Consumer Demand and the Cost of Living

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Consumer demand systems?

- A consumer demand system is the relationship

$$w_i^j = w^j(\mathbf{p}_i, x_i, \mathbf{z}_i) \tag{1}$$

  where $i = 1, ..., N$ indexes households, $j = 1, ..., J$ indexes commodities; $\mathbf{p}_i = [p_i^1, ..., p_i^J]'$ is the **price vector**, x_i is the **total-expenditure** (aka: "budget", "income") level, $\mathbf{z}_i = [z_i^1, ..., z_i^T]'$ is a $T - vector$ of **demographic characteristics**, and w_i^j is the budget share of (share of total expenditure commanded by) the j'th commodity for the i'th household. Denote $\boldsymbol{\epsilon}_i = [\boldsymbol{\epsilon}_i^1, ..., \boldsymbol{\epsilon}_i^J]'$ as an idiosyncratic effect—an error term—for the j'th expenditure share for the i'th household. We want to estimate the functions $w^j(\mathbf{p}, x, \mathbf{z})$.
- For convenience and terseness, one may use matrix notation and suppress the subscript i, writing

$$\mathbf{w} = \mathbf{w}(\mathbf{p}, x, \mathbf{z})$$

How are they useful?

Without Integrability

- predict the effects of policies that change prices, eg, sales tax changes.

With Integrability

- demands $w^j(\mathbf{p}, x, \mathbf{z})$ can inform us about cost and indirect utility functions.
- $C(\mathbf{p}, f(u, \mathbf{z}), \mathbf{z})$ the cost function giving the minimum cost of utility $f(u)$; $f(V(\mathbf{p}, x, \mathbf{z}), \mathbf{z})$ the indirect utility function
  - The function f is monotonically increasing but unobservable; just there to emphasise the ordinal nature of the utility function.
  - I will suppress f hereafter.

Uses

- household (type) specific price indices for poverty, inequality and social welfare measurement.

$$C(p_1, V(p_0, x, z), z) / C(p_0, V(p_0, x, z), z)$$

- Macroeconomic policy, vis the 'inflation rate' (for average or all households) (see Crossley and Pendakur 2009).
- consumer surplus (CV, EV) calculation

$$C(p_1, V(p_1, x, z), z) - C(p_1, V(p_0, x, z), z)$$
$$C(p_0, V(p_1, x, z), z) - C(p_0, V(p_0, x, z), z)$$

- reveal inter-household comparisons of well-being from behaviour—measure equivalence scales

$$C(p, u, z_1) / C(p, u, z_0)$$

- Integrability of estimated consumer demand system allows recovery of the cost function (up to f).

Integrability

- means that the consumer demand system can be generated by differentiating an indirect utility or cost function:
  - Sheppard's Lemma:

$$\omega^j(\mathbf{p}, u, \mathbf{z}) = \frac{\partial \ln C(\mathbf{p}, u, \mathbf{z})}{\partial \ln p^j} \tag{2}$$

  (replace u in ω^j with $V(\mathbf{p}, x, \mathbf{z}) \equiv C^{-1}(p, \cdot, z)$), or
  - Roy's Identity uses V to derive $\frac{\partial \ln C(\mathbf{p},u,\mathbf{z})}{\partial \ln p^j}$ as an implicit function: $\partial c/\partial p = \partial x/\partial p = -\partial V/\partial p/\partial V/\partial x$:

$$w^j(\mathbf{p}, x, \mathbf{z}) = -\frac{\frac{\partial V(\mathbf{p},u,\mathbf{z})}{\partial \ln p^j}}{\frac{\partial V(\mathbf{p},u,\mathbf{z})}{\partial \ln x}} \tag{3}$$

- If you faced a 10% increase in the price of, say, rent, and rent commanded half your budget, how much would your costs rise? The natural answer is 10%*50%=5%. This natural answer is due to the fact that for a small price change, we do not adjust our consumption choices: we just need 5% more money to buy exactly what we bought before. Essentially, this insight (going backwards) gives us Sheppard's Lemma. The proportionate change in cost due to a small price increase is equal to the budget share of that good.

More Integrability

- Properties of C (V) imply properties on w^j, typically testable properties.
- integrability requires **homogeneity** (cost is HD1 in prices). This implies that absence of money illusion.
  - homogeneity implies that
    $$w^j(\lambda \mathbf{p}, \lambda x, \mathbf{z}) = w^j(\mathbf{p}, x, \mathbf{z})$$
  - this is equivalent to a condition on the derivatives of demand equations
    $$\sum_{k=1}^{J} \frac{\partial w^j(\mathbf{p}, x, \mathbf{z})}{\partial \ln p^k} + \frac{\partial w^j(\mathbf{p}, x, \mathbf{z})}{\partial \ln x} = 0. \quad (4)$$
- integrability requires **symmetry**. Define the Slutsky Matrix as:
  $$\Gamma^{jk}(\mathbf{p}, x, \mathbf{z}) = \frac{\partial w^j(\mathbf{p}, x, \mathbf{z})}{\partial \ln p^k} + \frac{\partial w^j(\mathbf{p}, x, \mathbf{z})}{\partial \ln x} w^k(\mathbf{p}, x, \mathbf{z}).$$
  Symmetry is satisfied if and only if
  $$\Gamma^{jk}(\mathbf{p}, x, \mathbf{z}) = \Gamma^{kj}(\mathbf{p}, x, \mathbf{z})$$

More Integrability

- integrability requires **concavity**–the Slutsky matrix given by the following elements is negative semidefinite (cost is concave in prices):

$$\Gamma^{jk}(\mathbf{p}, x, \mathbf{z}) + w^j(\mathbf{p}, x, \mathbf{z})w^k(\mathbf{p}, x, \mathbf{z}) - d^{jj}w^j(\mathbf{p}, x, \mathbf{z}) \quad (5)$$

  where d^{jj} indicates $j = k$.

- We may state these conditions in matrix form as:
  - homogeneity:

$$\mathbf{w}(\lambda\mathbf{p}, \lambda x, \mathbf{z}) = \mathbf{w}(\mathbf{p}, x, \mathbf{z}).$$

  - Let $\mathbf{\Gamma}(\mathbf{p}, x, \mathbf{z}) = \nabla_{\ln p}\mathbf{w}(\mathbf{p}, x, \mathbf{z}) + \nabla_{\ln x}\mathbf{w}(\mathbf{p}, x, \mathbf{z})\mathbf{w}(\mathbf{p}, x, \mathbf{z})'$. Symmetry is satisfied if and only if

$$\mathbf{\Gamma}(\mathbf{p}, x, \mathbf{z}) = \mathbf{\Gamma}(\mathbf{p}, x, \mathbf{z})'$$

    .

  - concavity is satisfied if and only if

$$\mathbf{\Gamma}(\mathbf{p}, x, \mathbf{z}) + \mathbf{w}(\mathbf{p}, x, \mathbf{z})\mathbf{w}(\mathbf{p}, x, \mathbf{z})' - diag(\mathbf{w}(\mathbf{p}, x, \mathbf{z}))$$

    is negative semidefinite.

Integrability Matters

- Without integrability, the surplus measures, cost-of-living indices are not uniquely identified:
  - that is, you could get any cost-of-living index for a big price change, just by choosing a suitable path of little price changes.
- With integrability, the surplus measures and cost-of-living indices are uniquely identified from demands; the cost and indirect utility functions are identified up to a monotonic transformation of utility.
  - cost-of-living changes associated with a big price change are 'path-independent'.

Parametric estimation

- Ignore z for a while.
- Can estimate via specifying functional form for demand equations, eg,

$$w^j(\mathbf{p}, x) = a^j + \sum_{k=1}^{J} b^{jk} \ln p^k + b^{jx} \ln x,$$

  where a^j, b^{jk} and b^{jx} are parameters estimated by OLS.
- Fine for prediction. Not fine for doing surplus or welfare measurement.

Homogeneity

- homogeneity not too bad:

$$b^{jx} = -\sum_{k=1}^{J} b^{jk}.$$

It is a linear restriction in each equation.

  - could impose this restriction via substitution:

$$w^j(\mathbf{p}, x) = a^j + \sum_{k=1}^{J} b^{jk} \left(\ln p^k - \ln x \right),$$

where we simply substitute the restriction into the equation.

Symmetry

- symmetry implies

$$\begin{aligned} & b^{jk} + b^{jx}\left(a^k + \sum_{s=1}^{J} b^{ks}\ln p^s + b^{kx}\ln x\right) \\ = \quad & b^{kj} + b^{kx}\left(a^j + \sum_{s=1}^{J} b^{js}\ln p^s + b^{jx}\ln x\right) \end{aligned}$$

  which is a mess of cross-equation nonlinear restrictions, which is hard to substitute in.

- concavity is worse—a set of nonlinear cross-equation inequality restrictions.

A Good Trick

- Instead of making demand equations easy to look at, focus on utility or cost.
- Choosing a parametric function for cost or indirect utility and differentiate to get implied parametric functions for expenditure shares.
- eg, Deaton and Muelbauer's (1980) Almost Ideal (AI) model

$$\ln C(\mathbf{p}, u) = \ln a(\mathbf{p}) + b(\mathbf{p}) f(u) \quad (6)$$

where f is an unknown transformation, a is homogeneous of degree 1 and b homogeneous of degree 0.

  - gives via Sheppard's Lemma and substitution

$$w^j(\mathbf{p}, x) = \frac{\partial \ln a(\mathbf{p})}{\partial \ln p^j} + \frac{\partial \ln b(\mathbf{p})}{\partial \ln p^j} (\ln x - \ln a(\mathbf{p})) \quad (7)$$

  - homogeneity? did it on a and b.
  - symmetry? properties inherited in derivatives.
  - concavity? C is concave in p if a is concave in p. Easy to check, harder to impose, but do-able (Ryan and Wales 1999).

Almost Ideal

- Parametric structure
  - let a be a translog and b is cobb-douglas in prices.

$$\ln a(\mathbf{p}) = \sum_{k=1}^{J} a^k \ln p^k + \frac{1}{2} \sum_{k=1}^{J} \sum_{l=1}^{J} a^{kl} \ln p^k \ln p^l$$
$$\ln b(\mathbf{p}) = \sum_{k=1}^{J} b^k \ln p^k$$

then get

$$w^j(\mathbf{p}, x) = a^j + \sum_{k=1}^{J} a^{jk} \ln p^k + b^j \ln \overline{x} \qquad (8)$$

where

$$\ln \overline{x} = \ln x - \sum_{k=1}^{J} a^k \ln p^k + \frac{1}{2} \sum_{k=1}^{J} \sum_{l=1}^{J} a^{kl} \ln p^k \ln p^l.$$

  - 'slightly' nonlinear because a stuff gets multiplied by b stuff.

- Homogeneity is satisfied if $\sum_{k=1}^{J} a^k = 1$ and $\sum_{k=1}^{J} a^{jk} = 0$ for all j (makes $\ln \overline{x}$ HD0 by construction).
    - Could impose this restriction by substitution. The last element of a in each row is given by $a^{jJ} = -\sum_{k=1}^{J-1} a^{jk}$. So, you could instead write

$$w^j(\mathbf{p}, x) = a^j + \sum_{k=1}^{J-1} a^{jk} \left(\ln p^k - \ln p^J \right) + b^j \ln \overline{x}$$

      which imposes the restriction directly.
- Symmetry requires $a^{jk} = a^{kj}$ for all j, k.
    - in matrix notation, we have

$$\ln C(\mathbf{p}, u) = \mathbf{a}'\mathbf{p} + \frac{1}{2} \sum_{k=1}^{J} \sum_{l=1}^{J} \ln \mathbf{p}'\mathbf{A} \ln \mathbf{p} + \exp(\mathbf{b}' \ln \mathbf{p}) f(u),$$

      where $\iota'\mathbf{a} = 1$, $\iota'\mathbf{A} = 0_J$, $\mathbf{A} = \mathbf{A}'$, $\iota'\mathbf{b} = 0$.

Estimating Demand, Cost and Indirect Utility

- What can you do with this? Estimate demand, and reconstruct cost and indirect utility.
- What are the data?
- Aggregate or micro data on how households allocate expenditure across goods when facing different budget constraints define by prices and total expenditure.
- Assume that
$$w_i^j = w^j(\mathbf{p}_i, x_i, \mathbf{z}_i) + \epsilon_i^j$$
where ϵ_i^j is the disturbance term for individual i in equation j. Assume that these disturbances are exogenous, which implies that $E[\epsilon_i^j] = 0$ for all $j = 1, ..., J$.
- Since expenditure shares sum to 1, we can recover the function w^J if we have $w^1, .., w^{J-1}$ in our pocket.
- Thus, we only ever estimate $J - 1$ equations: $w^1, .., w^{J-1}$.

Estimating the Almost Ideal DS

- Given the AI demand system with no demographics, we get

$$w_i^j = a^j + \sum_{k=1}^{J} a^{jk} \ln p_i^k + b^j \ln \overline{x}_i + \epsilon_i^j \tag{9}$$

where

$$\ln \overline{x}_i = \ln x_i - \sum_{k=1}^{J} a^k \ln p_i^k - \frac{1}{2} \sum_{k=1}^{J} \sum_{l=1}^{J} a^{kl} \ln p_i^k \ln p_i^l.$$

- in matrix notation, this is

$$\begin{aligned} w_i &= \mathbf{a} + \mathbf{A} p_i + \mathbf{b} \ln \overline{x}_i + \epsilon_i, \\ \ln \overline{x}_i &= \ln x_i - \mathbf{a}'\mathbf{p}_i - \frac{1}{2} \ln \mathbf{p}_i' \mathbf{A} \ln \mathbf{p}_i, \end{aligned}$$

which is nicer to look at.

- Estimate (8) either by nonlinear methods (LS/ML or GMM), or iterative linear methods (Blundell and Robin 1999).

- Define $\epsilon_i = [\epsilon_i^1, ..., \epsilon_i^{J-1}]'$, and let $\Omega = E[\epsilon_i \epsilon_i']$, the expectation of the variance of disturbances across equations for an individual. All methods are system methods, because we have a system of equations. Nonlinear SUR subject to cross-equation symmetry restrictions:

$$\begin{aligned} & \min_{a^j, a^{jk}, b^j} \sum_{i=1}^{N} \epsilon_i' \Omega \epsilon_i' \\ st\ a^{jk} &= a^{kj} \text{ for all } j, k \end{aligned}$$

given an estimate of $\widehat{\Omega}$ such that plim $\widehat{\Omega} = \Omega$. Minimise sum of squared errors to get parameters subject to restrictions.

Hatting it Up

- This gives you a bunch of estimates $\widehat{a}^j, \widehat{a}^{jk}, \widehat{b}^j$, which imply a log-cost function

$$\begin{aligned} \ln C(\mathbf{p}, u) &= \ln a(\ln \mathbf{p}) + b(\ln \mathbf{p}) f(u) \\ &= \sum_{k=1}^{J} \widehat{a}^k \ln p^k + \frac{1}{2} \sum_{k=1}^{J} \sum_{l=1}^{J} \widehat{a}^{kl} \ln p^k \ln p^l + \prod_{j=1}^{J} \left(p^k\right)^{\widehat{b}^k} f(u) \end{aligned}$$

  this thing is all 'hatted up'—it is full of numbers now.
- *real expenditure* $\ln x^R = \ln R(\mathbf{p}, x)$ is the expenditure you need when facing some particular price vector $\overline{\mathbf{p}}$ to get the same utility as with x facing $\mathbf{p}$.
- Choose $\overline{\mathbf{p}} = 1_J$ for convenience.

$$\ln x^R = \ln R(\mathbf{p}, x) = \ln C(\overline{\mathbf{p}}, f(u))$$

The Cost-of-Living Index

- Then, log-cost at $\overline{\mathbf{p}}$ is given by

$$\begin{aligned}
\ln C(\overline{p}, f(u)) &= \ln C(\overline{\mathbf{p}}, f(V(\mathbf{p}, x))) \\
&= \ln a(\mathbf{p}) + b(\mathbf{p}) V(\mathbf{p}, x) \\
&= 0 + 1 \cdot f(V(\mathbf{p}, x)
\end{aligned}$$

indirect utility V is found by inverting C around u:

$$\ln C(\mathbf{p}, u) = \ln a(\mathbf{p}) + b(\mathbf{p}) f(u) \Leftrightarrow \quad (10)$$

$$f(V(\mathbf{p}, x) = \frac{\ln x - \ln a(\mathbf{p})}{b(\mathbf{p})} \quad (11)$$

Familiar? $f(V(\mathbf{p}, x)) = \ln \overline{x}$.

So,

$$\ln x^R = \ln R(\mathbf{p}, x) = \frac{\ln x - \ln a(\mathbf{p})}{b(\mathbf{p})}.$$

- The cost-of-living index I is the ratio of expenditure to real expenditure:

$$\begin{aligned}\ln I(\mathbf{p}, x) &= \ln x - \ln x^R \\ &= \ln x - \frac{\ln x - \ln a(\mathbf{p})}{b(\mathbf{p})} \\ &= \ln x - \frac{\ln x - \sum_{k=1}^{J} \widehat{a}^k \ln p^k + \frac{1}{2} \sum_{k=1}^{J} \sum_{l=1}^{J} \widehat{a}^{kl} \ln p^k \ln p^l}{\prod_{j=1}^{J} \left(p^k\right)^{\widehat{b}^k}}.\end{aligned}$$

- The log-cost-of-living is
  - computable
  - linear in $\ln x$
  - independent of $\ln x$ if and only if $\widehat{b}^k = 0$ for all k.
  - Canada produces us a single cost-of-living index for all people at all expenditure levels. Is this right?

Quadratic Extension to the AI

- Quadratic Almost Ideal (QAI) model

$$\ln C(\mathbf{p}, u) = \ln a(\mathbf{p}) + \frac{b(\mathbf{p})u}{1 + q(\mathbf{p})u} \tag{12}$$

  - gives budget shares with $\ln \overline{x} \equiv \ln x - \ln a(p)$

$$w^j(p,x) = \frac{\partial \ln a(\mathbf{p})}{\partial \ln p^j} + \frac{\partial \ln b(\mathbf{p})}{\partial \ln p^j} \ln \overline{x} + \frac{\partial q(\mathbf{p})}{\partial \ln p^j} \frac{(\ln \overline{x})^2}{b(\mathbf{p})}. \tag{13}$$

  - It really uglied up just to get that quadratic term. Slutsky symmetry is doing the uglifying.
  - Gorman (1963) showed that if budget shares are additive in functions of expenditure, then all J budget shares may depend on at most 3 functions of expenditure. So, quadratics are the limit. *Rank* may be defined as the dimension of the nonlinear subspace of expenditure occupied by the budget share equations (see Lewbel 1993). So, this is referred to as the *rank 3 limit*.

Parametric Estimation of Cost-of-Living Indices

- Pendakur (2001, 2002) uses the QAI to estimate poverty and inequality with household-specific price indices.
- This yields a price index which is different for rich and poor.
- In Canada, use of the QAI price index yields a different picture of inequality changing over time
  - standard indices showing falling inequality over late 1970s; expenditure dependent indices show rising inequality.
  - standard indices show slightly rising inequality over 1990s; expenditure dependent indices show slightly falling inequality.
- Donaldson (1992) showed that the inequality and poverty indices derived from 'real expenditure' calculated in this way depends on the choice of base price vector.
  - david: 'There is no substitute for utility'.
  - I agree, but that means we have to somehow identify f, which cardinalises utility.

Nonparametric Estimation of Engel Curves

- Rather than specifying a functional form and estimating its parameters, the idea in nonparametric estimation is to 'let the data speak for themselves' as to the shape of functions.
- local mean nonparametric estimation uses *nearby* data to estimate the height of the regression curve at every point.
- Pendakur (1999) and Blundell, Duncan and Pendakur (1998) use the 'shape invariance' restrictions to incorporate demographics into nonparametric estimation of Engel curves ($w^j(x)$) and to estimate equivalence scales.
  - local mean estimator is

$$w^j(x) = \frac{\sum_{i=1}^{N} K_h(x_i - x) w_i^j}{\sum_{i=1}^{N} K_h(x_i - x)}.$$

  - K_h is kernel function, eg, $K_h(u) = \varphi(\frac{u}{h})$ where φ is the standard normal density function.
  - At each x, $w^j(x)$ is the WLS estimate with w^j on the LHS and a constant on the RHS with weights $K_h(x_i - x)$.

- Blundell, Duncan and Pendakur show that partially additive effects are NOT integrable.
- get nonparametric estimates of Engel curves for each demographic type. Find the closest shape-invariance satisfying engel curves for all demographic types.
- maintaining the assumption of ESE implies shape-invariance, which is sufficient to identify equivalence scales even without imposing parametric structure on Engel curves.
- Pendakur (2004) uses similar restrictions to estimate *lifetime* equivalence scales without imposing parametric structure.

Nonparametric Estimation of Consumer Demand Systems

- an Engel curve is not a demand system. We need shares over $\mathbf{p}, x, \mathbf{z}$, and we need to impose integrability to get prices indices.
- Haag, Hoderlein and Pendakur (2009) develops a methodology for nonparametric estimation of integrable consumer demand systems and price indices.
- The trick is to use local polynomial estimation.
- Here, at each point in the domain, $w^j(\mathbf{p}, x, \mathbf{z})$ is the WLS estimate of a model with W^j on the LHS and a polynomial in $\mathbf{p}, x, \mathbf{z}$ on the RHS with weights given by the K_h.
- Recall that integrability is a set of conditions on derivatives. These are hard to impose on local mean estimators, but easy to impose on local polynomials.
  - The $r'th$ derivative of a local polynomial is just the coefficient on the $r'th$ term.

Where should we go?

- Measurement error (eg, Chesher et al, Review of Economic Studies, 2002) remains a big problem.
- Preference Heterogeneity (Lewbel and Pendakur 2009, 2015)
- non- and semi-parametric flexibility (Lewbel and Pendakur 2009; Haag, Hoderlein and Pendakur 2009; Pendakur and Sperlich 2009).
- Lack of price variation means price effects are largely gotten from their indirect rather than direct effects—in QAI, price effects come from intercepts and not from price derivatives.
- Cardinalise utility. Money metrics are not a good substitute for utility (Donaldson 1992; Roberts 1980), but we need to know the shape of utility over expenditure.
- Are nonparametrics really worth the trouble? I suspect that the best use of all this nonparametric demand stuff is as a diagnostic of parametric models.
  - Nonparametric estimators are local estimators only.
  - Nonparametric estimators require more data than we typically have.