Lecture 3

Life Annuity Valuation

Reading: SAIS, sec. 2.1

- Life Contingencies in Roman Law
- Census Contracts and the Prohibition on Usury
- De Witt's Theoretical Solution
- Halley's Life Table and Life Annuity Valuation
- De Moivre's Approximation: Simplified Pricing
- Bernoulli's Problem: Contingent claims versus annuities certain

Modern Importance of Life Annuities: Retirement Planning

- Two general types of **pension plans** provided by employers: Defined Benefit and Defined Contribution.
  - Possible for self-employed to purchase either type of plan from an insurance company
  - **Defined Benefit**:
    - In this type of plan, regular contributions (by employer and usually the employee) result in a fixed income stream to be received following retirement until the employee dies at which time payments cease. Similar to a life annuity.
      - The plan assumes the life contingency and investment risk
      - The fixed income payout is usually based on a formula that uses some fraction (60% is common) of the average of the best five years of income – based on some required number of years, e.g., 30 years. The payout is reduced pro rata when the number of years worked is less.
      - Defined benefit plans are disappearing from the corporate sector. Most important DB plans in BC are: the Municipal Pension Plan (firemen, nurses, municipal employees); and, the BC Teachers Pension Plan. Also, the federal Canadian Pension Plan (CPP) is defined benefit..

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The most common employer pension plan: Defined Contribution

- **Defined Contribution**:
  - In this type of plan, regular contributions (by employer and usually the employee) accumulate in a fund and the employee receives the accumulated value (contributions + investment returns) in the fund at retirement
    - The employee assumes the life contingency risk and investment management risk of the fund manager prior to retirement and needs to decide on investment management after retirement.
    - Possible for employee to purchase a life annuity (a type of 'variable annuity') at retirement – the uncertain value of the fixed income received at retirement will depend on the size of contributions, the rate of return earned and the age at retirement.
- RRSP (Registered Retirement Savings Plan) works like a defined contribution plan except that contributions and investment management depends on decisions made by the owner of the registered plan before and after retirement

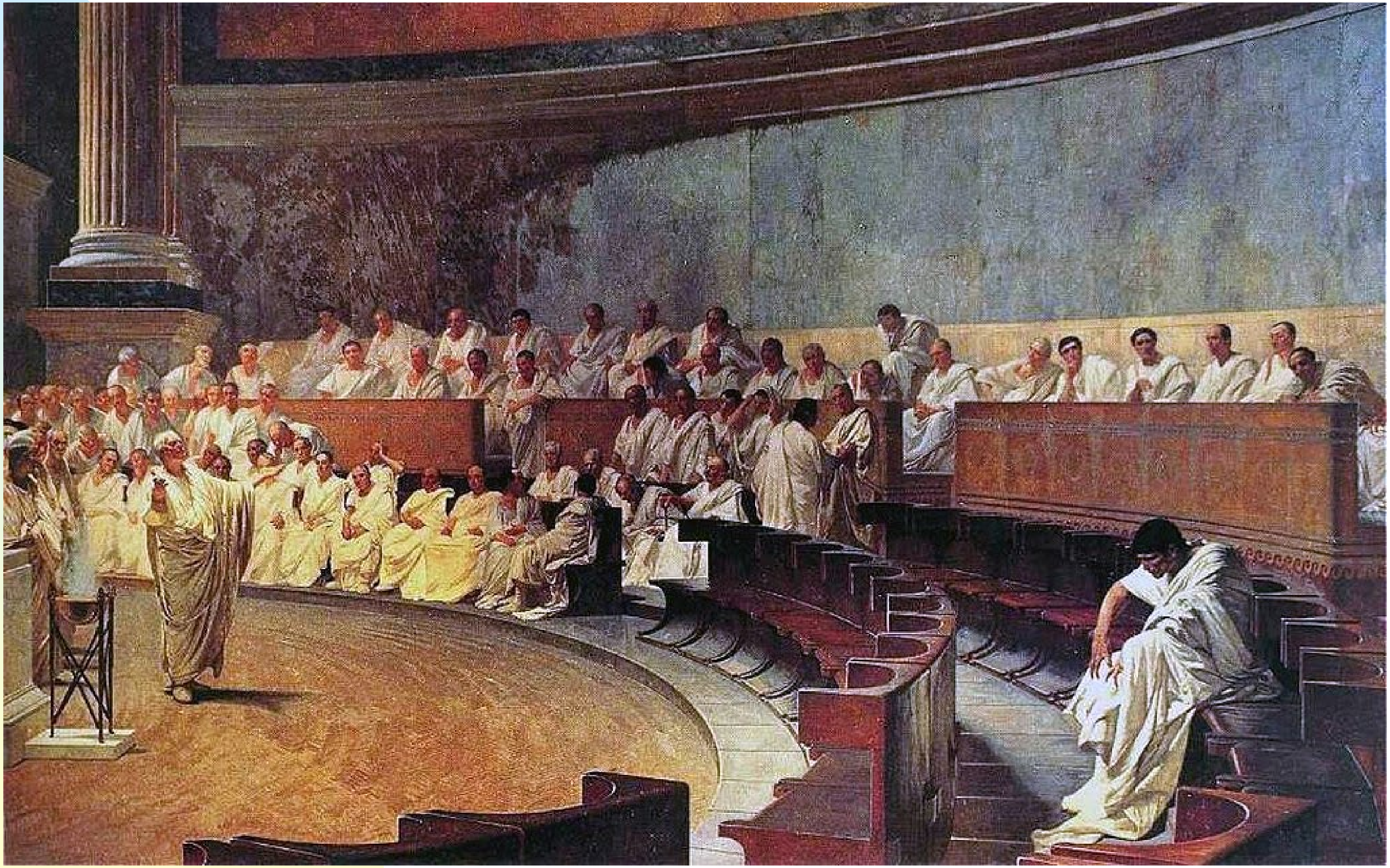
Early History of Life Contingencies

- The origins of life annuities and inheritance practices can be traced to ancient times; early Western practices inherited from Romans.
  - Socially determined rules of inheritance usually meant a sizable portion of the family estate would be left to a predetermined individual, often the first born son.
    - Early variation of Roman partnership (*societas*) that did not end when one partner died associated with legatees deciding not to divide the estate
  - Bequests such as usufructs, maintenances and life incomes were common methods of providing security to family members and others not directly entitled to inheritances.
  - The Falcidian law of ancient Rome, effective from 40 BC, maintained that the rightful heir(s) to an estate was entitled to not less than one quarter of the property left by a testator, the ***"Falcidian fourth"***

A Brief History of Roman Law

- Other than the 12 Tables (mid 5th century BC), sources on Roman law prior to appearance of *Institutes* of Gaius from the 2nd century (AD) are sketchy, much information about early laws is contained in the *Digest* of Justinian, prepared in the 6th century.
  - Sources from Roman times, not just those dealing with legal matters, are relatively scarce – most sources that have survived come down through copies of copies of originals that were prepared by scribes in the 9th to 12th centuries.
  - Roman law has the defining characteristic of being codified, with the twelve tables being the first variation of such a code.
- Specific laws are associated with the magistrate (such as a tribune) during the Republic or emperor (during the Empire) responsible for introducing the law (*Lex*)
  - The *Lex Falcidia de Legatis* was introduced in 40 BC by the tribune Publius Falcidius and dealt with inheritance, the law was still in place at the time of Justinian
  - A general reference such as *Lex Sempronia* could refer to laws introduced by the *Sempronia* and not with a specific law

‘Cicero attacking Catiline in the Senate for attempting to overthrow the Republic’ by the Italian painter Cesare Maccari (1888)



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The Judicial Quandary

- The judicial quandary is to determine a value for any other legacies: if the total legacy value exceeded three quarters of the value of the total estate, these bequests had to be reduced proportionately.
  - The relevant law regarding allegations that a legacy exceeds the ‘Falcidian fourth’ is discussed in *Digest* 35.3
- The Falcidian fourth created a legitimate valuation problem for jurists because many types of bequests did not have observable market values.
  - An alternative approach was proposed by Aemelius Macer, a judicial contemporary of Ulpian
- This was the case for bequests of life incomes. Some method was required to convert bequests of life incomes to a form that could be valued.

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Ulpian's Conversion Table

- To address the problem of the Falcidian fourth, the Roman jurist Ulpian (Domitianus Ulpianus, 170?-228) devised a table for the conversion of life incomes to annuities certain (*Digest*, 35.2.68)
  - The first historical instance of a rudimentary life contingency table
  - The alternative proposed by Aemelius Macer involved using 30 for all ages below thirty and reducing by 1 for each year after 29

- **Ulpian's Table**

Age of annuitant in years

0-19	20-24	25-29	30-34	35-39	40 ... 49	50-54	55-59	60-
30	28	25	22	20	19 ... 10	9	7	5

Comparable Term to maturity of an annuity certain in years (r = 0?)

Problems of application

- Whether Ulpian's Table was applied in Roman times with $r > 0$, using annuity valuation, or $r = 0$, by summing the payments, is unknown.
- In medieval practice, the legacy value was almost certainly determined by summing the payments, i.e., by taking the annual value of the legacy, and multiplying this by the term to maturity in Ulpian's Table to get the associated legacy value, resulting in a significant overvaluation
  - Nicholas Bernoulli in the *De Usu Artis Conjectandi in Jure*, (1709) identifies such overvaluation due to discounting at $r = 0$ being the common practice.
  - Summing the payments was the result of social prohibitions on interest payments associated with scholastic usury doctrine.

Usury and the *Census*

- Justice, not profit, was the basis of scholastic doctrine on economic relations
  - Doctrine governing exchange – just price – and taking of ‘interest’ – usury – and gambling/risk.
- The *census* contract has Carolingian (800-924AD) origins
  - Monasteries acquired bequests of lands, on condition that the donor/family/descendants receive an annual usufruct income from the land, in kind or money, for the rest of their life and (sometimes) for the lives of their heirs → ‘donation today in exchange for goods tomorrow’
  - The transaction did not have a market value basis but was associated with the Church as providing the social safety net and to establish ‘good works’
- Scholastic doctrine on usury and much of later canon law, in general, can trace roots back to Carolingian practices
  - Most famous (German Frankish) Carolingian leader was Charlemagne

Early Life Annuity Contracts

- **Unlike the forced loans of the Italian city states,** medieval towns in Northern European favoured annuities or *rentes* secured by urban taxes. As early as 1260, such early issues of *rentes heritables* (perpetual annuities) and *rentes vagieres* (life annuities) appeared in the French towns such as Calais.
  - This debt was marketed to wealthier citizens, instead of being a forced loan. In order to sell such debt, issues were typically under-priced – those setting prices were typically of the same social groupings as those purchasing the debt
- Municipal finance using life and fixed term annuities spread to the Low Countries and German towns
- Between 1275 and 1290 the city of Ghent in Flanders issued *lijfrenten* or life annuities followed by issues of *erfrenten* or redeemable *rentes*.
- Municipalities, particularly in Holland, Flanders and Brabant, continued to issue life and redeemable annuities leading to increasingly larger stocks of public debt and, ultimately, to repayment difficulties for some towns by the 16th century.

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Medieval Ghent

The Life Annuity Contract

- The life annuity usually was a contract involving three parties:
  - the subscriber who provided the initial capital
  - the shareholder who was entitled to receive the annuity payments
  - the nominee on whose life the payout was contingent
- Different variations are possible

Quoting Prices: *Years' Purchase*

- Annuity prices were quoted in '***years' purchase***',
  - The price of the annuity divided by the annual annuity payment is the years' purchase
    - For a perpetual annuity, years purchase is the inverse of the annual yield to maturity.
    - Related to the current yield *(CY)(*see SAIS, sec. 4.1):
    - *CY* = (Annual Bond Coupon) / (Bond Price)
  - Note:

Current Yield = Yield to Maturity for a Par Bond

Table 6.1 Expectation of life and present value of a life annuity at selected ages for selected life tables

	Expectation of Life					Present Value of a Life Annuity (Years' Purchase)				
Age:	0	5	10	20	50	0	5	10	20	50
Theoretical:										
Graunt	n/a	19.9	20.2	19.5	13.1	n/a	9.7	9.9	9.9	8.2
De Witt	n/a	32.1	29.6	24.9	12.3	n/a	13.6	13.2	12.1	7.8
De Moivre	n/a	40.5	38.0	33.0	18.0	n/a	14.9	14.6	13.9	10.3
Vital Registers:										
Halley	33.1	41.5	39.9	33.5	16.6	11.8	15.1	15.3	14.3	9.8
Smart-Simpson	19.2	36.1	34.7	28.7	15.3	7.4	14.2	14.3	13.1	9.2
Buffon	23.6	40.0	39.2	31.7	15.3	8.5	14.9	15.4	13.9	9.3
Price	25.2	40.9	39.8	33.4	18.0	8.9	14.8	15.1	14.0	10.3
Annuities and Tontines:										
Hudde	n/a	40.4	37.3	30.9	15.7	n/a	15.3	14.8	13.5	9.5
Kersseboom	n/a	44.5	42.7	36.3	19.4	n/a	15.4	15.6	14.6	10.9
Struyck:										
female	n/a	44.5	41.8	34.8	17.5	n/a	15.8	15.7	14.5	10.0
male	n/a	40.4	37.7	31.1	15.2	n/a	15.3	15.0	13.7	9.0

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Preliminaries for Jan de Witt's Solution

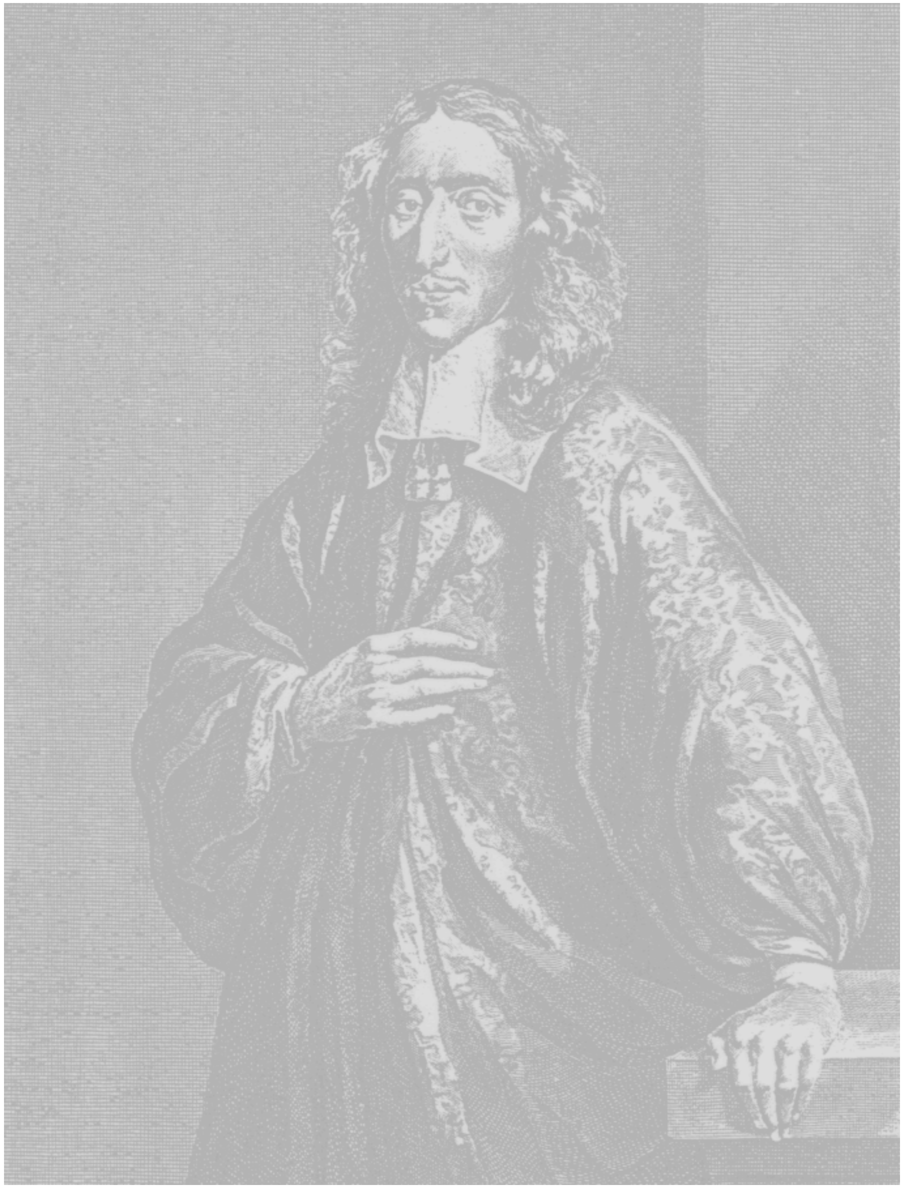
- Who was Jan de Witt?
- Some notation is needed for the A_n (the value of a annuity of $1 for *n* periods)
- For de Witt this was the present value of an annuity with a 4% annual rate to be paid at the end of the half year *n*
- Note that the method of determining the semi-annual interest rate for mathematical was more precise at the time of de Witt

$$A_n = \sum_{t=1}^{n} \frac{1}{(1+r)^t} = \frac{1}{r} - \frac{1}{r(1+r)^n} \qquad where \quad (1+r) = \sqrt{1.04}$$

Jan de Witt (1625-1672)
(also Johan de Wit, Jan de Wit, etc.)

Grand Pensionary of Holland (1653-1672)

Value of Life Annuities in Proportion to Redeemable Annuities (1671)



Determining the Theoretical Life Distribution

- Does **not** assume a uniform distribution where death at each age would be equally likely, i.e., equiprobable.
- De Witt divided the interval between 3 and 80 years into four subperiods: (3,53), (53,63), (63,73) and (73,80). Within each subperiod, an equal chance of mortality is assigned. This results in 154 possible cash flows. (First is zero)
- The weighting for chances assigned to each subgroup is 1, 2/3, 1/2, 1/3, e.g., the first 100 cash flows (in the first 50 years from age 3 to age 53) are 3 three times as likely as the last 14 cash flows (from age 73 to 80)
- The chance of living beyond 80 is assumed to be zero.
- Notice that if death occurs in the first period the cash flow is zero. Also it is possible to live to age 80 but not 80 1/2

De Witt's Solution for the Fair Value

To evaluate the expected present value of the life annuity, de Witt performs the calculation:

(**Why is the divisor 128**? This is because the number of chances is adjusted to account for each case not being the same as the equi-probable case (1/154)– verify that instead of 154 chances the adjusted number of chances is 128)

$$E[A_n] = \frac{\sum_{n=1}^{99} A_n + \frac{2}{3}\sum_{n=100}^{119} A_n + \frac{1}{2}\sum_{n=120}^{139} A_n + \frac{1}{3}\sum_{n=140}^{153} A_n}{128}$$

Edmond Halley: Demography and Life Annuity Valuation

- Edmond Halley (1656-1742)
  - published his influential paper “An Estimate of the Degrees of Mortality of Mankind, drawn from the curious Tables of the Births and Funerals at the City of Breslaw; with an Attempt to ascertain the Price of Annuities upon Lives” in 1694
  - At this time the English government was still selling life annuities at seven years' purchase, independent of age.

Edmond Halley
(1656-1742)

Discovered Halley's Comet and Mapped the Stars of the Southern Hemisphere

"An Estimate of the Degrees of Mortality" (1694) contains first application of a life table to the pricing of life annuities

The Breslau Data

- The absence of empirical distribution for population demographics – a life table -- is a problem in de Witt's life annuity valuation.
- Earlier 17th C, efforts at demographics in England (John Graunt) constructed a life table from the bills of mortality.
- From the end of the 16th century, Breslaw, a city in Silesia, had maintained a register of births and deaths, classified according to sex and age.
- The Breslau data is used in the preparation of Halley's "Estimate..."
  - For the purposes of constructing a precise life table, only the population size is missing.
- Breslau data is a **vital register**, i.e., applies to the whole population and not the sub-group that buys life annuities
  - Why does this matter?

TABLE IX.

Shewing the *Expectations* of Life in LONDON, according to the preceding Table. See Mr. *Simpson's Select Exercises*, p. 255.

Age.	Expectation.	Age.	Expectation.	Age.	Expectation.
1	27.0	28	24.6	55	14.2
2	32.0	29	24.1	56	13.8
3	34.0	30	23.6	57	13.4
4	35.6	31	23.1	58	13.1
5	36.0	32	22.7	59	12.7
6	36.0	33	22.3	60	12.4
7	35.8	34	21.9	61	12.0
8	35.6	35	21.5	62	11.6
9	35.2	36	21 1	63	11.2
10	34.8	37	20.7	64	10.8
11	34.3	38	20.3	65	10.5
12	33.7	39	19.9	66	10.1
13	33.1	40	19.6	67	9.8
14	32.5	41	19.2	68	9.4
15	31.9	42	18.8	69	9.1
16	31.3	43	18.5	70	8.8
17	30.7	44	18.1	71	8.4
18	30.1	45	17.8	72	8.1
19	29.5	46	17.4	73	7.8
20	28.9	47	17.0	74	7.5
21	28.3	48	16.7	75	7.2
22	27.7	49	16.3	76	6.8
23	27.2	50	16.0	77	6.4
24	26.6	51	15.6	78	6.0
25	26.1	52	15.2	79	5.5
26	25.6	53	14.9	80	5.0
27	25.1	54	14.5		

TABLE III.

Shewing the Probabilities of the Duration of Life, as deduced by Dr. *Halley* from Obſervations on the Bills of Mortality of BRESLAW.

Ages.	Perſons living.	Decr. of Life.	Ages.	Perſons living.	Decr. of Life.	Ages.	Perſons living.	Decr. of Life.
1	1000	145	31	523	8	61	232	10
2	855	57	32	515	8	62	222	10
3	798	38	33	507	8	63	212	10
4	760	28	34	499	9	64	202	10
5	732	22	35	490	9	65	192	10
6	710	18	36	481	9	66	182	10
7	692	12	37	472	9	67	172	10
8	680	10	38	463	9	68	162	10
9	670	9	39	454	9	69	152	10
10	661	8	40	445	9	70	142	11
11	653	7	41	436	9	71	131	11
12	646	6	42	427	10	72	120	11
13	640	6	43	417	10	73	109	11
14	634	6	44	407	10	74	98	10
15	628	6	45	397	10	75	88	10
16	622	6	46	387	10	76	78	10
17	616	6	47	377	10	77	68	10
18	610	6	48	367	10	78	58	9
19	604	6	49	357	11	79	49	8
20	598	6	50	346	11	80	41	7
21	592	6	51	335	11	81	34	6
22	586	7	52	324	11	82	28	5
23	579	6	53	313	11	83	23	4
24	573	6	54	302	10	84	19	4
25	567	7	55	292	10	85	15	4
26	560	7	56	282	10	86	11	3
27	553	7	57	272	10	87	8	3
28	546	7	58	262	10	88	5	2
29	539	8	59	252	10	89	3	2
30	531	8	60	242	10	90	1	1

Preliminaries for Halley’s Valuation

- The total number of annuities sold on a life starting at year x is R_x – simple case is to assume each individual buys only one life annuity, then d_{x+I} is the number dying i periods after the valuation starting age x (for de Witt x = 3)
  - R_x equals the sum of $d_x + d_{x+1} + \ldots\ldots + d_{w-1}$ where d_i is the number of annuities which terminate in period i due to the death of annuitant nominees in that half year
  - $d_i = 0$ for $x > w$.
  - Under the simplifying assumption above, taking R_{x+t} to be the number of nominees, starting in year x surviving in period $x + t$, it follows that: $d_{x+t} = R_{x+t} - R_{x+t+1}$ and that the probability of death in any given half year j is (d_{x+j} / R_x).

Halley's Fair Value Solution

- Halley used the Breslau data to provide empirical estimates for the probabilities used in the expectation.
- Halley observed the calculated solutions involved "a not ordinary number of Arithmetical operations"
- What this shows is that the formula used by de Witt (which involves multiplying the annuity value by the probability of dying in that period) gives the same solution as multiplying the cash flow received in any period by the fraction of the population that receives that cash flow)
- **Note: de Witt involves doing discounted cash flow directly using probability weights while Halley only does a transformed version where the cash flow weights are not probabilities**

$$E[A_n] = \frac{1}{\ell_x}\sum_{n=1}^{w-x-1} A_n\, d_{x+n} = \frac{1}{\ell_x}\sum_{n=1}^{w-x-1} d_{x+n} \sum_{t=1}^{n} \frac{1}{(1+r)^t}$$

$$= \frac{1}{\ell_x}\sum_{t=1}^{n} \sum_{n=1}^{w-x-1} d_{x+n} \frac{1}{(1+r)^t} = \frac{1}{\ell_x} \sum_{n=1}^{w-x-1} \ell_{x+n} \frac{1}{(1+r)^n}$$

De Moivre's Approximations

- Abraham de Moivre (1667-1754), an expatriate Frenchman transplanted to London following the Repeal of the Edict of Nantes
  - Signed in 1598 to end the wars of religion between Protestants (mostly Calvinist Huguenots) and Catholics in France, the Edict was repealed in 1685 by Louis XIV resulting in a diaspora of Protestants out of France to neighboring countries
- De Moivre laid the theoretical foundation for Richard Price, James Dodson and others to develop the actuarially sound principles required to implement modern life insurance
- Together with Laplace, one of two giants of probability theory in the 18th century

Abraham de Moivre
1667-1754
(there are alternative spellings)

18th century giant of mathematical statistics (first to derive the normal distribution)

Treatise of Annuities on Lives (1st ed. 1725)

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Solving the Valuation Formula

- De Moivre solved the valuation problem for both arithmetic (uniform death rates) and geometrically declining survival rates.
- Solving for the uniform case *(k = w – x):*

$$E[A_k] = \frac{k-1}{k}\frac{1}{1+r} + \frac{k-2}{k}\frac{1}{(1+r)^2}$$

$$+ \;....+\frac{k-(k-1)}{k}\frac{1}{(1+r)^{k-1}} + \frac{k-k}{k}\frac{1}{(1+r)^k}$$

$$= \sum_{t=1}^{k}\frac{1}{(1+r)^t}\left(1 - \frac{t}{k}\right) = \sum_{t=1}^{k}\frac{1}{(1+r)^t} - \sum_{t=1}^{k}\frac{t}{k\,(1+r)^t}$$

The Final Steps in the Solution

- After some further manipulation, de Moivre is able to find the solution:

$$E[A_k] = [A_k + \frac{1+r}{k}\frac{dA_k}{d(1+r)}]$$

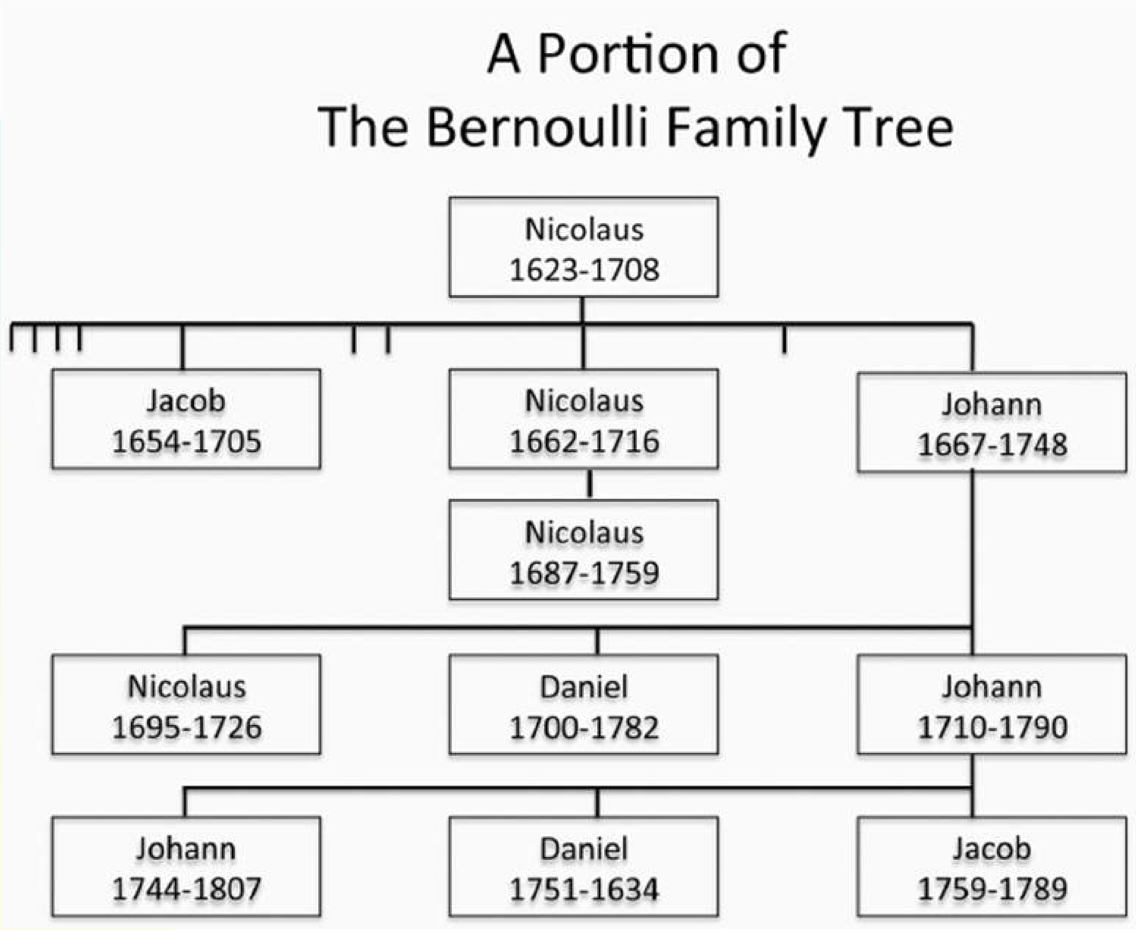
$$= \left(\frac{1}{r} - \frac{1}{r(1+r)^k}\right) + \frac{1+r}{k}\left[\frac{k}{r(1+r)^{k+1}} + \frac{1}{r^2(1+r)^n} - \frac{1}{r^2}\right]$$

$$= \frac{1}{r} - \frac{1+r}{k}\left[\frac{1}{r}\{\frac{1}{r} - \frac{1}{r(1+r)^k}\}\right] = \frac{1}{r}\{1 - \frac{1+r}{k}A_k\}$$

Bernoulli's problem

- "...I notice that the value of (life annuity) incomes is not correctly calculated by supposing that the return will last as many years as someone is supposed probably to live."
- The problem is to demonstrate that the fair value of a life annuity with differ from that for a term annuity with term to maturity equal to the expected duration of life
  - More precisely, it can be shown that the term annuity will have a higher price.

Nicolaus Bernoulli, author of *De Usu Artis Conjectandi in Jure* (Use of the Art of Conjecture in Law) (1709) was the head of a famous family of Swiss mathematicians



Some Important Formulas for this problem

- There are three values to consider: *D*, the expected duration of life; *E[A_n]* the fair value of a life annuity; and A_d the value of a term annuity with term to maturity equal to *D*. (See worked example on class webpage for derivation of duration of life from life table)

$$D = \sum_{t=1}^{w-x-1} t \frac{d_{x+t}}{\ell_x} \qquad E[A_n] = \sum_{n=1}^{w-x-1} A_n \frac{d_{x+n}}{\ell_x} \qquad A_d = \sum_{t=1}^{D} \frac{1}{(1+r)^t}$$

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Some Basic Results

- Comparing *D* with $E[A_n]$ and A_d it is apparent:
  - For $r > 0$, it follows that $D > E[A_n]$ and $D > A_d$, due to the impact of discounting on the terms in $E[A_n]$ and *Ad*.
  - Even if interest rates are zero and $D = A_d$, $E[A_n]$ and *D* are still not equal due to the $E[A_n]$ only crediting the cash flow if the end of period is reached. (This problem is avoided by making the simplifying assumption – see webpage example).

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Solving Bernoulli's Problem

- Though Bernoulli did not find the solution, using de Moivre's approximation (the formula given previously for de Moivre), it can be shown that in situations when the de Moivre formula is accurate:

$$A_d - E[A_n] = -\left\{ \frac{1}{r(1+r)^D} - [\frac{1+r}{n}.\frac{1}{r}]\{\frac{1}{r} - \frac{1}{r(1+r)^n}\} \right\}$$

$$= -\left\{ \frac{1}{r^2}\left\{\frac{r}{(1+r)^D} - [\frac{1+r}{n}\{1 - \frac{1}{(1+r)^n}\}]\right\} \right\}$$

$$= -\left\{ \frac{1}{r^2}\{\frac{nr\,(1+r)^{n-D} + (1+r) - (1+r)^{n+1}}{n(1+r)^n}\} \right\} > 0$$

Bernoulli's problem details

It follows that for the greater than condition it is necessary that:

$$\left\{ \frac{nr\,(1+r)^{n-D} + (1+r) - (1+r)^{n+1}}{n(1+r)^n} \right\} < 0$$

$$\Rightarrow \quad nr < (1+r)^{D+1}\left[1 - \frac{1}{(1+r)^n}\right]$$