SOME FIXED INCOME BASICS

Geometric Series, Perpetuity and Term Annuity

Geometric Series

$$f[x] = 1 + x + x^2 + x^3 + x^4 + \ldots\ldots\ldots\ldots\ldots \textit{forever} \ldots. = \frac{1}{1 - x} \qquad (\textit{why?})$$

$$\textit{where:} \qquad |\, x \,| < 1$$

Perpetuity ($\$A$ per year forever)

$$P_{perp} = \$A \left[\frac{1}{1 + r} + \frac{1}{(1 + r)^2} + \frac{1}{(1 + r)^3} + \ldots. + \frac{1}{(1 + r)^T} + \frac{1}{(1 + r)^{T+1}} + \ldots\ldots\ldots\ldots \textit{forever} \ldots. \right]$$

$$P_{perp} = \frac{\$A}{1 + r} \left[1 + \frac{1}{1 + r} + \frac{1}{(1 + r)^2} + \frac{1}{(1 + r)^3} + \ldots. + \frac{1}{(1 + r)^T} + \ldots\ldots\ldots\ldots \textit{forever} \ldots. \right]$$

Apply geometric series solution and set $x = \frac{1}{1 + r} < 1 \quad \textit{for} \quad r > 0$ It follows:

$$P_{perp} = \frac{\$A}{1 + r} \left[\frac{1}{1 - \frac{1}{1 + r}} \right] = \frac{\$A}{r}$$

Term Annuity

Consider two perpetuities: One perpetuity is valued today: $P_{perp} = \frac{\$A}{r}$

The other perpetuity is valued at time T with first payment at $T+1$. The present value of this perpetuity is:

$$P_{perp,\ T} = \left[\frac{\frac{\$A}{r}}{(1 + r)^T} \right] = \$A \left[\frac{1}{r\,(1 + r)^T} \right]$$

Subtracting the present value of a perpetuity that has first payment at $T+1$ from a perpetuity valued today eliminates all the cash flows beyond T, to get:

$$P_A = \sum_{t=1}^{T} \frac{\$A}{(1 + r)^t} = \$A \left[\frac{1}{r} - \frac{1}{r\,(1+ r)^T} \right]$$

In the de Witt formula P_A is given as A_n to recognize that **payments are semi-annual** and discounting is being done using the square root.

Why use the Square Root?

In 315, the method of valuing an annuity with semi-annual cash flows is:

$$P_{SA} = \sum_{t=1}^{2T} \frac{\frac{\$A}{2}}{\left(1 + \frac{r}{2}\right)^{t}}$$

where $\$A$ is the annual payment, r is the annual interest rate and T is the term to maturity in years. Using this formula, the *semi-annual* interest rate is calculated by dividing the annual interest rate by 2. But this creates the following problem:

$$\frac{1}{(1 + y)} \neq \frac{1}{\left(1 + \frac{y}{2}\right)^{2}}$$

This **effective interest rate** problem does not arise if the semi-annual interest rate is calculated as as square root:

$$\left[\sqrt{1 + r}\,\right]^{2} = (1 + r)$$

Taking the square root is the method of calculating the semi-annual interest rate in de Witt.

Years' Purchase and Current Yield

If the annual payment is $\$A$ (the semi-annual payment times 2), then the **years' purchase** *(YP)* is the market price of the life annuity P_{LA} divided by $\$A$. This number is easy to calculate and is closely related to the **yield to maturity.** For a perpetuity:

$$P_{perp} = \frac{\$A}{r} \qquad YP = \frac{P_{perp}}{\$A} = \frac{1}{r}$$

Example: If $P_{perp} = \$100$ and $\$A = 10$ then $YP = 10$ and $r = 10\%$

For a **par bond** $P = 100$ and $C/P = r$. In this case, agaiin $YP = \frac{1}{r}$