consequently, an equivalence scale is the ratio of household expenditure to equivalent expenditure (x/x^e) and we can write

$$s = S(p, x, z) = \frac{x}{X(p, x, z)}. \quad (2.9)$$

Because X is homogeneous of degree one in (p, x) , S is homogeneous of degree zero in (p, x) . In addition, $S(p, x, z^f) = 1$ for all (p, x) . It is usual to write equivalence scales as functions of utility levels. To do this, Eq. (2.8) is solved using expenditure functions and

$$s = \frac{E(u, p, z)}{E^r(u, p)}. \quad (2.10)$$

Without additional assumptions, neither equivalent-expenditure functions nor equivalence scales can be determined by household demand behaviour. The reason is that the same behaviour is implied by any two utility functions \hat{V} and \tilde{V} , satisfying Eq. (2.6). Theorem 1 shows that, unless the function ψ is independent of z , the associated equivalent-expenditure functions are different.⁹

In general, equivalence scales depend on expenditure or, equivalently, on utility, but those that do not are called exact. The scale is independent of expenditure ($S(p, x, z) = \bar{S}(p, z)$) and, therefore, exact if and only if the expenditure function is multiplicatively decomposable (Blackorby and Donaldson, 1991, 1993; Lewbel, 1991), with

$$E(u, p, z) = \bar{S}(p, z)E^r(u, p) \quad (2.11)$$

and we refer to this condition as equivalence-scale exactness (ESE). ESE is satisfied if and only if X is proportional to expenditure for each (p, z) , with

$$X(p, x, z) = \frac{x}{\bar{S}(p, z)}. \quad (2.12)$$

ESE is equivalent to a condition on the way interpersonal comparisons are related to expenditures called income-ratio comparability (Blackorby and Donaldson, 1991, 1993). If a household with arbitrary characteristics and a reference household facing the same prices have expenditures such that their utilities are equal, common scaling of their expenditures (which leaves the expenditure ratio unchanged) preserves utility equality. Thus, an increase in a household's expenditure of one per cent matched by an increase in the reference household's expenditure of one per cent preserves equality of well-being.

If $V(\cdot, \cdot, z)$ is differentiable for each $z \in \mathcal{Z}$ with $\partial V(p, x, z)/\partial x > 0$ for all $(p, x,$

⁹See Pollak and Wales (1979, 1981, 1992).

z), ESE implies that the expenditure-share function $W_j(\cdot, \cdot, z)$ (written as a function of $\ln x$) is related to the expenditure-share function of the reference household W_j^r by

$$W_j(p, x, z) = W_j^r(p, \ln x - \ln \bar{S}(p, z)) + \frac{\partial \ln \bar{S}(p, z)}{\partial \ln p_j}, \quad (2.13)$$

$j = 1, \dots, m$ (Pendakur, 1999).

ESE suffers from an important weakness: it handles children's goods poorly (see Browning, 1988, 1992; Pendakur, 1999). If c is any children's good, the reference share $W_c^r(p, \ln x) = 0$ and this, together with Eq. (2.13) implies

$$W_c(p, x, z) = \frac{\partial \ln \bar{S}(p, z)}{\partial \ln p_c}, \quad (2.14)$$

which implies that the expenditure elasticity of every children's good is one.

3. Generalized equivalence-scale exactness (GESE)

If equivalence-scale exactness (ESE) is satisfied, an increase in household expenditure must be matched by the same percentage increase in reference-household expenditure in order to preserve utility equality across households. We weaken ESE to generalized equivalence-scale exactness (GESE) by requiring the matching percentage increase to be positive and independent of expenditure while allowing it to depend on prices and household characteristics.¹⁰ Given this, there exist functions $\kappa: \mathcal{R}_{++}^m \times \mathcal{Z} \rightarrow \mathcal{R}_{++}$ and $\gamma: \mathcal{R}_{++}^m \times \mathcal{Z} \rightarrow \mathcal{R}_{++}$ such that

$$x^e = X(p, x, z) = \gamma(p, z)x^{\kappa(p, z)}. \quad (3.1)$$

Because the matching percentage increase $\kappa(p, z) > 0$ and X is increasing in x , $\gamma(p, z) > 0$ for all (p, z) . In addition, because $X(p, x, z^1) = x$ for all (p, x) , $\gamma(p, z^1) = 1$ and $\kappa(p, z^1) = 1$ for all p . $\kappa(p, z)$ is the expenditure-elasticity of X .¹¹

The function κ is homogeneous of degree zero in p (see Lemma 1 in Appendix A).

Eq. (3.1) is equivalent to

$$\ln x^e = \ln X(p, x, z) = \kappa(p, z) \ln x + \ln \gamma(p, z). \quad (3.2)$$

Defining $K(p, z) = 1/\kappa(p, z)$ and $G(p, z) = 1/\gamma(p, z)^{1/\kappa(p, z)}$, Eq. (3.2) can be rewritten as

¹⁰Thus $dx^e/x^e = \kappa(p, z) dx/x$.

¹¹Other functional forms for X are possible. Donaldson and Pendakur (1999) considers equivalent-expenditure functions that are affine in expenditure, with $X(p, x, z) = r(p, z)x + a(p, z)$.

$$\ln x^e = \ln X(p, x, z) = \frac{\ln x - \ln G(p, z)}{K(p, z)} \quad (3.3)$$

and, because $V(p, x, z) = V^r(p, x^e)$, it follows that

$$V(p, x, z) = V^r\left(p, \exp\left\{\frac{\ln x - \ln G(p, z)}{K(p, z)}\right\}\right), \quad (3.4)$$

and the expenditure function satisfies

$$\ln E(u, p, z) = K(p, z) \ln E^r(u, p) + \ln G(p, z). \quad (3.5)$$

Because $\gamma(p, z^r) = \kappa(p, z^r) = 1$, $G(p, z^r) = K(p, z^r) = 1$. In addition, because κ is homogeneous of degree zero in p , so is K .

Given GESE, the equivalence scale S satisfies

$$\ln S(p, x, z) = \frac{(K(p, z) - 1) \ln x + \ln G(p, z)}{K(p, z)}. \quad (3.6)$$

S is increasing (decreasing) in x if $K(p, z) > 1$ ($K(p, z) < 1$).

The functions G and K are not, in general, independent: if E^r , G and K are differentiable in p , K is functionally dependent on G but the converse is not true. In particular, Lemma 2 in Appendix A shows that

$$K(p, z) = 1 - \sum_{j=1}^m \frac{\partial \ln G(p, z)}{\partial \ln p_j}. \quad (3.7)$$

This means that K is functionally dependent on G : for every G there is exactly one K . As an example suppose, for $m = 2$, that

$$G(p, z) = \Gamma(z) p_1^{\delta_1(z)} p_2^{\delta_2(z)}. \quad (3.8)$$

Then

$$K(p, z) = 1 - \delta_1(z) - \delta_2(z). \quad (3.9)$$

K is the same for every choice of the function Γ .

If $V(\cdot, \cdot, z)$ is differentiable for each z with $\partial V(p, x, z) / \partial x > 0$ for all (p, x, z) , Roy's Theorem implies that, given GESE, the share equations W_j , $j = 1, \dots, m$, satisfy

$$\begin{aligned} W_j(p, x, z) = & K(p, z) W_j^r\left(p, \frac{\ln x - \ln G(p, z)}{K(p, z)}\right) \\ & + \frac{\partial K(p, z)}{\partial \ln p_j} \left(\frac{\ln x - \ln G(p, z)}{K(p, z)}\right) + \frac{\partial \ln G(p, z)}{\partial \ln p_j}, \end{aligned} \quad (3.10)$$

where $W_j^r := W_j(\cdot, \cdot, z^r)$, $j = 1, \dots, m$ is the reference household's share equation. Given ESE, $K(p, z) = 1$, and the second term of Eq. (3.10) vanishes. The

restrictions on share equations implied by GESE are thus a generalization of those implied by ESE.

The problem with children's goods possessed by ESE disappears with GESE. If good c is a children's good, $W_c^r(p, \ln x) = 0$ and Eq. (3.10) implies

$$W_c(p, x, z) = \frac{\partial K(p, z)}{\partial \ln p_c} \left(\frac{\ln x - \ln G(p, z)}{K(p, z)} \right) + \frac{\partial \ln G(p, z)}{\partial \ln p_c}. \quad (3.11)$$

These share equations are affine in the logarithm of expenditure and, as a consequence, expenditure elasticities are not forced to be one.

4. Theoretical identification

Blackorby and Donaldson (1993, 1994) investigate the theoretical identification of equivalence scales when ESE is accepted as a maintained hypothesis. Given one of two technical conditions, estimation of the scale from demand behaviour is possible if and only if the reference expenditure function is not PIGLOG (Blackorby and Donaldson, 1993, Theorems 5.1 and 5.3). E^r is PIGLOG if and only if it can be written as

$$\ln E^r(u, p) = C(p)f(u) + \ln D(p) \quad (4.1)$$

where C is homogeneous of degree zero, D is homogeneous of degree one, and f is increasing.

In this section, we investigate the relationship of behaviour and equivalent-expenditure functions when GESE is satisfied and show that the results of Blackorby and Donaldson (1993) can be generalized. Assuming that there is a characteristic, such as age, that is a continuous variable and that V is a continuous function of it, we show that the equivalent-expenditure function and equivalence scales are uniquely identified by behaviour if GESE is maintained and the reference expenditure function is not PIGLOG. Although the theorems are quite different, the requisite condition on the reference expenditure functions is the same as the one found by Blackorby and Donaldson for ESE.

The range condition used for the theorem in this section is global: it applies to all $p \in \mathcal{R}_{++}^m$. It need not hold globally, however. The theorem needs only two price vectors at which the range condition holds and, if the condition holds for one price vector, continuity ensures that it holds in a neighborhood. Consequently, the theorem can be applied locally.

The indirect utility functions \hat{V} and \tilde{V} , represent the same behaviour if and only if Eq. (2.6) holds. Equivalently, the expenditure functions \hat{E} and \tilde{E} , represent the same behaviour if and only if

$$\hat{E}(\psi(u, z), p, z) = \tilde{E}(u, p, z) \quad (4.2)$$

or, equivalently,

$$\hat{E}(u, p, z) = \tilde{E}(\rho(u, z), p, z) \tag{4.3}$$

where $\rho(\cdot, z)$ is the inverse of $\psi(\cdot, z)$ for each $z \in \mathcal{Z}$. Because $\hat{V}(\cdot, \cdot, z)$, $\tilde{V}(\cdot, \cdot, z)$, $\hat{E}(\cdot, \cdot, z)$ and $\tilde{E}(\cdot, \cdot, z)$ are continuous by assumption, $\phi(\cdot, z)$ and $\rho(\cdot, z)$ are continuous for all $z \in \mathcal{Z}$.

The functions \hat{V} and \tilde{V} , represent the same preferences when GESE is satisfied if and only if

$$\begin{aligned} \ln \hat{E}(u, p, z) &= \hat{K}(p, z) \ln \hat{E}^r(u, p) + \ln \hat{G}(p, z) \\ &= \tilde{K}(p, z) \ln \tilde{E}^r(\rho(u, z), p) + \ln \tilde{G}(p, z) \\ &= \ln \tilde{E}(\rho(u, z), p, z) \end{aligned} \tag{4.4}$$

for all (u, p, z) , where \hat{K} , \tilde{K} , \hat{G} and \tilde{G} , are defined in the obvious way. Eq. (4.2) implies that

$$\tilde{E}^r(\rho(u, z), p) = \hat{E}^r(\psi(\rho(u, z), z^r), p). \tag{4.5}$$

Defining $\sigma(u, z) = \psi(\rho(u, z), z^r)$ and rearranging, Eq. (4.4) implies

$$\ln \hat{E}^r(\sigma(u, z), p) = \frac{\hat{K}(p, z)}{\tilde{K}(p, z)} \ln \hat{E}^r(u, p) + \frac{\ln \hat{G}(p, z) - \ln \tilde{G}(p, z)}{\tilde{K}(p, z)}. \tag{4.6}$$

Defining the functions H and Q by $H(p, z) = \hat{K}(p, z)/\tilde{K}(p, z)$ and $Q(p, z) = (\ln \hat{G}(p, z) - \ln \tilde{G}(p, z))/\tilde{K}(p, z)$, Eq. (4.6) can be rewritten as

$$\ln \hat{E}^r(\sigma(u, z), p) = H(p, z) \ln \hat{E}^r(u, p) + Q(p, z). \tag{4.7}$$

Although \hat{K} and \tilde{K} are determined, in the differentiable case, by \hat{G} and \tilde{G} , the relationship, as noted in Section 4 is not one-to-one. We investigate the problem using a condition on the ranges of the functions \hat{G} , \tilde{G} , \hat{K} , and \tilde{K} .

Range Condition. \mathcal{Z} contains a subset $\hat{\mathcal{Z}}$ of continuous variables such as age and, for every $p \in \mathcal{R}_{++}^m$, the functions \hat{G} , \tilde{G} , \hat{K} , and \tilde{K} are continuous in and sensitive to $\hat{z} \in \hat{\mathcal{Z}}$. In addition, $Q(p, z)$ can be moved independently of $H(p, z)$ by changing \hat{z} .

Theorems 2 and 3 prove that the equivalent-expenditure function and equivalence scales are uniquely identified by behaviour if GESE holds and the reference expenditure function is not PIGLOG. Theorem 2 investigates the identification problem when $\hat{G} \neq \tilde{G}$ and $\hat{K} \neq \tilde{K}$.

Theorem 2. *Two indirect utility functions \hat{V} and \tilde{V} represent the same preferences*

and satisfy GESE and the range condition with $\hat{K} \neq \tilde{K}$ and $\hat{G} \neq \tilde{G}$ if and only if there exist functions $C: \mathcal{R}_{++}^m \rightarrow \mathcal{R}_{++}$, $D: \mathcal{R}_{++}^m \rightarrow \mathcal{R}_{++}$, $\hat{f}: \mathcal{R} \rightarrow \mathcal{R}$, $\tilde{f}: \mathcal{R} \rightarrow \mathcal{R}$, $h: \mathcal{Z} \rightarrow \mathcal{R}_{++}$ and $q: \mathcal{Z} \rightarrow \mathcal{R}$ such that, for all (u, p, z) ,

$$\ln \hat{E}^r(u, p) = C(p)\hat{f}(u) + \ln D(p), \quad (4.8)$$

$$\ln \tilde{E}^r(u, p) = C(p)\tilde{f}(u) + \ln D(p), \quad (4.9)$$

$$\hat{K}(p, z) = h(z)\tilde{K}(p, z), \quad (4.10)$$

and

$$\ln \hat{G}(p, z) = \ln \tilde{G}(p, z) + \tilde{K}(p, z)[C(p)q(z) + \ln D(p)(1 - h(z))], \quad (4.11)$$

where C is homogeneous of degree zero, D is homogeneous of degree one, \hat{f} and \tilde{f} are increasing and continuous, $h(z^r) = 1$ and $q(z^r) = 0$.

Proof. See Appendix A.

Theorem 3 considers the case in which there are two different G 's, \hat{G} and \tilde{G} , but only one function K . In that case, $H(p, z) = 1$ for all (p, z) .

Theorem 3. Two indirect utility functions \hat{V} and \tilde{V} represent the same preferences and satisfy GESE and the range condition with $\hat{K} = \tilde{K} = :K$ and $\hat{G} \neq \tilde{G}$ if and only if there exist functions $C: \mathcal{R}_{++}^m \rightarrow \mathcal{R}_{++}$, $D: \mathcal{R}_{++}^m \rightarrow \mathcal{R}_{++}$, $\hat{f}: \mathcal{R} \rightarrow \mathcal{R}$, $\tilde{f}: \mathcal{R} \rightarrow \mathcal{R}$, $h: \mathcal{Z} \rightarrow \mathcal{R}_{++}$ and $q: \mathcal{Z} \rightarrow \mathcal{R}$ such that, for all (u, p, z) ,

$$\ln \hat{E}^r(u, p) = C(p)\hat{f}(u) + \ln D(p), \quad (4.12)$$

$$\ln \tilde{E}^r(u, p) = C(p)\tilde{f}(u) + \ln D(p) \quad (4.13)$$

and

$$\ln \hat{G}(p, z) = \ln \tilde{G}(p, z) + K(p, z)C(p)q(z), \quad (4.14)$$

where C is homogeneous of degree zero, D is homogeneous of degree one, \hat{f} and \tilde{f} are increasing and continuous, and $q(z^r) = 0$.

Proof. See Appendix A.

Because, in the differentiable case, K is functionally dependent on G , we need not consider the case where $\hat{G} = \tilde{G}$ and $\hat{K} \neq \tilde{K}$.

If GESE is satisfied, Theorems 2 and 3 together imply that: (1) if the reference expenditure function is PIGLOG, there are infinitely many log-affine equivalent-

expenditure functions that are consistent with behaviour; and (2) if the reference expenditure function is *not* PIGLOG, there are infinitely many equivalent-expenditure functions that are consistent with behaviour but only one of them is log-affine. It follows that, in order to identify equivalent-expenditure functions and equivalence scales from behaviour alone, the reference expenditure function must not be PIGLOG. If this condition is met, the functions K and G are unique and can, therefore, be estimated from behaviour.

We believe that it is appropriate to maintain GESE for the following reasons: (1) some structure must be imposed in order to identify equivalent-expenditure functions and equivalence scales; (2) GESE is a generalization of ESE and so provides more flexibility; and (3) GESE is not inconsistent with commonly held intuitions and beliefs about equivalence scales (see Koulovatianos et al., 2001).

5. Empirical model

Estimation of equivalent-expenditure functions requires estimation of a demand system. Recently, many scholars have chosen non-parametric or semiparametric approaches to the estimation of consumer demand and equivalence scales (see, for example, Gozalo, 1997; Blundell et al., 1998; Pendakur, 1999, 2000) because such approaches restrict the shapes of Engel curves less than parametric approaches do. Unfortunately, non-parametric and semiparametric methods are not well-suited to the problem at hand for two reasons. First, identification of equivalent-expenditure functions comes from *both* price and expenditure responses. Non-parametric and semiparametric methods are not well-developed for estimation of consumer demand as a function of both prices and expenditure; rather the usual strategy is to hold prices constant and look at variation across expenditure only. Second, the parameters of the equivalent-expenditure function are not independent— K is a function of G . Estimating functionally dependent parameters in a semiparametric environment is extremely cumbersome.

As noted above, the form of equivalent-expenditure functions (for example, ESE or GESE) must be assumed a priori in order to estimate them. Given the assumed GESE functional form for the equivalent-expenditure function, Theorems 2 and 3 show that demand estimation is sufficient to reveal its parameters. In addition, it is possible to assess whether or not the restrictions that GESE places on behaviour are satisfied by testing the restriction on the demand system given by Eq. (3.10).

Parametric estimation of equivalent-expenditure functions requires specification of the demand system and parametric expressions of the GESE restrictions. The econometric strategy we employ exploits the following convenient characteristic of GESE. If any household type has expenditure-share equations that are quadratic in the logarithm of expenditure, then all household types do. Thus, we estimate

equivalent-expenditure functions using the Quadratic Almost Ideal demand system (QAI) due to Banks et al. (1997) (see also Pendakur, 2001).

5.1. *The data*

We use expenditure data from the 1969, 1974, 1982, 1984, 1986, 1990, 1992 and 1996 Canadian Family Expenditure Surveys and the 1997, 1998 and 1999 Surveys of Household Spending (Statistics Canada a, b) to estimate demand systems given GESE and recover equivalent-expenditure functions. These data contain annual expenditures in approximately 100 categories for 5 000 to 15 000 households per year. We use only: (1) households in cities with 30 000 or more residents (to match commodity price data and to minimize the effects of home production); (2) households with rental tenure (to avoid rent imputation); (3) households whose members are all full-year members under the age of 65; and (4) households whose heads are aged 25 to 64.¹²

We estimate a demand system composed of the following ten expenditure categories: (1) food purchased from stores; (2) food from restaurants; (3) total rent, including utilities; (4) household operation (including child care); (5) household furnishing and equipment; (6) clothing for adults; (7) clothing for children; (8) private transportation operation (does not include capital expenditures); (9) public transportation; and (10) personal care. Personal care is the 'left-out' equation in all estimation. Price data for all these commodity groups except rent are available from Browning and Thomas (1999) for 1969 to 1996 in five regions of Canada: (1) Atlantic Canada; (2) Québec; (3) Ontario; (4) the Prairies; and (5) British Columbia. Prices for 1997 to 1999 are taken from Pendakur's (2001) update of these price series. Rent prices are from CMHC (1996) and CANSIM (see Pendakur, 2001 for details). Prices are normalized so that residents of Ontario, who form the largest population subgroup in the sample, face the prices (100, . . . , 100) in 1986.

We use several household demographic characteristics in our estimation. For each household, we include: a dummy indicating that the household is a childless adult couple; the natural logarithm of household size for households with no children; the natural logarithm of household size for households with children; a single parent indicator; the age of the household head less 40 for households without children; and the age of the household head less 40 for households with children. For the reference household type, a single childless adult aged 40, these variables are all equal to zero.

¹²The exclusion of owner-tenure households may induce some selection bias because these households are richer, on average, than rental-tenure households. This is especially true if tenure decisions are correlated with liquidity constraints and liquidity constraints raise child costs (see Pashardes, 1995).

Summary statistics on the data are given in Table 1.

5.2. Parametric demand-system specifications

We use the Quadratic Almost Ideal (QAI) model (see Banks et al., 1997) in which

$$V(p, x, z) = \left(\left(\frac{\ln x - \ln a(p, z)}{b(p, z)} \right)^{-1} - q(p, z) \right)^{-1}, \quad (5.1)$$

where a is homogeneous of degree one in p and b and q are homogeneous of degree zero in p . Denoting $a^r(p) = a(p, z^r)$, $b^r(p) = b(p, z^r)$ and $q^r(p) = q(p, z^r)$, and assuming that the reference indirect utility function is QAI, GESE implies that

Table 1
The data

| No. of households: 19 276 | Mean | S.D. | Min | Max |
|--|---------|---------|-------|---------|
| Total expenditure | 14 283 | 8658 | 799 | 110 555 |
| Food from stores | 0.217 | 0.103 | 0 | 0.809 |
| Restaurant food | 0.061 | 0.063 | 0 | 0.643 |
| Rent | 0.33433 | 0.127 | 0.001 | 0.949 |
| Household operation | 0.082 | 0.049 | 0 | 0.636 |
| Household furnishing and equipment | 0.049 | 0.057 | 0 | 0.646 |
| Adult clothing | 0.092 | 0.061 | 0 | 0.585 |
| Children's clothing | 0.009 | 0.019 | 0 | 0.240 |
| Private transportation operation | 0.093 | 0.083 | 0 | 0.591 |
| Public transportation | 0.030 | 0.039 | 0 | 0.452 |
| Personal care | 0.034 | 0.021 | 0 | 0.219 |
| Childless single adults | 0.361 | 0.480 | 0 | 1 |
| Childless adult couples | 0.216 | 0.412 | 0 | 1 |
| Log household size (children not present) | 0.213 | 0.368 | 0 | 1.946 |
| Log household size (children present) | 0.464 | 0.636 | 0 | 2.398 |
| Single parent indicator | 0.055 | 0.228 | 0 | 1 |
| Age of head less 40 (children not present) | 1.111 | 9.731 | – 15 | 24 |
| Age of head less 40 (children present) | – 1.417 | 5.047 | – 15 | 24 |
| Food from stores price | 4.426 | 0.539 | 3.193 | 4.942 |
| Restaurant food price | 4.499 | 0.602 | 3.147 | 5.139 |
| Rent price | 4.356 | 0.498 | 3.289 | 4.971 |
| Household operation price | 4.424 | 0.54006 | 3.208 | 4.924 |
| Household furnishing and equipment price | 4.470 | 0.378 | 3.666 | 4.803 |
| Adult clothing price | 4.599 | 0.38649 | 3.737 | 5.032 |
| Children's clothing price | 4.528 | 0.396 | 3.652 | 4.982 |
| Private transportation operation price | 4.371 | 0.595 | 3.072 | 5.133 |
| Public transportation price | 4.452 | 0.71128 | 3.025 | 5.292 |
| Personal care price | 4.479 | 0.463 | 3.491 | 4.942 |

$$\begin{aligned}
 V(p, x, z) &= V^r(p, x^e) = V^r\left(p, \exp\left\{\frac{\ln x^e - \ln G(p, z)}{K(p, z)}\right\}\right) \\
 &= \left(\left(\frac{\ln x - \ln G(p, z) - K(p, z) \ln a^r(p)}{K(p, z)b^r(p)}\right)^{-1} - q^r(p)\right)^{-1}. \quad (5.2)
 \end{aligned}$$

Thus, if reference preferences satisfy QAI then, under GESE, all households have QAI preferences. In this case, GESE implies that

$$\ln a(p, z) = K(p, z) \ln a^r(p) + \ln G(p, z), \quad (5.3)$$

$$b(p, z) = K(p, z)b^r(p), \quad (5.4)$$

and

$$q(p, z) = q^r(p). \quad (5.5)$$

Thus, we can estimate equivalent-expenditure functions given GESE by requiring $q(p, z) = q^r(p)$ and calculating

$$K(p, z) = \frac{b(p, z)}{b^r(p)} \quad (5.6)$$

and

$$\ln G(p, z) = \ln a(p, z) - K(p, z) \ln a^r(p). \quad (5.7)$$

In addition to estimating equivalent-expenditure functions, use of the QAI demand system allows a simple parametric test of the behavioural restrictions of GESE against an unrestricted QAI alternative. In particular, if preferences do not satisfy $q(p, z) = q^r(p)$, then GESE cannot hold. It is also possible to test down from GESE in this framework. We can test ESE against GESE by asking whether $K(p, z) = 1$ or, equivalently, $b(p, z) = b^r(p)$.

To estimate the demand systems, we specify the functions a , b and q as follows:¹³

$$\ln a(p, z) = a_0(z) + \sum_{k=1}^m a_k(z) \ln p_k + \frac{1}{2} \sum_{k=1}^m \sum_{l=1}^m a_{kl} \ln p_k \ln p_l, \quad (5.8)$$

where $\sum_{k=1}^m a_k(z) = 1$, $\sum_{l=1}^m a_{kl} = 0$ for all k , and $a_{kl} = a_{lk}$ for all k, l ;

$$b(p, z) = \frac{1}{1 - b_0(z)} \exp\left\{\sum_{k=1}^m b_k(z) \ln p_k\right\}, \quad (5.9)$$

¹³The function $a(p, z)$ is a translog in prices and demographics whose quadratic terms are independent of demographics. We also estimated models whose quadratic parameters, a_{kl} , depend on demographics. Such models have similar, but much less precise, estimates of equivalence scale parameters.

where $\sum_{k=1}^m b_k(z) = 0$; and

$$q(p, z) = \sum_{k=1}^m q_k(z) \ln p_k, \quad (5.10)$$

where $\sum_{k=1}^m q_k(z) = 0$.

The functions a_k , b_k and q_k depend on z , and we assume that:

$$a_k(z) = a_k^r + a_k^{\text{coup}} \text{couple} + a_k^{\text{nlhs}} \text{nlhsiz} + a_k^{\text{clhs}} \text{clhsiz} \\ + a_k^{\text{spar}} \text{singlepar} + a_k^{\text{nage}} \text{nheadage} + a_k^{\text{cage}} \text{cheadage}, \quad (5.11)$$

for $k = 0, \dots, m$;

$$b_k(z) = b_k^r + b_k^{\text{coup}} \text{couple} + b_k^{\text{nlhs}} \text{nlhsiz} + b_k^{\text{clhs}} \text{clhsiz} \\ + b_k^{\text{spar}} \text{singlepar} + b_k^{\text{nage}} \text{nheadage} + b_k^{\text{cage}} \text{cheadage}, \quad (5.12)$$

for $k = 0, \dots, m$; and

$$q_k(z) = q_k^r + q_k^{\text{coup}} \text{couple} + q_k^{\text{nlhs}} \text{nlhsiz} + q_k^{\text{clhs}} \text{clhsiz} \\ + q_k^{\text{spar}} \text{singlepar} + q_k^{\text{nage}} \text{nheadage} + q_k^{\text{cage}} \text{cheadage}, \quad (5.13)$$

for $k = 1, \dots, m$. couple indicates a childless married couple, nlhsiz is the natural logarithm of household size for households with no children, clhsiz is the natural logarithm of household size for households with children, singlepar is a dummy indicating that the household has children present but only has one adult aged 18 or greater, nheadage is the age of the household head less 40 for households with no children, and cheadage is the age of the household head less 40 for households with children. Children are defined as persons less than 18 years of age.¹⁴

Two parameters are set rather than estimated. We set a_0^r so that $\ln a(p, z^r)$ is equal to the average expenditure of the reference household type in Ontario 1986 (the base price situation). This gives an equivalence scale that is a simple function of the parameters at this expenditure level (see below). We set $b_0^r = 0$ for identification purposes (see below).

With the GESE-restricted QAI system (where $q_k(z) = q_k^r$), the GESE functions K and G take on relatively simple forms in terms of the parameters if evaluated at an m -vector of equal prices $\hat{p} \mathbf{1}_m = (\hat{p}, \dots, \hat{p})$, with

$$K(\hat{p} \mathbf{1}_m, z) = \frac{1}{(1 - b_0(z))} \quad (5.14)$$

and

¹⁴There is some variation in the age definition of children in the data. In all years prior to 1996, children are defined as persons less than 16 years old. In 1996, they are defined as persons less than 15 years old. In 1997, 1998 and 1999, they are defined as persons less than 18 years old.

$$\ln G(\hat{p}\mathbf{1}_m, z) = a_0(z) + \ln \hat{p} - \frac{1}{(1 - b_0(z))} (a_0^r + \ln \hat{p}). \tag{5.15}$$

Substituting these expressions into Eq. (3.6) and manipulating, we get a simple expression for the log-equivalence scale evaluated at a vector of equal prices:

$$\ln S(\hat{p}\mathbf{1}_m, x, z) = b_0(z)(\ln x - a_0(z) - \ln \hat{p}) + (a_0(z) - a_0^r). \tag{5.16}$$

The first term is zero if $\ln x = a_0(z) + \ln \hat{p}$, which by construction is true for households whose equivalent expenditure is equal to the average expenditure of reference households in the base year. Thus, $\ln S(\hat{p}\mathbf{1}_m, a(\hat{p}\mathbf{1}_m, z), z) = a_0(z) - a_0^r$. If ESE is true, then $b_0(z) = 0$, so that the equivalence scale takes this form at all levels of expenditure.

Application of Roy’s Theorem to Eq. (5.1) generates the log-quadratic expenditure-share equations

$$\begin{aligned} W_j(p, x, z) &= \frac{\partial \ln a(p, z)}{\partial \ln p_j} + \frac{\partial \ln b(p, z)}{\partial \ln p_j} (\ln x - \ln a(p, z)) \\ &+ \frac{\partial q(p, z)}{\partial \ln p_j} \frac{(\ln x - \ln a(p, z))^2}{b(p, z)}, \end{aligned} \tag{5.17}$$

$j = 1, \dots, m$. Substituting Eqs. (5.8)–(5.13) into Eq. (5.17), we get

$$\begin{aligned} W_j(p, x, z) &= \left(a_j(z) + \sum_{k=1}^m a_{jk} \ln p_k \right) + b_j(z)(\ln x - \ln a(p, z)) \\ &+ q_j(z)(1 - b_0(z)) \frac{(\ln x - \ln a(p, z))^2}{\exp \left\{ \sum_{k=1}^m b_k(z) \ln p_k \right\}}, \end{aligned} \tag{5.18}$$

$j = 1, \dots, m$. Adding an error term to the right-hand side produces an estimable demand system.

To assess the properties of GESE with respect to children’s goods, we pay special attention to the children’s-clothing share equations. For childless households, we code children’s clothing expenditures as zero (4% of childless households have non-zero children’s clothing expenditures, presumably due to gift-giving) and restrict the demand system so that the demands for children’s clothing are zero in childless households. This means that given ESE, expenditure shares for children’s clothing are independent of x and given GESE, expenditure shares for children’s clothing are affine in $\ln x$.¹⁵

GESE requires $q_k(z) = q_k^r$ for all $k = 1, \dots, m$. Thus, given QAI demands, GESE

¹⁵Let c denote the children’s clothing equation. We restrict $a_c^r = a_c^{\text{coup}} = a_c^{\text{nlhs}} = a_c^{\text{nage}} = 0$, $b_c^r = b_c^{\text{coup}} = b_c^{\text{nlhs}} = b_c^{\text{nage}} = 0$, and $q_c^r = q_c^{\text{coup}} = q_c^{\text{nlhs}} = q_c^{\text{nage}} = 0$.

restricts preferences in such a way that the coefficients on $(\ln x)^2$ are proportional across household types. To test up from GESE, we estimate a model in which GESE is not maintained so that these proportionality restrictions are relaxed. In that case, the share equations are homogeneous in $(b_0(z), q_1(z), \dots, q_m(z))$, so that these functions are not separately identifiable. To identify $q_k(z)$ in the unrestricted QAI, we impose the restriction $b_0(z) = 0$. However, under GESE, this restriction is not necessary because $q_k(z) = q_k^r$ and we use the weaker restriction that $b_0(z^r) = b_0^r = 0$.

GESE can be also be tested down in two interesting ways. One can test whether or not the function K actually depends on prices by testing whether or not $b_j(z) = b_j^r$ for all $j = 1, \dots, m$. In addition, one can test whether or not $K(p, z) = 1$ —which gives ESE—by testing the additional restriction that $b_0(z) = b_0^r = 0$ in a GESE-restricted model.

Given GESE, equivalent-expenditure functions are uniquely identifiable if and only if GESE is maintained a priori and the reference expenditure function is not PIGLOG. The QAI model is PIGLOG if and only if $q(p, z) = 0$. Thus, we may test the identification restrictions of GESE by testing $q_k^r = 0$, $k = 1, \dots, m$, in a GESE-restricted model.

All reported estimates are by standard maximum likelihood and do not correct for the possible endogeneity of total expenditure. Estimations using UK data (e.g. Banks et al., 1997; Blundell et al., 1998) commonly correct for endogeneity of total expenditure because those data are for 2-week time-spans, leading to problems of lumpy durables consumption, which induces correlations between total expenditure and the error terms. In contrast, Canadian expenditure data are reported at the annual level which may mitigate this problem. In addition, we do not include owned-accommodation expenses or automobile-purchase expenses in the demand system.¹⁶

6. Results

All results in this section are based on estimation where the demand system is given by Eq. (5.18). Selected coefficients are presented in the tables and equivalence scales for several household types in the figures. Complete and detailed parameter estimates are available from the authors on request. The

¹⁶Because we are interested in equivalent-expenditure functions corresponding to unconditional rather than conditional expenditure functions, it is important to take possible endogeneity of household characteristics seriously (see Browning, 1983; Browning and Meghir, 1991 for a full treatment of conditional versus unconditional expenditure functions). We take household composition (age and number of members) as exogenous because children are not simply consumption goods, parents do not have complete control of fertility and are not fully informed about the long-term consequences of fertility decisions, and because evolutionary theory predicts that self-interest may have little to do with the desire to have children.

reference household type for all estimation is a childless single adult aged 40. All models use 19 276 observations of households with 60 relative price situations (12 years and five regions). Asymptotic standard errors are provided in Table 3.

Table 2 presents model statistics for five models. The models are: (1) unrestricted QAI; (2) GESE-restricted QAI; (3) GESE-restricted PIGLOG-restricted (almost ideal) QAI; (4) GESE-restricted QAI with K independent of p ; and (5) ESE-restricted QAI.

Table 2 can be summarized simply. Every testable hypothesis of interest is rejected at conventional levels of significance. Using the QAI model, we reject the hypothesis that GESE is true against an unrestricted QAI alternative. Further, given GESE and QAI, we reject the hypotheses that the GESE equivalent-expenditure function is not identified, that K is independent of p , and that $K = 1$ which is required by ESE.

We see three important observations in these tests of hypotheses. First, the restrictions imposed by GESE do less violence to the data than those imposed by ESE. The likelihood ratio test statistic for the hypothesis that ESE is true against a GESE alternative where K is independent of prices is 24 with a 1 percent critical value of 17. The test statistic for the hypothesis that ESE is true against a GESE alternative where K may depend on prices is 1242 with a 1 percent critical value of 85. That ESE is rejected against a GESE alternative does not surprise us, given the restrictions that ESE places on the shapes of Engel curves and on the demand for children's goods in particular.

Second, Theorems 2 and 3 show that the GESE equivalent-expenditure function is only identifiable if the reference expenditure function is not PIGLOG (affine in the natural logarithm of expenditure). The likelihood ratio test statistic for the hypothesis that GESE is true and the reference expenditure function is PIGLOG against a GESE-restricted QAI alternative is 786 and has a 1 percent critical value of 29. Thus, under the maintained assumption that GESE is true, the parameters of the GESE equivalent-expenditure function are identified.

Third, we reject the restrictions of GESE against an unrestricted QAI alternative. The likelihood ratio test statistic for the hypothesis that q is independent of z is equal to 448 with a 1 percent critical value of 70. Thus, the observable restrictions imposed by GESE on the QAI model are rejected: demographic effects are more complex than those permitted by GESE. That an unrestricted model

Table 2
Model statistics

| | Log likelihood | <i>df</i> |
|---------------------------------|----------------|-----------|
| Unrestricted | 308 134 | 219 |
| GESE (K depends on p) | 307 910 | 174 |
| PIGLOG | 307 517 | 160 |
| GESE (K independent of p) | 307 301 | 123 |
| ESE | 307 289 | 117 |

dominates a GESE-restricted model in a statistical sense does not imply that the differences between the unrestricted and GESE-restricted models are important. In Section 6.3, we assess the economic importance of the GESE and ESE restrictions by considering how cost-of-living indices for different demographic groups are affected by the restrictions.

6.1. Estimated equivalent-expenditure functions and equivalence scales

Table 3 shows selected parameter estimates from QAI demand systems. The leftmost column shows estimated parameters from an ESE-restricted QAI demand system where $q(p, z) = q(p)$ and $b(p, z) = b(p)$. The middle column shows estimated parameters from a GESE-restricted QAI demand systems where $q(p, z) = q(p)$ and K is independent of p (i.e. $b_k(z) = b_k^r$ for $k = 1, \dots, m$). The rightmost column shows estimated parameters from a GESE-restricted QAI demand system where $q(p, z) = q(p)$ and K may depend on p .

The parameters of the GESE equivalent-expenditure function are difficult to identify. The estimated standard errors for the parameters in $a_0(z)$ are about three times as large in the rightmost column as they are for the other two specifications. This is because the price elasticities of K and G and the level of G affect the demand system in similar ways. Empirically, these are distinguished by the fact that the price elasticities of K and G are tied to the levels of K and G in different price regimes.

Table 3
Selected parameter estimates

| | ESE | | GESE | | GESE | |
|---------------------|---------------|-----------|------------------------|-----------|------------------|-----------|
| | $K(p, z) = 1$ | | $K(p, z) = \bar{K}(z)$ | | K unrestricted | |
| | Coefficient | Std error | Coefficient | Std error | Coefficient | Std error |
| a_0^r | 4.837 | @ | 4.837 | @ | 4.837 | @ |
| a_0^{coup} | -0.121*** | 0.039 | -0.126*** | 0.043 | -0.276*** | 0.099 |
| a_0^{nlhs} | 0.598*** | 0.044 | 0.600*** | 0.051 | 0.759*** | 0.120 |
| a_0^{clhs} | 0.253*** | 0.022 | 0.294*** | 0.022 | 0.484*** | 0.039 |
| a_0^{spar} | -0.404*** | 0.038 | -0.319*** | 0.038 | 0.387*** | 0.083 |
| a_0^{nage} | 0.004*** | 0.001 | 0.004*** | 0.001 | 0.016*** | 0.003 |
| a_0^{cage} | 0.034*** | 0.002 | 0.031*** | 0.002 | 0.042*** | 0.003 |
| b_0^r | 0.000*** | @ | 0.000 | @ | 0.000 | @ |
| b_0^{coup} | | | 0.043 | 0.167 | -0.066 | 0.130 |
| b_0^{nlhs} | | | -0.091 | 0.206 | -0.053 | 0.158 |
| b_0^{clhs} | | | -0.169* | 0.094 | -0.109* | 0.059 |
| b_0^{spar} | | | -0.419** | 0.170 | -0.275*** | 0.124 |
| b_0^{nage} | | | -0.013*** | 0.005 | -0.009*** | 0.003 |
| b_0^{cage} | | | 0.018** | 0.008 | 0.004 | 0.005 |

* Significant at the 10% level; ** significant at the 5% level; *** significant at the 1% level; @ indicates a parameter that has been set rather than estimated.

At the base price vector, the natural logarithm of the equivalence scale given ESE is equal to $a_0(z) - a_0^r$. Given GESE, it is equal to $b_0(z)(\ln x - \ln a(p, z)) + (a_0(z) - a_0^r)$, so that it simplifies to the same functional form as the ESE scale if $x = a(p, z)$ which is in the middle of the equivalent-expenditure distribution by construction. Table 4 gives equivalence scales and their expenditure elasticities evaluated at base prices and $x = a(p, z)$ for several household types given the estimates in Table 3. Asymptotic standard errors are calculated using Cramer's Rule.

We draw three conclusions from Table 3. First, ESE scales and GESE scales where K is independent of p are very similar to each other in the middle of the distribution of well-being, but they are quite different from the GESE scales where K depends on p . In particular, the GESE scales reported in the rightmost column of the table show costs that are (insignificantly) lower for childless couples and higher for all other household types. Second, not all household types have reasonable estimated equivalence scales. In particular, single parent households, who comprise 5 percent of households in the sample, have estimated scales outside the plausible range. Third, the lower panel of Table 3 which shows the estimated expenditure elasticity of equivalence scales suggests that households with children have significantly decreasing equivalence scales. We note that many of the standard errors are rather large, so that significance is only at the 10 percent level.

Fig. 1 shows the GESE equivalence scale where K is independent of p and Fig. 2 shows the GESE equivalence scale where K depends on p for several household

Table 4
Equivalence scales at $x = a(p, z)$

| | ESE $K(p, z) = 1$ | | GESE $K(p, z) = \bar{K}(z)$ | | GESE K unrestricted | |
|-----------------------------|----------------------|---------|--------------------------------|---------|--------------------------|---------|
| | Scale | Std err | Scale | Std err | Scale | Std err |
| Couple: no children | 1.34 | 0.04 | 1.34 | 0.04 | 1.29 | 0.09 |
| Three adults | 1.93 | 0.06 | 1.93 | 0.10 | 2.30 | 0.30 |
| Single parent: two children | 0.88 | 0.04 | 1.00 | 0.09 | 2.50 | 0.29 |
| Couple: one child | 1.32 | 0.03 | 1.38 | 0.03 | 1.70 | 0.08 |
| Couple: two children | 1.42 | 0.04 | 1.50 | 0.05 | 1.96 | 0.11 |
| Couple: three children | 1.50 | 0.05 | 1.60 | 0.06 | 2.18 | 0.14 |
| | | | elast† | Std err | elast† | Std err |
| Couple: no children | | | -0.02 | 0.11 | -0.10 | 0.08 |
| Three adults | | | -0.10 | 0.22 | -0.06 | 0.17 |
| Single parent: two children | | | -0.60*** | 0.20 | -0.40*** | 0.14 |
| Couple: one child | | | -0.19* | 0.10 | -0.12* | 0.06 |
| Couple: two children | | | -0.23* | 0.13 | -0.15* | 0.08 |
| Couple: three children | | | -0.27* | 0.15 | -0.18* | 0.10 |

† $\partial \ln S(p, x, z) / \partial \ln x$; * significant at the 10% level; ** significant at the 5% level; *** significant at the 1% level.

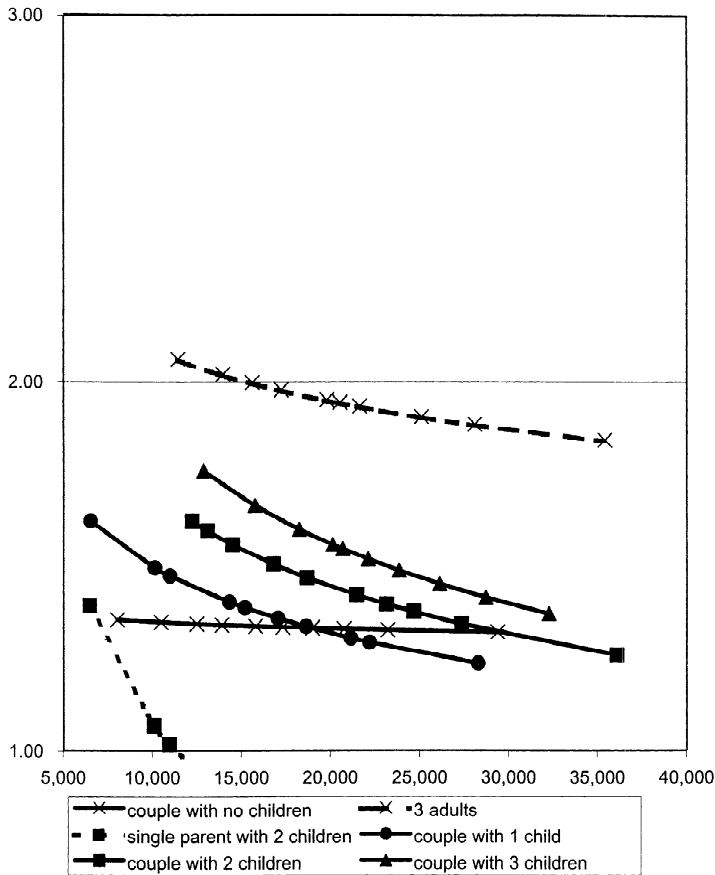


Fig. 1. Equivalence scales, K independent of p .

types. The equivalence scales are evaluated at equally spaced percentiles from the fifth to ninety-fifth percentiles of the total expenditure distribution in Ontario in 1986 for each household type. They are evaluated at base prices.

The figures reveal that estimated expenditure-dependent equivalence scales are fairly similar across the specifications and fairly reasonable. Excepting single parent households, the estimated equivalence scales lie in the plausible range, that is, they are bounded by 1 and household size. Regardless of whether or not K is allowed to depend on prices, the estimated equivalence scale declines with total expenditure for households with children. However, the equivalence scales for households with children are larger if K is allowed to depend on p . The point estimates also suggest equivalence scales that decline (slowly) with total expenditure for childless households, but this feature is not statistically significant.

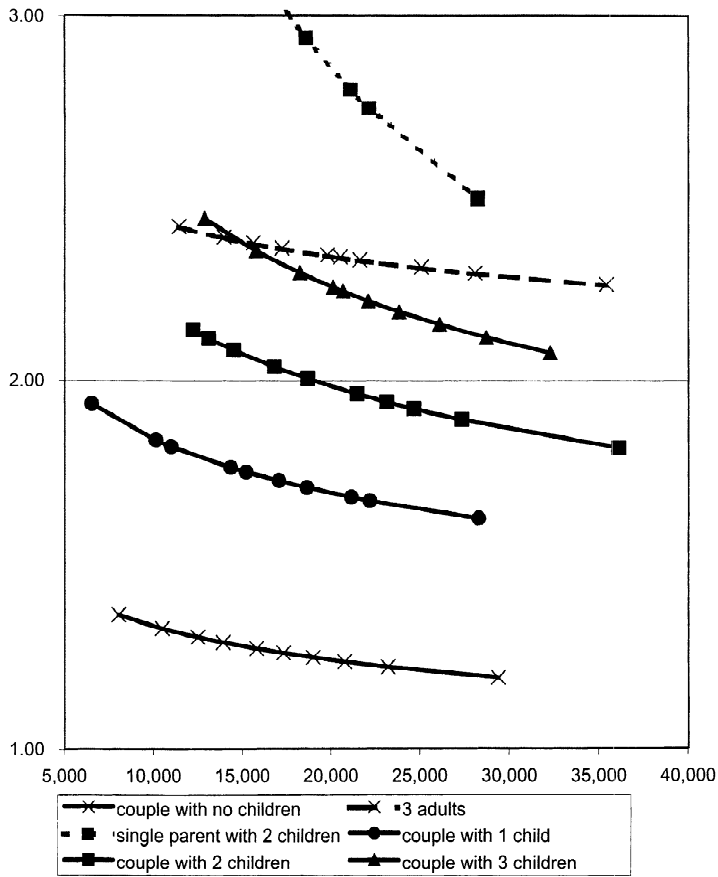


Fig. 2. Equivalence scales, K depends on p .

For households with children, the declines are marginally statistically significant and economically important. For example, consider the estimated equivalence scales for couples with two children. If K is assumed to be independent of p , then at low expenditure (fifth percentile) such households have an equivalence scale value of 1.62, and at high expenditure (ninety-fifth percentile) the value is 1.26. If K depends on p , then the equivalence scale is higher at all expenditure levels, with a value at low expenditure of 2.14 and a value at high expenditure of 1.81.

In contrast, the estimated equivalence scales for childless couples are insignificantly declining and the point estimates suggest a smaller change. If K is assumed to be independent of p , the equivalence scale ranges from 1.35 at low expenditure to 1.32 at high expenditure. If K depends on p , then the equivalence scale ranges from 1.36 at low expenditure to 1.19 at high expenditure. We note that

the imprecision of the estimates does not allow the conclusion that all childless households have equivalence scales which are flatter than all households with children. For example, from Table 4, we see that the difference between the equivalence-scale expenditure elasticities of couples with two children and childless couples is -0.05 . However, the standard error for this difference is 0.09 , yielding an insignificant test on the difference between these elasticities.

We take these results as suggesting that equivalence scales may be declining for households with children, and that these declines may be economically important in, for example, the measurement of poverty or inequality. Regarding the measurement of poverty, these results highlight the importance of using equivalence scales which reflect the needs of households at the bottom of the distribution of expenditures. The equivalence scale at the middle of the distribution reflects lower child costs than the equivalence scale at the bottom, and its use to scale poverty lines across household types would bias estimated poverty rates downwards. Regarding the measurement of inequality, equivalence scales which are downward sloping for many household types would imply more inequality for those household types than would equivalence scales which are independent of expenditure.

The expenditure dependence of equivalence scales may also affect tax/transfer policy. If, for example, equivalence scales were found to increase with expenditure, this might temper the optimal amount of progressivity for households with children. In contrast, if equivalence scales are declining with total expenditure, as we find, then the optimal amount of progressivity might be higher because rich households with children face lower (relative) costs of children than poor households do.

6.2. The demand for children's goods

In contrast to the case of ESE, the expenditure share for children's goods given GESE is affine in the logarithm of expenditure. Table 5 shows selected parameter estimates that reveal how the demand for children's clothing—denoted c —changes

Table 5
Expenditure responses in children's clothing shares

| Parameter | Estimate | Standard error |
|---------------------|------------------|----------------|
| b_c^f | 0 | @ |
| b_c^{coup} | 0 | @ |
| b_c^{nlhs} | 0 | @ |
| b_c^{ehs} | -0.00045^{***} | 0.00004 |
| b_c^{spar} | -0.00429^{***} | 0.00079 |
| b_c^{nage} | 0 | @ |
| b_c^{cage} | 0.00016^{***} | 0.00004 |

*** Significant at the 1% level; @ indicates a parameter that has been set rather than estimated.

with expenditure. Given GESE, the derivative of the children's clothing expenditure share equation with respect to $\ln x$ is equal to $b_c(z)$, the parameters of which are reported in the table.

The table shows that the demand for children's clothing responds fairly strongly to expenditure. For couples with children, the derivative of the expenditure share equation for children's clothing is significantly decreasing in household size and significantly increasing in the age of the household head. For a couple with two children where the household head is aged 40 (the reference age), the slope of the expenditure share equation in $\ln x$ is -0.0006 . Since the span of $\ln x$ is approximately 2, this means that the richest two-child households allocate 0.12 percentage points less of their expenditure to children's clothing than do the poorest two-child households. This suggests that the relaxation of restrictions on consumer demand imposed by ESE may be a valuable feature of GESE.

6.3. ESE, GESE and cost of living indices

As we have seen above, GESE structures both interpersonal comparisons of utility and demand behaviour. We may relax the interpersonal comparisons restriction, however, without affecting behaviour. In this case, GESE does not hold, but the restrictions it imposes on behaviour are maintained. In this environment, we cannot say anything about the equivalence scale, but we can assess how the behavioural restrictions associated with GESE affect another tool of welfare analysis, the cost-of-living index.

Blundell and Lewbel (1991) argue that although equivalence scales cannot be identified from behaviour without maintaining (arbitrary) assumptions about interpersonal comparisons, they do have an effect on something observable—they structure cost-of-living indices. Blundell and Lewbel define the relative equivalence scale, R , as the ratio of cost-of-living indices for a pair of household types at a given level of total expenditure: $R(p, x, z) = I(p, x, z)/I(p, x, z')$ where I is a cost-of-living index. We may define the cost-of-living index at a particular level of base-period expenditure x for a household with characteristics z as $I(p, x, z) = E(V(p^b, x, z), p, z)/x$, where p^b is the base price vector.

Table 6 gives the estimated cost-of-living index for the reference household type and the estimated relative equivalence scales for other types given the estimated demand systems under ESE, GESE and unrestricted QAI. They are reported for three levels of expenditure, approximately the 10th, 50th and 90th percentiles of the base-period total-expenditure distribution, and they evaluate the cost difference between base prices (Ontario 1986) and the price situation in Ontario in 1999. The top panel in gives the cost-of-living index for the reference household type at expenditure levels \$10 000, \$20 000 and \$30 000. The lower panel gives the estimated relative equivalence scales multiplied by 100 (if the reference and

Table 6
Cost of living indices and relative equivalence scales for Ontario 1999

| Household type | Exp | ESE | GESE ¹ | GESE ² | Unrest QAI |
|---|--------|--------|-------------------|-------------------|------------|
| Reference cost of living index for Ontario 1999 (Ontario 1986=100) | | | | | |
| Single adult: no children | 10 000 | 142.67 | 142.64 | 142.77 | 142.84 |
| Single adult: no children | 20 000 | 144.29 | 144.20 | 144.15 | 144.27 |
| Single adult: no children | 30 000 | 145.73 | 145.52 | 145.48 | 145.78 |
| Relative equivalence scales for Ontario 1999 (reference cost of living=100) | | | | | |
| Couple: no children | 10 000 | 98.88 | 98.87 | 98.93 | 99.02 |
| Couple: no children | 20 000 | 98.58 | 98.62 | 98.76 | 98.55 |
| Couple: no children | 30 000 | 98.41 | 98.48 | 98.71 | 98.17 |
| Couple: one child | 10 000 | 97.81 | 97.85 | 98.14 | 98.00 |
| Couple: one child | 20 000 | 97.54 | 97.56 | 97.53 | 97.43 |
| Couple: one child | 30 000 | 97.37 | 97.45 | 97.21 | 96.95 |
| Couple: two children | 10 000 | 97.27 | 97.33 | 97.75 | 97.47 |
| Couple: two children | 20 000 | 96.92 | 96.95 | 96.97 | 96.85 |
| Couple: two children | 30 000 | 96.71 | 96.79 | 96.56 | 96.29 |

1. K independent of p
2. K unrestricted

non-reference cost-of-living indices are identical, then the relative equivalence scale equals 100).¹⁷

The cost-of-living index reported for the reference household type shows that, given the unrestricted QAI estimates, costs in Ontario 1999 were 42.7 percent higher than they were in Ontario 1986 for households spending 10 000 dollars, 44.3 percent higher for households spending 20 000 dollars, and 45.7 percent higher for households spending 30 000 dollars. Obviously, the reference price index rises with total expenditure. This feature is reproduced whether or not ESE or GESE is imposed and, at all expenditure levels, the imposition of ESE or GESE does not seem to affect this cost-of-living index by much.

The relative equivalence scales give the ratios of cost-of-living indices for non-reference households. The key question is: does the imposition of ESE or GESE distort our assessment of relative equivalence scales? Taking the unrestricted QAI estimates as the 'true' cost-of-living indices and relative equivalence scales, looking across columns reveals how much ESE and GESE distort our assessment.

Considering first the relative equivalence scales reported for childless couples, at low expenditure, GESE pulls the estimated relative equivalence scale towards the unrestricted QAI estimates. However, at medium and high expenditure, this is

¹⁷We also made a table similar to Table 6 showing cost-of-living indices and relative equivalence scales for Ontario 1969. The results are similar in spirit but the proportional deviation of GESE relative equivalence scales is somewhat larger.

not the case. For households with children, however, the relaxation of ESE to GESE seems to help slightly in the estimation of accurate relative equivalence scales. At all expenditure levels for couples with one child and at high expenditure for couples with two children, the relative equivalence scales given GESE are closer to the unrestricted QAI relative equivalence scales than are those given ESE.

We see two main findings in the table. First, GESE seems to perform slightly better than ESE in getting close to the relative equivalence scales that come from the unrestricted QAI estimates. Second, neither the point estimates given ESE nor GESE exactly reproduce the cost-of-living indices (and relative equivalence scales) of the unrestricted model. Thus, there is some cost to imposing the GESE structure on consumer demand systems.

7. Conclusion

The equivalent-expenditure functions introduced in this paper have expenditure elasticities that are independent of expenditure but may depend on prices and household characteristics. This generalisation of equivalence-scale exactness (ESE), which we call generalised equivalence-scale exactness (GESE), ameliorates some of the difficulties associated with ESE: (1) the equivalence scale can depend on expenditure so that its value can be different for rich and poor; (2) the restriction on preferences implied by ESE is weakened; and (3) the demand functions for children's goods may have expenditure elasticities that are different from one.

If the reference expenditure function is not PIGLOG and GESE is a maintained hypothesis, the equivalence expenditure function and associated equivalence scale can be identified from demand behaviour (Theorems 2 and 3). We estimate both using Canadian data and find that equivalence scales given GESE are larger than those given ESE. In addition, our estimates suggest that equivalence scales for households with children decline significantly with expenditure, a result that may be of some importance in the measurement of poverty and inequality and in the determination of tax/transfer policy.

For poverty measurement, equivalence scales should reflect the needs of households at the bottom of the expenditure distribution. Because we find declining equivalence scales for households with children, this suggests that the use of equivalence scales from the middle of the distribution of well-being understates poverty rates. In the case of inequality measurement, equivalence scales that decrease with expenditure for some household types imply more inequality for those types than expenditure-independent equivalence scales do. In addition, equivalence scales that decline with expenditure increase the optimal amount of progressivity because rich households with children face lower (relative) costs of children than poor households do.

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Appendix A

Theorem 1. *Two indirect utility functions \hat{V} and \tilde{V} represent the same reference preferences and have the same associated equivalent-expenditure functions \hat{X} and \tilde{X} if and only if Eq. (2.7) holds for all $(p, x, z) \in \mathcal{R}_{++}^{m+1} \times \mathcal{L}$.*

Proof. If Eq. (2.7) holds for all $(p, x, z) \in \mathcal{R}_{++}^{m+1} \times \mathcal{L}$, then, using the definition of equivalent expenditure,

$$\begin{aligned}
 \hat{V}^r(p, \hat{X}(p, x, z)) &= \hat{V}(p, x, z) \\
 &= \phi(\tilde{V}(p, x, z)) \\
 &= \phi(\tilde{V}^r(p, \tilde{X}(p, x, z))) \\
 &= \hat{V}^r(p, \tilde{X}(p, x, z))
 \end{aligned}
 \tag{A.1}$$

for all (p, x, z) . Consequently, $\hat{X} = \tilde{X}$.

If \hat{V} and \tilde{V} represent the same reference preferences, there exists an increasing function ϕ such that $\hat{V}^r(p, x) = \phi(\tilde{V}^r(p, x))$ for all $(p, x) \in \mathcal{R}_{++}^{m+1}$. If, in addition, $\hat{X} = \tilde{X} = :X$, then, using the definition of equivalent expenditure,

$$\begin{aligned}
 \hat{V}(p, x, z) &= \hat{V}^r(p, X(p, x, z)) \\
 &= \phi(\tilde{V}^r(p, X(p, x, z))) \\
 &= \phi(\tilde{V}^r(p, x, z))
 \end{aligned}
 \tag{A.2}$$

for all $(p, x, z) \in \mathcal{R}_{++}^{m+1} \times \mathcal{L}$ which implies that Eq. (2.7) holds for all $(p, x, z) \in \mathcal{R}_{++}^{m+1} \times \mathcal{L}$. \square

Lemma 1. *If GESE is satisfied, κ is homogeneous of degree zero in p .*

Proof. Suppose that GESE is satisfied. Then Eq. (3.1) holds and

$$X(p, \lambda x, z) = \lambda^{\kappa(p, z)} X(p, x, z)
 \tag{A.3}$$

for all $\lambda > 0$. Because X is homogeneous of degree one in (p, x) ,

$$X(\mu p, \mu \lambda x, z) = \mu X(p, \lambda x, z) = \mu \lambda^{\kappa(p,z)} X(p, x, z), \quad (\text{A.4})$$

and

$$X(\mu p, \mu \lambda x, z) = \lambda^{\kappa(\mu p, z)} X(\mu p, \mu x, z) = \mu \lambda^{\kappa(\mu p, z)} X(p, x, z) \quad (\text{A.5})$$

for all $\lambda > 0$, $\mu > 0$. Consequently, $\kappa(\mu p, z) = \kappa(p, z)$ for all $\mu > 0$, and κ is homogeneous of degree zero in p . \square

Lemma 2. *If E^τ , G and K are differentiable in p ,*

$$K(p, z) = 1 - \sum_{j=1}^m \frac{\partial \ln G(p, z)}{\partial \ln p_j} \quad (\text{A.6})$$

for all $p \in \mathcal{R}_{++}^m$, $z \in \mathcal{Z}$.

Proof. Recall that if a function f is homogeneous of degree ρ ,

$$\sum_{j=1}^m w_j \frac{\partial f(w)}{\partial w_j} = \rho f(w), \quad (\text{A.7})$$

which implies

$$\sum_{j=1}^m w_j \frac{\partial \ln f(w)}{\partial w_j} = \sum_{j=1}^m \frac{\partial \ln f(w)}{\partial \ln w_j} = \rho. \quad (\text{A.8})$$

GESE implies that

$$\ln E(u, p, z) = K(p, z) \ln E^\tau(u, p) + \ln G(p, z). \quad (\text{A.9})$$

Differentiating,

$$\begin{aligned} \sum_{j=1}^m p_j \frac{\partial \ln E(u, p, z)}{\partial p_j} &= \sum_{j=1}^m p_j \frac{\partial K(p, z)}{\partial p_j} \ln E^\tau(u, p) + K(p, z) \sum_{j=1}^m p_j \frac{\partial \ln E^\tau(u, p)}{\partial p_j} \\ &\quad + \sum_{j=1}^m p_j \frac{\partial \ln G(p, z)}{\partial p_j}. \end{aligned} \quad (\text{A.10})$$

Because $E(u, \cdot, z)$ and $E^\tau(u, \cdot)$ are homogeneous of degree one and $K(\cdot, z)$ is homogeneous of degree zero, Eq. (A.10) becomes

$$1 = K(p, z) + \sum_{j=1}^m p_j \frac{\partial \ln G(p, z)}{\partial p_j} \quad (\text{A.11})$$

and, using Eq. (A.8), Eq. (A.6) results. \square

Proof of Theorem 2. In Eq. (4.7), set $p = \tilde{p}$ to get

$$\ln \hat{E}^r(\sigma(u, z), \tilde{p}) = H(\tilde{p}, z) \ln \hat{E}^r(u, \tilde{p}) + Q(\tilde{p}, z). \quad (\text{A.12})$$

Defining $h(z) = H(\tilde{p}, z)$, $\hat{f}(u) = \ln \hat{E}^r(u, \tilde{p})$, and $q(z) = Q(\tilde{p}, z)$,

$$\hat{f}(\sigma(u, z)) = h(z)\hat{f}(u) + q(z), \quad (\text{A.13})$$

so that

$$\sigma(u, z) = \hat{f}^{-1}(h(z)\hat{f}(u) + q(z)) = \phi(h(z)\hat{f}(u) + q(z)) \quad (\text{A.14})$$

where $\phi = \hat{f}^{-1}$. Therefore Eq. (4.7) becomes

$$\ln \hat{E}^r(\phi(h(z)\hat{f}(u) + q(z)), p) = H(p, z) \ln \hat{E}^r(u, p) + Q(p, z). \quad (\text{A.15})$$

Defining $w = \hat{f}(u)$ and $F(\cdot, p) = \ln \hat{E}^r(\phi(\cdot), p)$,

$$\ln \hat{E}^r(u, p) = \ln \hat{E}^r(\hat{f}^{-1}(w), p) = \ln \hat{E}^r(\phi(w), p) = F(w, p), \quad (\text{A.16})$$

and Eq. (A.15) becomes

$$F(h(z)w + q(z), p) = H(p, z)F(w, p) + Q(p, z) \quad (\text{A.17})$$

for any $p \in \mathcal{R}_{++}^m$, $z \in \mathcal{Z}$.

Suppressing p in Eq. (A.17) and defining the functions \check{f} , \check{h} , and \check{q} in the obvious way, Eq. (A.17) is

$$\check{f}(h(z)w + q(z)) = \check{h}(z)\check{f}(w) + \check{q}(z). \quad (\text{A.18})$$

Defining $a = h(z)$, $b = q(z)$, Eq. (A.18) becomes

$$\check{f}(aw + b) = \check{h}(z)\check{f}(w) + \check{q}(z). \quad (\text{A.19})$$

Set $w = 0$ in Eq. (A.19) to get

$$\check{f}(b) = \check{h}(z)\check{f}(0) + \check{q}(z) \quad (\text{A.20})$$

so

$$\check{q}(z) = \check{f}(b) - \check{c}\check{h}(z) \quad (\text{A.21})$$

where $\check{c} = \check{f}(0)$. Consequently, Eq. (A.19) can be written as

$$\begin{aligned} \check{f}(aw + b) &= \check{h}(z)\check{f}(w) + \check{f}(b) - \check{c}\check{h}(z) = \check{h}(z)(\check{f}(w) - \check{c}) + \check{f}(b) \\ &= \check{h}(z)\check{f}(w) + \check{f}(b) \end{aligned} \quad (\text{A.22})$$

where $\bar{f}(w) = \check{f}(w) - \check{c}$. Now set $w = 1$ in Eq. (A.22) to get

$$\check{f}(a + b) = \check{h}(z)\bar{f}(1) + \check{f}(b) \quad (\text{A.23})$$

which implies

$$\check{h}(z) = \hat{h}(a, b) \tag{A.24}$$

for some function \hat{h} . Consequently, Eq. (A.22) is

$$\check{f}(aw + b) = \hat{h}(a, b)\bar{f}(w) + \check{f}(b). \tag{A.25}$$

For any b , define

$$\hat{f}_b(aw) = \check{f}(aw + b) - \check{f}(b) \tag{A.26}$$

and

$$\hat{h}_b(a) = \hat{h}(a, b). \tag{A.27}$$

Then Eq. (A.25) can be written as

$$\hat{f}_b(aw) = \hat{h}_b(a)\bar{f}(w), \tag{A.28}$$

a Pexider equation (Eichhorn, 1978) whose solution is

$$\hat{h}_b(a) = \hat{C}(b)a^{R(b)}, \tag{A.29}$$

$$\bar{f}(w) = \bar{C}(b)w^{R(b)}, \tag{A.30}$$

and

$$\hat{f}_b(t) = \hat{C}(b)\bar{C}(b)t^{R(b)} = \tilde{C}(b)t^{R(b)} \tag{A.31}$$

where $\tilde{C}(b) = \bar{C}(b)\hat{C}(b)$.

\bar{f} is independent of b . Setting $w = 1$ in Eq. (A.30), this implies that \bar{C} is independent of b which, in turn, implies that R is independent of b as well. Defining $\rho = R(b)$ for all b , Eqs. (A.26) and (A.31) imply

$$\check{f}(aw + b) = \hat{f}_b(aw) + \check{f}(b) = \tilde{C}(b)(aw)^\rho + \check{f}(b). \tag{A.32}$$

Now set $z = z^\Gamma$, so that $a = h(z^\Gamma) = H(p, z^\Gamma) = 1$ and $b = q(z^\Gamma) = Q(p, z^\Gamma) = 0$ so that Eq. (A.32) implies

$$\check{f}(w) = \tilde{C}(0)w^\rho + \check{f}(0) = cw^\rho + d \tag{A.33}$$

where $c = \tilde{C}(0)$ and $d = \check{f}(0)$. Because \check{f} is increasing, $c \neq 0$ and $\rho \neq 0$.

Define $\zeta = aw$ so that Eq. (A.32) can be written as

$$\check{f}(\zeta + b) = \tilde{C}(b)\zeta^\rho + \check{f}(b). \tag{A.34}$$

This implies, given Eq. (A.33), that

$$c(\zeta + b)^\rho + d = \tilde{C}(b)\zeta^\rho + cb^\rho + d \tag{A.35}$$

and

$$c(\zeta + b)^\rho = \tilde{C}(b)\zeta^\rho + cb^\rho. \tag{A.36}$$

Because $c \neq 0$, Eq. (A.36) can be rewritten as

$$\frac{\tilde{C}(b)}{c} = \frac{(\zeta + b)^\rho - b^\rho}{\zeta^\rho} \tag{A.37}$$

for all $\zeta \neq 0$. Because the left side of Eq. (A.37) is independent of ζ , the right side must be as well. It is if $\rho = 1$ and, in that case, $\tilde{C}(b) = c$. To show that $\rho = 1$ is necessary, define $\psi(\zeta)$ to be the right side of Eq. (A.37) for any fixed value of b , $b \neq 0$. ψ is differentiable and, because it is independent of ζ , $\psi'(\zeta) = 0$ for all ζ . Because $\rho \neq 0$,

$$\begin{aligned} \psi'(\zeta) = 0 &\Leftrightarrow \rho(\zeta + b)^{\rho-1}\zeta^\rho - ((\zeta + b)^\rho - b^\rho)\rho\zeta^{\rho-1} = 0 \\ &\Leftrightarrow (\zeta + b)^{\rho-1}\zeta - (\zeta + b)^\rho + b^\rho = 0 \\ &\Leftrightarrow (\zeta + b)^{\rho-1}(\zeta - \zeta - b) + b^\rho = 0 \\ &\Leftrightarrow (\zeta + b)^{\rho-1} = b^{\rho-1}. \end{aligned} \tag{A.38}$$

Because Eq. (A.38) is true for all ζ , $\rho = 1$. As a consequence, $\tilde{C}(b) = c$. This, together with Eq. (A.34), implies

$$\check{f}(w) = cw + d. \tag{A.39}$$

Because \check{f} is increasing, $c > 0$.

In the functional equation, p was suppressed so c and d can depend on p . Because $w = \hat{f}(u)$ and $\check{f}(w) = F(w, p) = \ln \hat{E}^r(\hat{f}^{-1}(w), p) = \ln \hat{E}^r(u, p)$, Eq. (A.39) implies that

$$\ln \hat{E}^r(u, p) = C(p)\hat{f}(u) + \ln D(p) \tag{A.40}$$

which is Eq. (4.8), and its necessity is established. Eq. (4.9) follows from Eqs. (A.40) and (4.2) with $\tilde{f}(u) = \psi(\hat{f}(u), z^r)$. Because $c > 0$ and f is increasing, $C(p)$ must be positive. Homogeneity properties of C and D follow from the fact that \hat{E}^r is homogeneous of degree one in p .

Eq. (A.15) requires

$$\ln \hat{E}^r(\phi(h(z)f(u) + q(z)), p) = H(p, z) \ln \hat{E}^r(u, p) + Q(p, z). \tag{A.41}$$

Because of Eq. (A.40) and the fact that $\phi = \hat{f}^{-1}$, this becomes

$$C(p)[h(z)\hat{f}(u) + q(z)] + \ln D(p) = H(p, z)[C(p)\hat{f}(u) + \ln D(p)] + Q(p, z), \tag{A.42}$$

so

$$C(p)h(z)\hat{f}(u) + C(p)q(z) + \ln D(p) = H(p, z)C(p)\hat{f}(u) + H(p, z) \ln D(p) + Q(p, z). \tag{A.43}$$

It follows that

$$H(p, z) = h(z) \tag{A.44}$$

which results in Eq. (4.10). Because $K(p, z) \in \mathcal{R}_{++}$, $h(z) \in \mathcal{R}_{++}$ for all $z \in \mathcal{Z}$ and, because $\hat{K}(p, z^r) = \tilde{K}(p, z^r) = 1$, $h(z^r) = 1$.

We know from Eq. (A.19) that

$$\check{f}(aw + b) = \check{h}(z)\check{f}(w) + \check{q}(z). \tag{A.45}$$

Because $\check{f}(t) = ct + d$ and the fact that $\check{h}(a) = h(z) = a$ (from Eq. (A.44)),

$$c[aw + b] + d = a[cw + d] + \check{q}(z), \tag{A.46}$$

so, because $b = q(z)$,

$$\check{q}(z) = cq(z) + d[1 - a], \tag{A.47}$$

and, because $\check{q}(z) = Q(p, z)$ with p suppressed,

$$Q(p, z) = C(p)q(z) + \ln D(p)[1 - h(z)] \tag{A.48}$$

which implies Eq. (4.11). Because, in Eq. (4.11), $\hat{G}(p, z^r) = \tilde{G}(p, z^r) = 1$, $\tilde{K}(p, z^r) = 1$ and $h(z^r) = 1$, $q(z^r) = 0$.

Sufficiency can be checked as follows. Suppose that Eqs. (4.8)–(4.11) and GESE are satisfied, with

$$\begin{aligned} \ln \hat{E}(u, p, z) &= \hat{K}(p, z) \ln \hat{E}^r(u, p) + \ln \hat{G}(p, z) \\ &= h(z)\tilde{K}(p, z)[C(p)\hat{f}(u) + \ln D(p)] + \ln \tilde{G}(p, z) \\ &\quad + \tilde{K}(p, z)[C(p)q(z) + \ln D(p)(1 - h(z))] \\ &= \tilde{K}(p, z)[C(p)[h(z)\hat{f}(u) + q(z)] + \ln D(p)] + \ln \tilde{G}(p, z) \\ &= \tilde{K}(p, z) \ln \tilde{E}^r(\tilde{f}^{-1}(h(z)\hat{f}(u) + q(z)), p) + \ln \tilde{G}(p, z) \\ &= \tilde{E}(\tilde{f}^{-1}(h(z)\hat{f}(u) + q(z)), p, z) \end{aligned} \tag{A.49}$$

which satisfies Eq. (4.3), implying that \hat{E} and \tilde{E} represent the same preferences. \square

Proof of Theorem 3. Because $\hat{K} = \tilde{K}$, $H(p, z) = 1$ for all $z \in \mathcal{Z}$. In the proof of Theorem 2, this implies $h(z) = \check{h}(z) = 1$ for all $z \in \mathcal{Z}$, and Eq. (A.19) becomes

$$\check{f}(w + b) = \check{f}(w) + \check{q}(z). \tag{A.50}$$

Set $w = 0$ to get

$$\check{q}(z) = \check{f}(b) - \check{f}(0), \tag{A.51}$$

and rewrite Eq. (A.50) as

$$\check{f}(w + b) = \check{f}(w) + \check{f}(b) - \check{f}(0), \tag{A.52}$$

which is equivalent to

$$[\check{f}(w + b) - \check{f}(0)] = [\check{f}(w) - \check{f}(0)] + [\check{f}(b) - \check{f}(0)]. \tag{A.53}$$

Defining $\mathring{f}(t) = \check{f}(t) - \check{f}(0)$, Eq. (A.53) becomes

$$\mathring{f}(w + b) = \mathring{f}(w) + \mathring{f}(b), \tag{A.54}$$

a Cauchy equation whose solution is (Eichhorn, 1978)

$$\mathring{f}(w) = cw \tag{A.55}$$

for some $c \in \mathcal{R}$. As a consequence,

$$\check{f}(w) = cw + \check{f}(0) = :cw + d \tag{A.56}$$

as in Eq. (A.39). The rest of the proof, including sufficiency, follows from the proof of Theorem 2. \square

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