Lecture 6

Fixed Income Portfolio Management: Duration and Convexity

- Classical immunization theory
- Use of Taylor Series Expansions
- Demonstrating the role of convexity
- Decomposition of multivariate Taylor series expansion for the bond price function

Fixed Income Portfolio Management

- Possible strategies for fixed income portfolio management:
  - ***classical immunization***, where the interest rate sensitivity of the cash flows from the fixed income assets and liabilities is duration matched with higher asset convexity
  - ***dedication*** or ***cash flow matching***, where the cash flow from the fixed income assets is structured to match the requirements of a portfolio of predetermined liabilities; difficult to implement in large applications
  - ***"dollar duration matching"***, which extends immunization to positive surplus situations; classical immunization is zero surplus methodology
  - ***horizon matching*, **a hybrid method which combines dedication and immunization methods by dividing the assets and liabilities being managed into two parts, one part being managed with dedication (when possible) and the other with classical immunization

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The Investment Horizon

- A key parameter governing the immunization problem is the date at which an obligation is to be discharged, i.e., the ***planning*** or ***investment horizon.***
  - ***Examples of investment horizon:***
    - ***Retirement date for an individual in a pension plan***
    - ***Expected payout by a life insurance company in a given future year based on the life table***
    - ***The maturity date for a GIC***
  - Most funds have ***multiple*** planning periods that are difficult to administer using techniques that treat each individual liability separately by creating dedicated asset portfolios for each liability, i.e., using ***cash-flow matching***.
- The immunization approach handles the different funds associated with the different possible planning periods by managing these funds as a single investment fund with a single investment horizon.

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Classical Immunization Rules

- Frank Redington (1952), the British actuary that first proposed classical immunization, posed the following problem: What allocation of assets and liabilities would minimize a life insurance company's possibility of losses from an unexpected (**instantaneous!**) change in market rates of interest?

- **Using a Taylor series expansion**, Redington derived ***Redington's rules***, known as the ***classical immunization conditions***, are derived as**:**
  - ***Duration matching***: Duration of cash inflows equals the duration of the outflows.
  - ***Higher Convexity of Assets***: When there is more than one planning period for the fund to satisfy, the value of the cash inflows should be more "dispersed" around the duration than the value of the cash outlfows.

Using Taylor Series Expansions

- Classical Immunization theory employs a Taylor series methodology → limitations of immunization theory associated with the use of Taylor series
- Taylor series are one of the most important analytical tools in applied mathematics
- **Simplifications** required to use the basic formulas to determine the bond price/yield:
  - Bond is valued on the issue date or coupon payment date
    - No need to take account of accrued interest.
    - It is possible to specify more complicated, exact formulas for price between payment dates.
  - The bond has no embedded options.
  - ‘Straight bonds’ with a bullet maturity are used.

Taylor Series Expansions

- The basic idea is a specific solution to the more general problem of approximation of functions, a topic that has occupied mathematicians for centuries.
  - The basic idea is solve a more complicated function by using an approximation based on a sequence of simpler functions
  - In the case of the bond price function, these simpler functions are Duration and Convexity
- ***Taylor series expansion:*** for a function of one variable, *f[x],* the expansion takes the general form:

$$f[x] = f[a] + \frac{df[a]}{dx}(x-a) + \frac{1}{2!}\frac{d^2f[a]}{dx^2}(x-a)^2 + \frac{1}{3!}\frac{d^3f[a]}{dx^3}(x-a)^3 + \ldots\ldots$$

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Evaluating the Taylor Series

- The Taylor series expands the univariate function *f*[*x*] about the point fixed point *a*, where *b # a # c.* Each of the derivatives in the expansion are evaluated by setting *x* = *a*. For this expansion to be valid, the function *f*[*x*] must have derivatives of all orders over [*b*,*c*] (some of which can be zero).
  - There are restrictions on *a* requiring the fixed point to be in the 'radius of convergence' of the Taylor series

- Term by term inspection of the Taylor reveals how the function *f*[*x*] is approximated.
  - The first term *f*[*a*] is a point, the value of the function evaluated at the point *x*=*a*.
  - The sum of the first term and the second term is the linear approximation to the function about the point *a*.

Zero order, First order, Second order and Third order approximations

- Zero order approximation is simply the fixed point around which the function is expanded. In this case, the derivative of the function $f[x]$ is evaluated at the point $x = a$. The first order approximation is a line with slope equal to the first derivative **evaluated at the point a** (this results in a number); the second order expansion is a quadratic, etc. → see the 'Taylor series approximation example' file on class webpage for application to the 'closed form' of the geometric series

$$f[x] = f[a]$$

$$f[x] = f[a] + \frac{df}{dx}\Big|_a (x - a)$$

$$f[x] = f[a] + \frac{df}{dx}\Big|_a (x - a) + \frac{1}{2}\frac{d^2f}{dx^2}\Big|_a (x - a)^2$$

$$f[x] = f[a] + \frac{df}{dx}\Big|_a (x - a) + \frac{1}{2}\frac{d^2f}{dx^2}\Big|_a (x - a)^2 + \frac{1}{6}\frac{d^3f}{dx^3} (x - a)^3$$

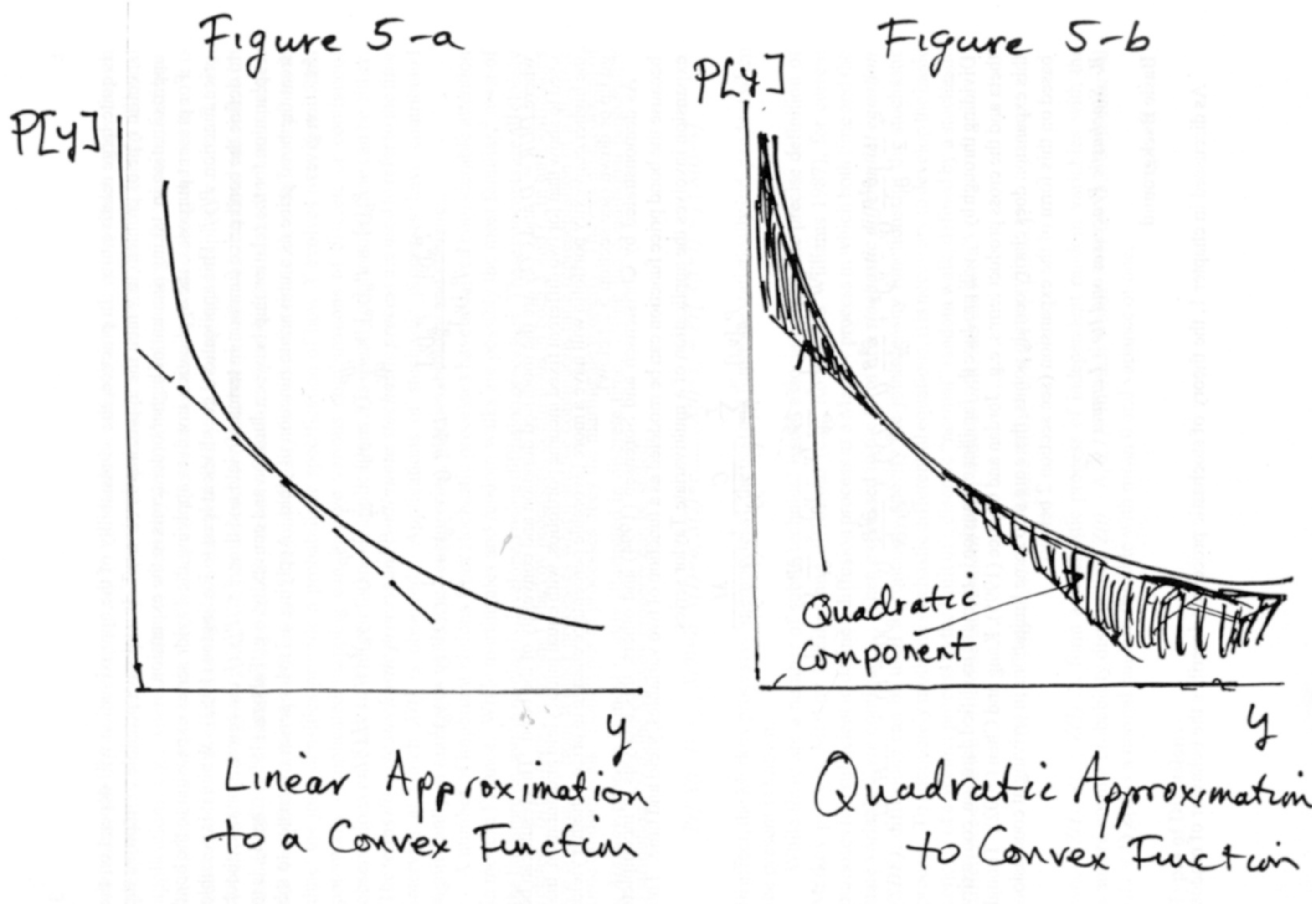
More on Taylor Series

- The quadratic approximation is achieved by adding the squared term in the Taylor series to the linear approximation. (Will this ***necessarily*** improve the approximation? Raises the problem of monotonic versus uniform convergence for a series)

- See Figures 5.1 and 5.2 → the tangent line would be the duration (*DUR*) approximation while the convexity (*CON*) component is represented by the shaded area

- Application to fixed income valuation: For default and option free bonds, the price function is convex in the yield to maturity, a condition that easily satisfies the conditions required for a Taylor series to be used.

Figures 5.1 and 5.2 (5.a + 5.b)



Application to Fixed Income Valuation

- It is possible to define the bond price as a multivariate function in terms of yield and time, i.e., $P_B[y,t]$. However, this is not the case with the classical immunization (textbook) results
  - For a par bond y_0 can be most conveniently chosen as the coupon rate, the facilitates interpretation of the results as % changes.

- The textbook explanation for the relationship between duration and convexity is to treat the bond price as a univariate function of yield, $P[y]$. Applying a Taylor series expansion to this function at some initial yield y_0 gives:

$$P_B[y] = P_B[y_0] + \frac{dP}{dy}(y - y_0) + \frac{1}{2}\frac{d^2P}{dy^2}(y - y_0)^2 + \dots H.O.T.$$

$$\frac{P_B[y] - P_B[y_0]}{P_B[y_0]} \cong -DUR\,(y - y_0) + \frac{1}{2}\,CON\,(y - y_0)^2$$

A Tabular Example of the Duration + Convexity approximation to the instantaneous % change in the price of the par bond with C = *10%* as interest rates change

- In the next slide, actual % change in the price of a 20 year **par bond** as interest rates change by 1 bp, 10bp, 50 bp, 100 bp (1%), 200 bp, 300 bp is calculated
  - These values are then compared with the approximation provided by modified duration (calculated to be 8.58) $(\Delta P / P) = -(D)(\Delta y)$, *e.g.*, when r increases by 200 bp then –(8.58)(2%) = 17.16% is the change estimated by duration (actual change computed directly with bond price formula is 15.05%
  - The line below modified duration gives the convexity term added

Table 5-4
Example of the Percentage Price Change Using Modified Duration and Convexity for a 20-Year 10% Coupon Bond Selling at Par to Yield 10%

			Increase in Required Yield			
Required yield	10.01%	10.10%	10.50%	11.00%	12.00%	13.00%
Change in basis points	1	10	50	100	200	300
Estimated %						
Duration	-0.09	-0.86	-4.29	-8.58	-17.16	-25.74
Convexity	0.00	0.01	0.14	0.57	2.27	5.10
Total	-0.09	-0.85	-4.15	-8.01	-14.89	-20.64
Actual % change	-0.09	-0.85	-4.15	-8.02	-15.05	-21.22
			Decrease in Required Yield			
Required yield	9.99%	9.90%	9.50%	9.00%	8.00%	7.00%
Change in Basis points	-1	-10	-50	-100	-200	-300
Estimated %						
Duration	0.09	0.86	4.29	8.58	17.16	25.74
Convexity	0.00	0.01	0.14	0.57	2.27	5.10
Total	0.09	0.87	4.43	9.15	19.43	30.84
Actual % change	0.09	0.86	4.44	9.20	19.79	32.03

Calculating the modified Duration for the Example

- Recall that the formula for the annual coupon Macaulay duration was derived as:

$$\frac{1+y}{y}\left[1-\frac{1}{(1+y)^{T}}\right]$$

- Divide this value by (1 + *y*) and solve using Mathematica to get the modified duration:

```
In[1]:= y = .10
        (1 / y) - (1 / (y ((1 + y) ^ 20)))
Out[1]= 0.1
Out[2]= 8.51356
```

Calculating the Convexity for the example

- Deriving the convexity of an annual coupon **par bond** gives:

$$\frac{2}{y}\left[\frac{1}{y} - \frac{1}{y(1+y)^{T}}\right] - \frac{2T}{y(1+y)^{T+1}}$$

Solving for a value using Mathematica

```
In[1]:= y = .10
        (1 / y) - (1 / (y ((1 + y) ^20)))
Out[1]= 0.1

Out[2]= 8.51356

In[5]:= ((2 / y) * ((1 / y) - (1 / (y ((1 + y) ^20))))) - ((2 * 20) / (y ((1 + y) ^ (21))))
Out[5]= 116.219
```

Evaluating the convexity term gives:

$(\frac{1}{2} CON (\Delta y)^2) = \frac{1}{2}\, 116\, (.01)^2 = 0.58\%$

The Role of Convexity

- Consider two bond portfolios (*A* and *B*), with values P_A and P_B . These portfolios are constructed to have equal duration ($D_A = DUR_A = DUR_B = D_B$), equal initial yield to maturity (y_0) and $CON_A > CON_B$.
- The impact of an instantaneous interest rate change on these two portfolios would be approximately (ignoring *H.O.T., i.e.,* higher order terms in the expansion):

$$\%\Delta P_A - \%\Delta P_L = ½(CON_A - CON_L)(y - y_0)^2$$

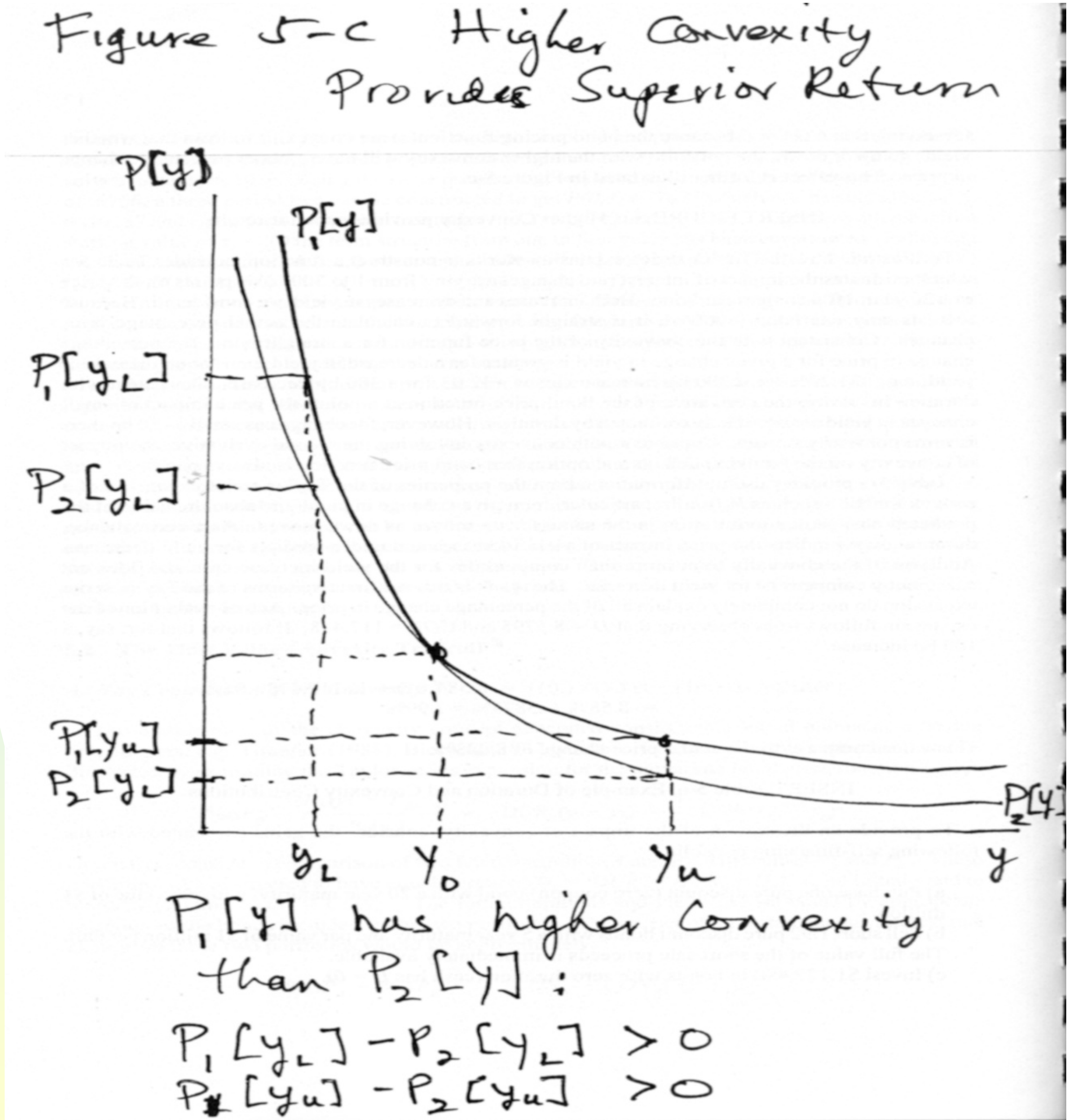
- EXERCISE: Derive this result from the Taylor series expansion of *P* involving *DUR* and *CON*

More on Convexity

- Observing that *CON > 0* because, in the simple case, the bond pricing function(s) are convex, it follows that whether yields go up or down, the portfolio with the higher convexity will have a better percentage change in price.
  - Application to fixed income portfolios is that **the duration of the portfolio is the value weighted sum of the individual fixed income asset durations**
  - **The same result holds for the convexity**
  - When the conditions for the simple case are relaxed, e.g., for bonds with contingencies, then the classical immunization conditions need to be adjusted, if possible.

Figure 5.4 (5-c)

A graphical illustration that a bond (or simple fixed income portfolio) with higher convexity will outperform whether yields increase or decrease.



Bond Trader Example Illustrating Convexity

- SAIS, p.277-78 (p.26 in ‘Supplementary Material for Lecture 5’) (see worked example on class webpage in Cost or convexity .zip file/ Convexity example).

- The bond trader example: the net investment in the position at $t = 0$ is zero.
- Comparison of a (*barbell)* cash + twenty year zero coupon assets with five year zero liability.
- The example demonstrates that whether interest rates go up or down, the higher convexity position does better.

Description of the Bond Trader Example

a) Purchase one pure discount (zero coupon) bond with a 20 year maturity, with par value of $1 million.
b) Sell short 1.92 pure discount bonds with a 5 year maturity and par value of $1 million per unit. The full value of the short sale proceeds is immediately available.
c) Invest $1,127,490 in bonds with zero duration (cash has *D = 0)*.

Without loss of generality, assume that the yield to maturity on all bonds is 5% initially. What is the initial investment in the portfolio? From an inspection of the balance sheet, it is apparent that no investment is required in the portfolio:

Assets:		Liabilities Plus Net Worth:	
20 Year bond	$376,890 = 1 million/(1.05)^{20}$	Short sale	$1,504,370 = 1.92 million/(1.05)^{5}$
Cash	$1,127,480	Net Worth = $0 (Self Financing)	
Total:	$1,504,370		

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Macaulay Duration of a Zero Coupon Bond

$$P[y] = \frac{1}{(1 + y)^T} \qquad \frac{dP}{dy} = -\frac{T}{(1 + y)^{T+1}} \qquad -\frac{(1 + y)}{P}\frac{dP}{d(1+y)} = T = D^*$$

Modified Duration and Convexity of a Zero Coupon Bond

$$P[y] = \frac{1}{(1 + y)^T} \qquad \frac{dP}{dy} = -\frac{T}{(1 + y)^{T+1}} \qquad -\frac{1}{P}\frac{dP}{dy} = \frac{T}{(1+y)} = D$$

$$\frac{d^2P}{dy^2} = \frac{T(T+1)}{(1 + y)^{T+2}} \qquad \frac{1}{P}\frac{d^2P}{dy^2} = \frac{T(T+1)}{(1 + y)^2} = CON$$

Duration and Convexity for Bond Trader Example from Chapter 5, p.276 (p.29)/Lecture 5

Liability: $D^* = T = 5$ $\qquad CON_L = (5 * 6) / (1.05)^2 = 27.21$

Asset: D* = .75 Cash + .25 (20) = 5 $\qquad CON_A = 0 + ((.25)(20 * 21)) / (1.05)^2 = 95.24$
Note: 376,890/1,504,370 = .25

$CON_A > CON_L$

Observe that the theta for the barbell is .75 (0) + .25 (5%) = 1.25%

Whether Interest rates go up or down, the net worth in the higher convexity asset portfolio increases → is this a 'free lunch'?

Assets:		Liabilities Plus Net Worth:	
20 Year bond	$\$311,800 = 1 \text{ million}/(1.06)^{20}$	Short sale	$\$1,434,740 = 1.92 \text{ million}/(1.06)^{5}$
Cash	$1,127,480	Net Worth	4,540
Total:	$1,439,280		$1,439,280

Now consider what happens if the yield to maturity of all assets and liabilities immediately falls to 4%:

Assets:		Liabilities plus Net Worth:	
20 Year bond	$\$456,390 = 1 \text{ million}/(1.04)^{20}$	Short sale	$\$1,573,100 = 1.92 \text{ million}/(1.04)^{5}$
Cash	$1,127,480	Net worth	10,770
Total:	$1,583,870		$1,583,870

Assets:			Liabilities plus Net Worth:		
20 Year bond	\$395,734 =	1 million/$(1.05)^{19}$	Short sale	\$1,579,590 =1.92 million/$(1.05)^{4}$	
Cash	\$1,127,480		Net Worth	(\$55,276)	
Total:	\$1,524,214			\$1,524,214	

Observe what happens if interest rates do not change but time is allowed to change. In this case, the net worth of the portfolio falls!

This illustrates the importance of Time Value → this point will be developed to explain the role of time value/convexity tradeoff in the shape of the term structure of interest rates

This result also illustrates the limitations of Taylor series → interest rate changes are instantaneous, the introduction of time into the bond price function $P[\,y,t\,]$ is necessary to theoretically incorporate the role of time.

The Role of Time Value

- The example demonstrates that when interest rates do not change then the time value (referred to as the **theta**) will favor the lower convexity portfolio.
- Time value is the return earned on portfolio as time passes and interest rates don't change → this would be the price increase for a zero coupon bond or the coupon income for a par bond
  - The use of **theta** corresponds to the theta of an option → fundamental PDE for an option relates delta, gamma and theta
  - The upshot is that the bond price function is multivariate, $P[y,t]$ instead of univariate $P[y]$. Need multivariate Taylor series to investigate approximation of this type of function.

Multivariate Taylor Series

- When evaluating derivatives of multivariate functions a change in the notation for derivative from *d* in the univariate case to ∂ in the multivariate case
  - ∂ refers to the 'partial derivative' of the function *P* with respect to *y* holding *t* constant or *t* holding *y* constant.
- The multivariate Taylor series expansion of the bond price function $P_B[y,t]$ can be manipulated to produce the result:

$$\frac{P_B[y,t] - P_B[y_0,\ t_0]}{P_B[y_0,\ t_0]} = \frac{1}{P}\frac{\partial P}{\partial y}(y - y_0) + \frac{1}{P}\frac{\partial P}{\partial t}(t - t_0)$$

$$+ \frac{1}{2}\left\{\frac{1}{P}\frac{\partial^2 P}{\partial y^2}(y - y_0)^2 + \frac{1}{P}\frac{\partial^2 P}{\partial t^2}(t - t_0)^2\right\} + \frac{1}{P}\frac{\partial^2 P}{\partial y\, \partial t}(y - y_0)(t - t_0) + \ldots\ H.O.T.$$

$$\%\Delta P \simeq -DUR\ (y - y_0) + \theta\ (t - t_0)$$

$$+ \frac{1}{2}\left[\ CON\ (y - y_0)^2 + \Theta_2\ (t - t_0)^2\ \right] + CROSS\ (y - y_0)(t - t_0)$$

Evaluating the Expansion

- Tables 5.5, 5.6 and 5.7 (SAIS p.281-2; Convexity and Time Value link on webpage)
  - This is an example illustrating the relative contributions of the terms in the Taylor series to the %Δ in *P*

- Compare Table 5.5 with 5.4; See 16-1 Midterm Convexity-Time Value Solution on webpage

- The upshot of the information contained in Table 5-7 and the Midterm Solution is that the shape of the yield curve imbeds a cost of convexity that appears as a time value.
  - **Setting duration** for two portfolios to be equal will have a time value (lower yield) cost that will be traded off against the convexity value.

These results are for semi-annual coupon par bonds with annual coupon of 8% → this requires adjusting the simplified annual coupon par bonds formulas

Table 5-d
Decomposition of the Taylor Series Expansion: Default Free Par Bonds from One Year to Thirty Years to Maturity, Δy = 10 Basis Points, $\Delta t = 0$

BondMaturity (years)	1	5	15	30
Duration (D^*)	0.9808	4.2177	8.9919	11.7642
Convexity	1.3512	20.819	104.978	214.24
Theta (θ)	0.04079	0.04079	0.04079	0.04079
New Price	99.9058	99.5955	99.1406	98.8795
%ΔP	-0.0009	-0.0041	-0.0086	-0.0112
$-(D\,\Delta y)\,/\,\%\Delta P$	100.0717	100.259	100.608	100.9486
$(\frac{1}{2}\,CON\,\Delta y^2)\,/\,\%\Delta P$	-0.0717	-0.2573	-0.6108	-0.956

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Calculating the Macaulay duration for a semi-annual coupon bond produces for the 30 year case:

(Observe that the formulas are slightly different than for the annual coupon modified duration and convexity given previously!)

```
In[57]:= y = (.04)
         ((1 + y) / y) - (1 / (y ((1 + y) ^59)))

Out[57]= 0.04

Out[58]= 23.5284

In[59]:= % / 2

Out[59]= 11.7642

In[75]:= ((1 / y) * ((1 / y) - (1 / (y ((1 + y) ^60))))) - ((60) / (y ((1 + y) ^ (61))))

Out[75]= 428.481

In[76]:= % / 2

Out[76]= 214.24
```

Interpreting the table:
-- Theta gives the 'time value', the coupon return adjusted for the semi-annual coupon paying half of the coupon before the end of period (would be .04 for annual)
-- The "New Price" is calculated directly from the bond price formula (as before)
-- The values given below %ΔV give the % contribution to the predicted change in the value of the bond from the different components in the multivariate Taylor series expansion
-- Notice that the cross product terms and the second derivative for time are, for practical purposes equal to zero!

Table 5-e
Decomposition of the Taylor Series Expansion: Default Free Par Bonds from One Year to Thirty Years to Maturity, Δy = 10 Basis Points, Δt = 1 day

Bond Maturity (years)	1	5	15	30
Duration (D^*)	0.9808	4.2177	8.9919	11.7642
Convexity	1.3512	20.819	104.978	214.24
Theta (θ)	0.04079	0.04079	0.04079	0.04079
New Price	99.9275	99.6174	99.1622	98.901
%ΔV	-0.0007	-0.0038	-0.0084	-0.011
$-(D\ \Delta y)$ / %ΔV	130.0734	106.006	103.1982	102.9245
$(\theta\ \Delta t)$ / %ΔV	-29.642	-5.646	-2.5651	-1.9554
$-(Cross\ \Delta y\ \Delta t)$ / %ΔV	-0.3354	-0.040	-0.0093	-0.0019
$(½\ CON\ \Delta y^2)$ / %ΔV	-0.0932	-0.2637	-0.625	-0.9747
$(½\ \Theta_2\ \Delta t^2)$ / %ΔV	-0.0032	-0.0006	-0.0003	-0.0002

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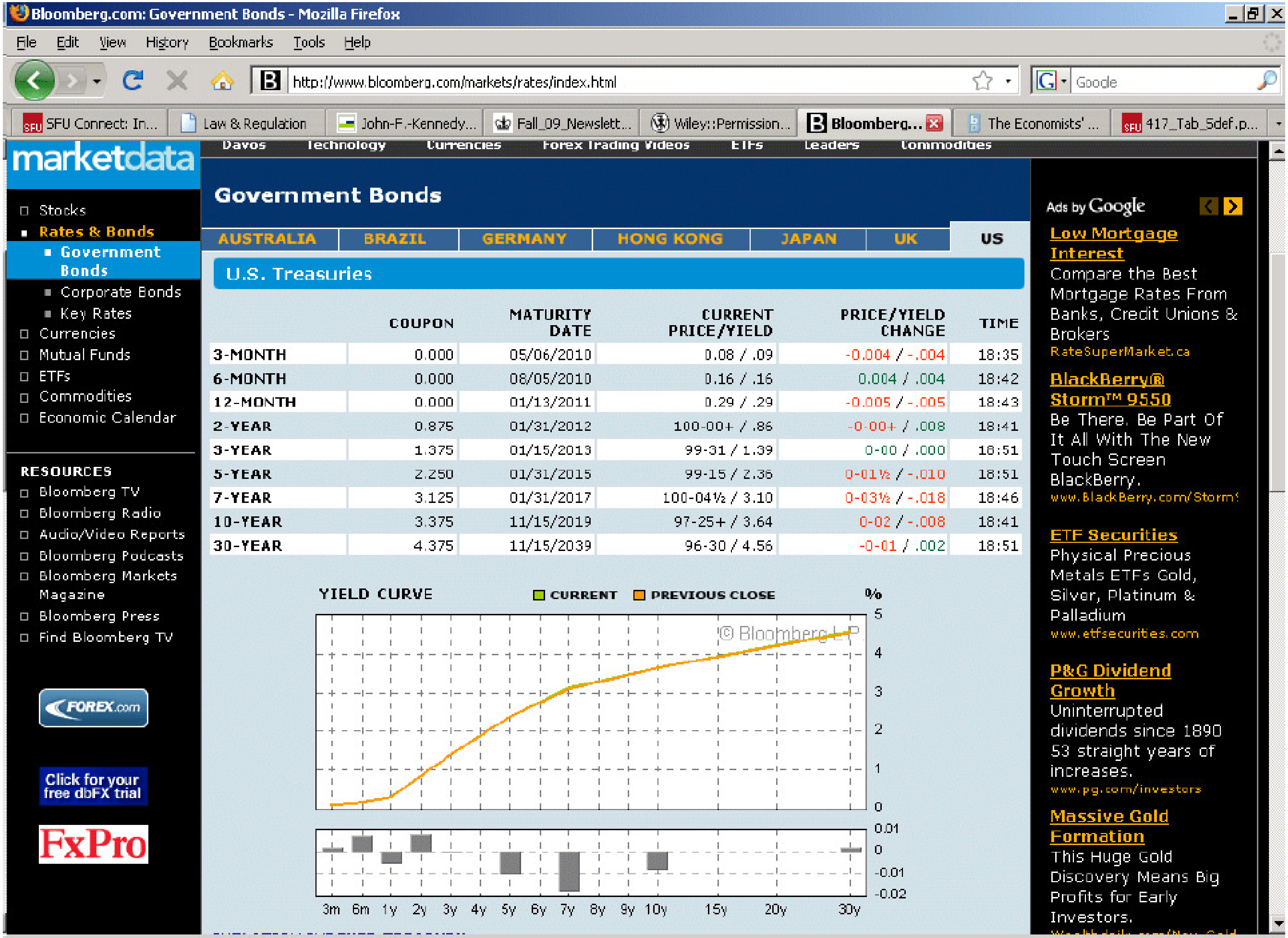
Calculating the Cost of Convexity

- The cost of convexity is the loss of time value associated with the gains associated with higher convexity when interest rates changes
  - The basic example concerns a fixed income tportfolio with a ‘bullet’ liability – a cash flow occurring on a single date – combined with a ‘barbell’ asset combination combining assets with a shorter and longer term to maturity than the liability
  - Given the asset maturities the value weights (fraction of portfolio invested in each asset) are calculated to ensure the **that the duration of assets equals the duration of the liabilities *(see midterm question!)***

Consider the following yield curve information from Bloomberg for Feb. 2, 2010



Government Bonds

U.S. Treasuries

	COUPON	MATURITY DATE	CURRENT PRICE/YIELD	PRICE/YIELD CHANGE	TIME
3-MONTH	0.000	05/06/2010	0.08 / .09	-0.004 / -.004	18:35
6-MONTH	0.000	08/05/2010	0.16 / .16	0.004 / .004	18:42
12-MONTH	0.000	01/13/2011	0.29 / .29	-0.005 / -.005	18:43
2-YEAR	0.875	01/31/2012	100-00+ / .86	-0-00+ / .008	18:41
3-YEAR	1.375	01/15/2013	99-31 / 1.39	0-00 / .000	18:51
5-YEAR	2.250	01/31/2015	99-15 / 2.36	0-01½ / -.010	18:51
7-YEAR	3.125	01/31/2017	100-04½ / 3.10	0-03½ / -.018	18:46
10-YEAR	3.375	11/15/2019	97-25+ / 3.64	0-02 / -.008	18:41
30-YEAR	4.375	11/15/2039	96-30 / 4.56	-0-01 / .002	18:51

Calculating the convexity cost for the 2/2/2010 yield curve for a 5 + 30 asset mix against a 15 year liability → need to calculate the value weights to ensure that the portfolio duration is equal to zero → compare time values

```
In[1]:= (.56 * (4.56)) + (.44 * (2.36)) // N

Out[1]= 3.592

In[3]:= ((1.0456) / (.0456)) * (1 - (1 / ((1.0456) ^ 30)))  // N

Out[3]= 16.9121

In[4]:= ((1.04) / (.04)) * (1 - (1 / ((1.04) ^ 15)))  // N

Out[4]= 11.5631

In[5]:= ((1.0236) / (.0236)) * (1 - (1 / ((1.0236) ^ 5)))  // N

Out[5]= 4.7747

In[6]:= 11.5631 - (.56 * (16.9121)) - (.44 * (4.77)) // N

Out[6]= -0.006476
```

The first lines calculate the time value of the assets using the correct value weights → can be compared with the time of the 15 year reading off the yield curve of approximate 4% → cost of convexity is about 40 basis points

The next group of the lines calculate the durations for the liabilities and assets and the final lines demonstrate that the weights chosen result in a zero duration portfolio

Notice that the interest rate level affects the duration calculations, if the values from the previous table are used incorrectly, the value weights are different and the time value of the assets is calculated incorrectly as 3.75%

D^* 5 year par bond 4.2177 15 year par bond 8.9919 30 year par bond 11.7642

Solving for the weights that set the duration of the high convexity 5 + 30 barbell to the 15 year par bond

$8.9915 = (1 - w)\, 4.2177 + w\, 11.7642 \rightarrow w = .632585 \qquad 1 - w = .367415$

The Shape of the Yield Curve provides an Estimation of the Market's Expectation of Interest rate Volatility

$$\frac{\sigma_r^2}{2}[\,CON_1 - CON_2\,] = \theta_2 - \theta_1$$

See 'Time Value-Convexity Tradeoff: Derivation of the Formula" and 16-1 Midterm Solution on Class Webpage for method of deriving the volatility estimate

Observe that the *CON* and θ values are determined from the type of portfolio selected (giving *CON*) and the shape of the yield curve (giving *θ*), leaving the ratio as the market prediction for future (short rate) interest rate volatility