EXAMPLE: VaR is not Sub-additive

The sub-additive property requires that a risk measure \( \rho[X] \) is sub-additive if:

\[
\rho[A + B] \leq \rho[A] + \rho[B]
\]

\[
\rho[A + B + C] \leq \rho[A] + \rho[B] + \rho[C]
\]

etc.

Let there be three securities A, B and C – zero coupon bonds selling at par value (100) are good – for each security there are three possible future outcomes for the price \{ X = 110, Y = 100, Z = 10 \} with Probabilities \( \text{Prob}[X] = 0.9 \) (90%) \( \text{Prob}[Y] = .095 \) (9.5%) \( \text{Prob}[Z] = .005 \) (0.5%)

The 99% VaR\[A\] = 100 - 100 (Initial Price) = 0 = 99% VaR\[B\] = 99% VaR\[C\]

\[
\text{VaR}[A] + \text{VaR}[B] + \text{VaR}[C] = 0
\]

Consider the outcomes for the portfolio with three securities

\[
\text{Prob}[X/A] \times \text{Prob}[X/B] \times \text{Prob}[X/C] = (.9)\times(.9)\times(.9) = .729
\]

\[
\text{Prob}[X/A] \times \text{Prob}[X/B] \times \text{Prob}[Y/C] = (.9)\times(.9)\times(.095) = .07695
\]

\[
\text{Prob}[Y/A] \times \text{Prob}[X/B] \times \text{Prob}[X/C] = .23085 \quad (3 \text{ cases of } 2X \text{ with } Y)
\]

\[
\text{Prob}[Y/A] \times \text{Prob}[Y/B] \times \text{Prob}[X/C] = .02436
\]

\[
\text{Prob}[X/A] \times \text{Prob}[Y/B] \times \text{Prob}[Y/C] = (.9)\times(.095)\times(.095) = .0081225
\]

\[
\text{Prob}[Y/A] \times \text{Prob}[Y/B] \times \text{Prob}[Y/C] = \text{VaR}[0] = .095 \times .095 \times .095 = .008574
\]

\[
.729 + .23085 + .008574 = .9607
\]

\[=>\text{ ALL RETURNS BELOW THIS AMOUNT HAVE A VaR > 0} \text{ (Remember VaR is a positive number when there is a loss)}\]

Therefore, VaR is not sub-additive