BUS 316

NOTES AND ANSWERS
BINOMIAL OPTION PRICING

3. Suppose there are only two possible future states of the world. In state 1 the stock price rises by 50%. In state 2, the stock price drops by 25%. The current stock price S(0) = \$50. If a call option has an exercise price of \$50 and the risk-free rate (r) for the period is 5%: (a) Calculate the call option hedge ratios; (b) Use the binomial option pricing model to value the call option.

This question provides a good introduction to binomial option pricing. For more indepth discussion see Dubofsky, Options and Financial Futures (Chapter 6) or Chance (5th ed. Chapter 4)

The binomial model starts with the (one stage) BINOMIAL PROCESS for the stock price. From the values given this can be specified as:

S+ = Su   C+ = max[0, S+ - X]   X = \$50
\$75   = \$25

S(0) = \$50

S- = Sd   C- = max[(S-) - X]   X = \$50
\$37.50   = 0

In this scheme, S+ = Su is the stock price following an up move and S- = Sd is the stock price following a down move. In this model the option expires at the end of the first stage, so C+ (C-) is the expiration date call option value associated with an up (down) move.

Calculating the Hedge Ratio:

The portfolio of interest combines a certain number of units of stock with a written call option. As the cash outflow from the stock purchase will be greater than the cash inflow from the written option position, this portfolio will have a positive value.

NOTE: The hedge ratio can be interpreted in two different ways (see p. 389-90 of the text), as the number units of stock to purchase to hedge a written call, or the number of units of call options to write to hedge a share of stock. (If the latter approach is used, the portfolio value equation is V(t) = S(t) - (hr) C(t)).

In both cases, the objective is to determine a hedge ratio which will produce the same value for the portfolio in all future states of the world. In other words, the portfolio is hedged when its value does not change as the random variable, the stock price, changes.

Let $V(0) = (HR)\ S(0) - C(0)$

where:

$HR = (C+ - (C-)) \div (S+ - (S-)) = (25 - 0) \div (75 - 37.50) = 25 \div 37.5 \quad (25/37.5)$

Checking the Hedge Ratio:

$V(1)+ = (25/37.5)\ 75 - 25 = HR\ S+ - (C+) = 25$
$V(1)- = (25/37.5)\ 37.5 - 0 = HR\ (S-) - (C-) = 25$

The value of the portfolio is the same in both future states of the world.

Determining the Call Price:

It follows that because the value of the hedge portfolio is the same in all future states of the world, the value of the portfolio at t = 0 is just the value at t=1 discounted at the riskless rate of interest:

$V(0) = 25/ 1.05 = HR\ S(0) - C(0)$

Hence: $C(0) = (25/37.5)\ (50) - (25/1.05) = \$9.52$

Determining the Probabilities:

The binomial pricing problem embeds probabilities for the binomial process. This follows because the stock price at t = 0 has to be related to the values at t = 1. More precisely:

$S(0) = \{ Prob(up)\ (S+) + Prob(down)\ (S-)\} / (1 + r)$

Observing that in the binomial model $Prob(up) + Prob(down) = 1$, it follows

$50\ (1.05) = (1 - Prob(down))\ 75 + Prob(down)\ 37.5$

Solving gives $Prob(down) = .6 \quad (1 - Prob(down)) = .4 = Prob(up)$

Verifying the Price of C(0):

$C(0) = \{Prob(up)\ C+ + Prob(down)\ C-\}/ (1 + r)$

$= (.4\ (25) + 0) / 1.05 = \$9.52$

The value of C(0) obeys the valuation equation because the valuation equation was the basis upon which the HR was determined.

EXERCISE: Redo the question assuming that the interest rate is continuous. (This was the approach used in tutorial).

TWO STAGE BINOMIAL:

NOTE: In the two stage binomial, the stock price process is permitted to undergo two changes or moves. Unlike the one stage binomial, there are two types of outcomes which can occur. If the process is **recombining** then the outcome from an up and then down move will give the same result as from a down and then up move. In a recombining process, there will be three outcomes at the end of two steps. If the process does not recombine then there will be four outcomes. The question below is for a recombining process.

As the number of stages increases, recombining processes are decidedly easier to handle as the number of final nodes increases less quickly than for a non-recombining process.

Question from Assignment:

1. You are currently in period 0. Consider the binomial option pricing model when the stock price is permitted to progress **two** periods into the future. The current (period 0) stock price is \$100. The stock price evolves by either rising 50% or dropping by 25% each period. The risk free interest rate for each period is 10%. Assume that a European call is written on this stock with exercise price X = \$120 and expiration date at the end of period 2.

a) What are all the possible values for the stock price at the end of the first period and at the end of the second period? (Hint: It would be easiest to write down the appropriate two-step binomial tree.)

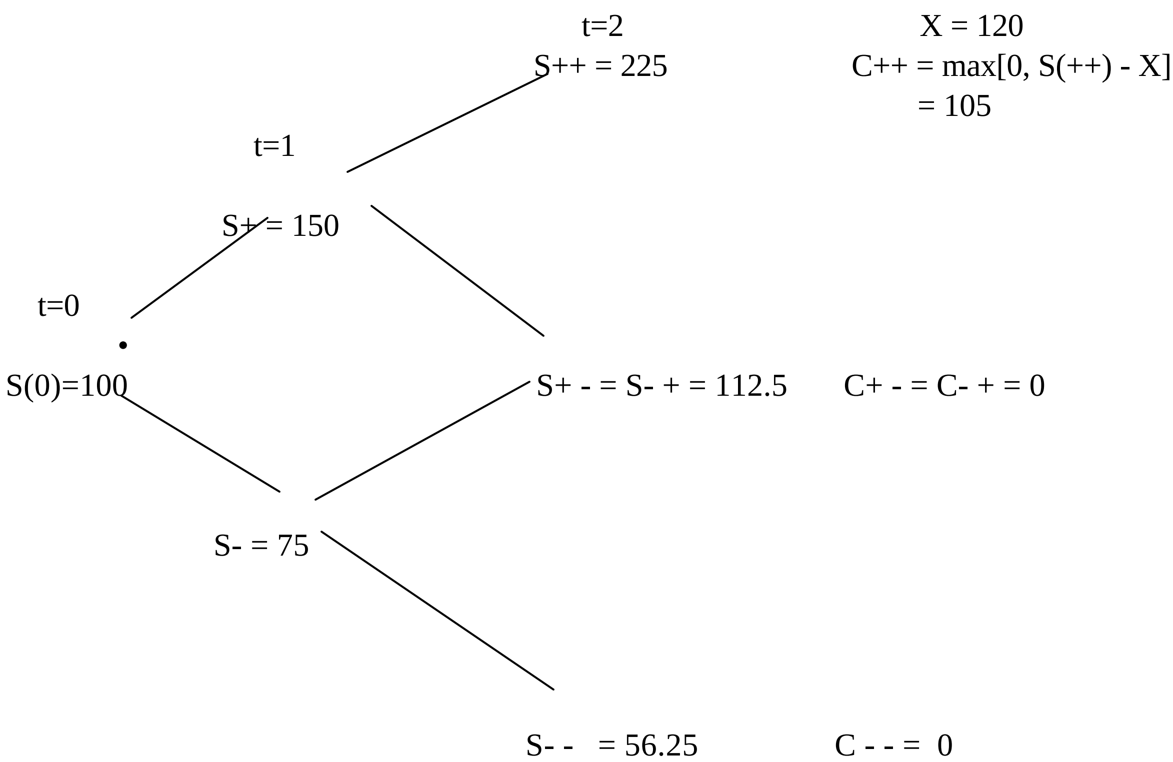
b) Using the period 2 expiration date call option prices and stock prices, calculate the call option hedge ratio needed at end of the first period if the stock price increases in the first period. Calculate the call option hedge ratio needed at the end of the first period if the stock price declines in the first period. What are the call option prices applicable at the end of the first period?

c) Calculate the period 0 call option price.

The two stage Binomial model is a straight forward extension of the one stage binomial. The objective is to value the call option at the second to last step, using the method for the one stage binomial. Having determined C+ and C- the discounted expected value of the option price is then calculated using the appropriate probabilities (which have to be calculated in this problem).

NOTE: If the probabilities have been given then it is not necessary to solve for the riskless interest rate.

The Two Stage Binomial Process and Associated Outcomes:



Calculating the Hedge Ratio at t = 1:

Notice that there are two hedge ratios that have to be calculated, one for S+ and one for S-. Fortunately, the hedge ratio for S- is irrelevant and the two option values of relevance are both zero, so the HR is zero. Only the up state requires calculation:

V(1)+ = HR+ (S+) - C+ = 105/112.5 (150) - C+

where HR+ = (C+ - C-)/(S+ - S-) = (105 - 0)/(225 -112.5) = 105/112.5

Checking the Hedge Ratio:

V++ = 105/112.5 (225) - 105 = 105 V+ - = 105/112.5 (112.5) = 105

Solving for the Values C+ and C-:

V(1)+ = (105)/1.1 = (105/112.5) 150 - C(1)+
C(1)+ = (105/112.5)(150) - (105/1.1) = $44.55

C(1)- = 0 (Why?)

Calculating the Probabilities:

$$S(0) = 100 = \{150\,(1 - Prob(down)) + 75\,Prob(down)\} / 1.1$$

$$150 - 110 = 75\,Prob(down) \quad Y \quad Prob(down) = (40/75) \quad Y \quad Prob(up) = (1 - (40/75))$$

Solving for the Call Price (C(0)):

$$C(0) = \{44.55\,(1 - (40/75)) + 0\}/\,(1.1) = \$18.90$$

Checking the Call Price:

$$C(0) = \{C+\,(1 - Prob(down)\} + C-\,(Prob(down))\}/\,1.1$$

$$18.90\,(1.1) = \{\,44.55\,(1 - (40/75)) + 0\} \quad Y \quad 20.79 = 20.79$$

In the two stage model the process of checking the call price is redundant.

Note: The methodology generalizes to three stages and higher of the binomial process. Calculate the hedge ratios/call prices at the next to last node. Using the probabilities and interest rates provided, either implicitly or explicitly, calculate the discounted expected value of the call price for t=0. In most situations, the last node corresponds to the maturity date of the option.

The binomial model can be adjusted to calculate American option prices on dividend paying stocks.