Skewness Preference, Mean-Variance and the Demand for Put Options

1. Introduction

This paper explores the relationship between optimal solutions derived from mean-variance and mean-variance-skewness objective functions. The practical problem being examined is a stylized risk management decision problem for a portfolio containing an asset with a negatively skewed return distribution and an associated put option. Because a mean-variance-skewness objective function explicitly values positive skewness, it is intuitively expected that the optimal demand for a negative skewness reducing security, the put option, would be higher than the optimal demand for the put option derived using mean-variance as the objective function. However, it is demonstrated that introducing positive skewness preference into the decision-making process will typically result in a **reduction** in the optimal amount of the put option demanded compared to the mean-variance solution. This reduction result is derived assuming that the put option is priced on an actuarially fair basis such that it will have no impact on expected return. Recognizing that insurance can be modelled as a put option, this theoretical result is illustrated using a state dependent example associated with determining the optimal level of crop insurance.

In the following, Section 2 provides an overview of the moment preference approximation to expected utility as an approach to decision making under uncertainty. The mean-variance and mean-variance-skewness objectives are motivated using a Taylor series expansion of a general expected utility function. Section 3 develops the wealth process required to specify the manager's objective function. The approach used admits the possibility that both price and quantity can be uncertain at the decision horizon date. Section 4 provides theoretical analysis of the optimal solution for the put option decision problem using a mean-variance-skewness objective function and compares this solution with the mean-variance case. In Section 5, comparison between the mean-variance and mean-variance-skewness solutions is facilitated by the use of a plausible state dependent example arising in a stylized farm risk management problem: what is the optimal level of crop insurance to purchase? In this example, crop insurance is modelled as a put option on crop yield. Finally, Section 6 summarizes the contents of the paper.

2. Expected Utility and Moment Preference

The relationship between moment preference and expected utility has received considerable attention. Important topics have included: the conditions under which mean-variance analysis is consistent with maximizing expected utility, e.g., Kroll, Levy and Markowitz (1984), Ormiston and Quiggin (1994), Bell (1995); and, the implications of introducing skewness preference into the mean-variance framework, e.g., Kraus and Litzenberger (1976), Hassett et al. (1985), Lim (1989), Diacogiannis (1994). Brockett and Kahane (1992), among others, have shown that there is not a direct correspondence between the derivatives of the expected utility function and moments of the return distribution. The implication is that maximization of a function defined over moments, such as mean-variance or mean-variance-skewness, may not give the same solution as directly maximizing expected utility. Yet, Meyer (1987), Ormiston and Quiggin (1994) and others demonstrate that the conditions on the random variables sufficient for mean-variance rankings to provide solutions consistent with expected utility rankings are relatively weak. Extensions providing the conditions on random variables required for mean-variance-skewness ranking to be consistent with expected utility ranking are currently unavailable.

As discussed in numerous sources, e.g., Loistl (1976) and Levy and Markowitz (1979), the relationship between expected utility and moment preference objective functions can be motivated using a Taylor series expansion of U[W], the decision maker's utility function (U) for wealth (W), evaluated at the expected value for terminal wealth [Ω]. Modelling the risk management decision as a discrete two period problem involving the decision date, t, and the terminal (decision horizon) date, t+ 1, gives ($E[W_{t+1}] = \Omega$):

$$U[W_{t+1}] = U[\Omega] + U'[\Omega](W_{t+1}-\Omega) + \frac{U''[\Omega]}{2!}(W_{t+1}-\Omega)^2 + \frac{U'''[\Omega]}{3!}(W_{t+1}-\Omega)^3 + \ldots.. \quad (1)$$

Exploiting this type of expansion requires certain technical conditions be satisfied. For example, convergence of the power series within the interval of interest is needed.[1] In addition, desirable properties for utility functions require: U'[W] > 0, non-satiation; U''[W] < 0, risk aversion;

and, U'''[W] > 0, preference for positive skewness.

With relatively weak distributional restrictions, e.g., Hassett et al. (1985), the Taylor series representation of U[W] can be transformed into an approximation for a general expected utility function based on the moments of the distribution for W_{t+1}. The relevant approximation is derived by taking conditional expectations and ignoring terms associated with moments higher than the second, for a mean-variance approximation, and moments higher than the third, for a mean-variance-skewness approximation. The general notation EU[·] will be used to denote such a moment preference functional. From (1), taking expectations for the mean-variance-skewness case gives:

$$EU_{MVS}[W_{t+1}] \equiv EU_{MVS} = U[\Omega] + 0 + \frac{U''[\Omega]}{2!} var[W_{t+1}] + \frac{U'''[\Omega]}{3!} skew[W_{t+1}]$$

$$= U[\Omega] - b\ var[W_{t+1}] + c\ skew[W_{t+1}] \qquad (2)$$

where $var[W_{t+1}]$ is the variance of terminal wealth, $skew[W_{t+1}]$ is the skewness or centralized third moment for terminal wealth. Restrictions imposed by assuming risk aversion and positive skewness preference permit the coefficients in (2) to be immediately signed as b,c > 0. Further restrictions on b and c, as well as the admissible range of W, can be derived by developing derivative properties of (1)-(2) and invoking Jensen's Inequality. If c= 0 in (2), then the mean-variance-skewness moment preference function will reduce to the mean-variance function, EU_{MV}.

What are the implications of introducing this additional skewness term into the moment preference objective function? Currently, little information is available comparing solutions from mean-variance and mean-variance-skewness approximations. Information about such comparisons would be relevant for a range of decision making situations, especially those involving skewness altering securities such as options and insurance. The few studies that do compare the mean-variance and mean-variance-skewness objective functions illustrate some confusion as to the implications of introducing skewness. In particular, Prakash et al. (1996, p.240) claim to "show how a risk-averse manager with sufficient preference for positive skewness may undertake projects with skewed payoff distributions that appear to be unfair gambles." Horowitz (1998) correctly takes exception to the Prakash et al. claim, arguing that there is no underlying utility function which is consistent with the central theoretical condition which Prakash et al. use, i.e., $3U''/U''' > \text{skew}[W_1]/\text{var}[W_1]$. Horowitz refutes the Prakash et al. claim by demonstrating that it is not possible for an expected utility function to conform to the Prakash et al. restrictions.

Studies examining the impact of skewness have been largely concerned with asset pricing and portfolio theory, e.g., Kraus and Litzenberger (1976), Sears and Trennepohl (1983), Lim (1989), Simaan (1993). However, combining of securities into portfolios almost certainly reduces the skewness of the portfolio relative to the value weighted sum of the individual asset skewness values. The structure of the decision problem under consideration here is decidedly different, being concerned only with transforming a negatively skewed return distribution for an exogenously determined amount of a single asset into a more symmetric distribution using a put option. This type of problem is typical of many risk management situations, e.g., farming or mining, where the size of the spot position is predetermined by production considerations and the decision problem is to solve for the size of the hedge position. Allowing both the quantity of the risky asset and the number of put options to be endogenous substantively complicates the analysis without adding significantly to the usefulness of the solutions, e.g., Poitras (1993). More importantly, practical situations where put options would be purchased often involve having the level of the risky asset fixed prior to assessing the amount of option to be purchased.

Finally, while one obvious potential benefit of introducing skewness preference into the objective function is enhanced ability to model certain types of decisions problems, this gain is not without some costs. Compared to the mean-variance approach, the introduction of skewness significantly increases complexity of the solutions, permitting only complicated preference dependent closed form solutions to be derived. This is due to the presence of quadratic terms in the first order conditions arising from skew[W]. Fortunately, intuitive results can still be obtained by fully solving for the mean-variance part of the solution, leaving an additional unresolved coskewness (cosk[λ]) term which is associated with the quadratic terms in the first order conditions. This unresolved term will be utility function dependent, as it will contain the parameters b and c. Hence, solving for the mean-variance-skewness optimal demand requires b/c to be specified before an optimal solution can be obtained. In this process, the mean-variance optimal solution acts a control variate against which the mean-variance-skewness optimal solution can be compared. Properties of this comparison are developed in detail in Sec. 4.

3. The Wealth Process

In order to obtain applicability to a range of decision-making situations, the approach taken is to specify the wealth process, admitting the possibility of two random variables, both price and yield uncertainty at the decision horizon date. The representative decision maker purchases an asset at time t and sells it at time t+ 1, and purchases put options to provide protection against downside movements, either in price or yield or both. The price and the yield at t+ 1, the end of the investment horizon, can both be unknown at time t, the date the relevant risk management decision is initiated. In some types of decision problems, such as the typical problem of investment in domestic assets, this level of generality is more than is required because there is only one random variable in this problem. However, where the problem involves investment in foreign assets, there are two random variables involved, the exchange rate and the yield on foreign assets (denominated in foreign currency terms). In other problems, such as a farmer subject to random crop yield or a mine subject to random ore quality, both price and quantity are uncertain. Given that price and yield can be uncertain, the optimization problem does not permit the amount of initial wealth to invest in the asset to vary.[2] Starting from the given initial level of wealth, the investor's objective is to maximize a moment preference function for the value of terminal wealth assuming that the balance (possibly negative) of initial wealth which is not allocated to the risky asset will earn (pay) the riskfree rate of interest.

Initially, consider the wealth process for a decision maker not having access to any put options. Once the initial structure of the terminal wealth function is specified, usage of put options will be introduced. Following Poitras (1993) and others, allowing for **both** the quantity and the price to be random leads to the underlying wealth process:

$$W_{t+1} = A\, Y_{t+1} P_{t+1} + [W_t - C(A)]\,(1+r) \qquad (3)$$

where: W_{t+1} is wealth at time t+ 1 and W_t is the known level of initial wealth; A is the fixed initial size of the asset, e.g., acres planted for a farmer; Y_{t+1} is the possibly random quantity per unit or yield per unit of the asset observed at t+ 1; P_{t+1} is the random spot price at t+ 1; C(A) is the given cost function associated with purchasing A; r is the riskfree interest rate.[3] Manipulation of (3) gives the more conventional form of the wealth process for a single risky asset:

$$W_{t+1} = W_t\,(x(1+R) + (1-x)(1+r)) = W_t\,((1+r) + x(R-r)) = W_t + \pi_{t+1} \qquad (4)$$

where: π_{t+1} is the profit defined by (3) realised at time t+ 1, x is $(C(A)/W_t)$ the given fraction of initial wealth invested in the risky asset, and (1+ R) is $[(A\, Y_{t+1}P_{t+1})/C(A)]$ one plus the rate of return on the risky asset. For simplicity of exposition, it will be assumed that $x > 0$ in what follows.[4]

The basic specification for the decision maker's terminal wealth function given in (4) now requires some additional terms to capture the payoffs associated with introducing the put option. While the terminal wealth function with put options included follows appropriately from (4), some motivation is required. In particular, in the absence of some form of put option to provide asset insurance, there is a natural minimum on R, the rate of return on the investment. Either a complete catastrophic loss occurs where $Y_{t+1}= 0$, or a spot price of zero occurs at time t+ 1, both cases corresponding to the result (1+ R)= 0. Significantly, three possible variants of put option payout are possible, each aimed at dealing with the different types of risks faced by the risk manager. More precisely, put option payouts can depend on the deviation of **price**, **yield** or **revenue** from a stated exercise value. Payouts based on revenue provide protection against $Y_{t+1}P_{t+1}$ falling below a given floor. In contrast, payouts based on yield or price cannot guarantee a minimum return higher than (1+ R) = 0. For the farmer example of Section 5, while put payouts based on revenue set a lower level for farm income, put option payouts guaranteeing a price of \$K per bushel cannot prevent a 100% loss due to crop failure, nor can a put payout based on yield providing for, say, $\underline{Y}$ bushels an acre prevent the future spot price falling to zero. However, put payouts based on either price and yield do reduce the probability of the total return attaining low values and, as a result, do alter the distribution for terminal wealth.

In practice, conventional exchange traded put options are structured with payouts based on price. Other types of put options, such as multiple peril crop insurance schemes, are a type of yield insurance. Still other types of put options, such as some types of real options, provide revenue or income protection. The case where the put payout is based on revenue insurance produces a wealth process similar to the yield insurance case. Introduction of a put option based on price into (3) produces:

$$W_{t+1}^{z} = AY_{t+1}P_{t+1} + (W_t - C(A))(1+r) + Q_z(\max[0,\ K - P_{t+1}] - z)$$

$$= W_t\ \{x(1+R) + (1-x)(1+r) + \frac{Q_z P_t}{W_t}\ [\max[0,\ \frac{K - P_{t+1}}{P_t}] - \frac{z}{P_t}]\} \qquad (5')$$

$$= W_t\ \{(1+r) + x(R-r) + \gamma(\max[0,\ -R_P] - \frac{z}{P_t})\} \qquad (5)$$

where K is exercise price on the put option <u>which in going from (5') to (5) is assumed to be "at the money" (where K = P_t)</u>, z is the price per unit of output of the put, Q_z is the number (in output units) of puts purchased, with the ratio γ being the asset value covered by the option position divided by initial wealth.[5]

This specification can be contrasted with that for put option payouts based on yield where, instead of the number of options to purchase, it is the fraction of A to insure which is the decision variable:

$$W_{t+1}^{y} = AY_{t+1}P_{t+1} + (W_t - C(A))(1+r) + Q_y\ (P_{t+1}\ \max[0,\ \underline{Y} - Y_{t+1}] - L)$$

where L is the price (put premium) per unit of A for the yield put option, Q_y is the number of units covered by the yield put option and $\underline{Y}$ is the yield floor provided by the put option or insurance plan. Defining the optimization problem by allowing the risk manager to choose the fraction of A to insure leads to:

$$W_{t+1}^{y} = W_t\{(1+r) + x[R-r] + x\lambda[\max[0,\underline{RR}-R] - l]\} \qquad (6)$$

where l equals (LA/C(A)), $\lambda = (Q_y/A)$ is the fraction of A, e.g., the total planted acreage, covered or insured with the physical yield put option and $\underline{RR} = \{P_{t+1}\ \underline{Y}\ A\}/C(A)$.[6] Assuming actuarially fair pricing requires insurance to impact the decision problem through its effect on downside risk and skewness.

4. The Risk Management Decision Problem

The results in this section require assuming that: the risk manager optimizes a moment preference approximation to a general expected utility function of the form, $EU_{MVS} = U\{E[W_{t+1}]\} - b\ \mathrm{var}[W_{t+1}] + c\ \mathrm{skew}[W_{t+1}]$, where b and c are measures of the sensitivity of EU to changes in var[·] and skew[·], with $b,c > 0$; and, that all options/insurance premiums are "fairly priced". It is also assumed that R has a negative skewed probability density function, where skew[R] < 0.[7] The following now applies:[8]

Proposition: **The Optimal Demand for Put Options**[9]

Assuming that the risk manager optimizes a moment preference objective function, defined over the mean, variance and skewness of terminal wealth as given by (2) and (6), then the optimal demand for a put option with payoff depending on **yield** is:[10]

$$\lambda^* = -\frac{\sigma_{Rq}}{\sigma_q^2} + x\,\frac{3c\ W_t}{2b\ \sigma_q^2}\,\{cosk[\lambda^*;\underline{RR}]\} \qquad (7)$$

where: the subscript q corresponds to the random variable max[0, $\underline{RR}$ - R], λ^* is the optimal solution to the mean-variance-skewness objective function and the subscript R refers to the rate of return on the asset. The coskewness term, cosk[λ;$\underline{RR}$], has the interpretation:

$$cosk[\lambda;\underline{RR}] \equiv cosk[\lambda] = E\{\lambda^2 (\max[0,\underline{RR}-R] - E[\max[0,\underline{RR}-R]])^3$$
$$+ (\max[0,\underline{RR}-R] - E[\max[0,\underline{RR}-R]])(R - E[R])^2$$
$$+ 2\lambda (\max[0,\underline{RR}-R] - E[\max[0,\underline{RR}-R]])^2 (R-E[R])\}$$
$$\equiv \lambda^2\, skew[q] + \sigma_{qR^2} + 2\{\lambda\, \sigma_{q^2R}\}$$

The mean-variance solution (λ_{MV}) is given by ignoring the second term on the rhs of λ^* in (7). The closed form solution associated with (7) is:

$$\lambda^* = \frac{(\sigma_q^2 - 2B\,\sigma_{q^2R}) - \sqrt{(\sigma_q^2 - 2B\,\sigma_{q^2R})^2 - 4B\, skew[q](B\,\sigma_{R^2q} - \sigma_{qR})}}{2B\, skew[q]} \qquad (8)$$

where B = $x(3c\, W_t)/2b$.

The stated closed form solution is one of two roots of the quadratic equation in λ specified in (7). Using results contained in the following corollaries, it possible to verify by differentiating the first order condition that the stated closed form solution corresponds to a maximum, while the other root corresponds to a minimum.

Closer analysis of the Proposition can be used to identify conditions for which the optimal put option demand derived from the mean-variance-skewness objective, λ^*, is less than the mean-variance optimal demand, λ_{MV} . A result such as $\lambda_{MV} \geq \lambda^*$ is interesting because it is seemingly counter-intuitive since the mean-variance-skewness function explicitly values positive skewness and the put option is a security which reduces the negative skewness in asset returns. However, the λ which maximizes skewness of W is typically less than the λ which minimizes the variance of W, making the solutions considerably more complicated than simple intuition would suggest.[11] While it is possible to work with the closed form (8) to derive the relevant conditions, it is more transparent to derive results from (7) and then verify the results by making the appropriate substitutions in (8).[12] For example, from (7) it is apparent that negative cosk[·] at the λ^* optimum is required for $\lambda_{MV} \geq \lambda^*$ to apply. Observing that cosk[·] is a quadratic function of λ leads to consideration of three points associated with cosk[·] function: the two roots of the quadratic which are associated with cosk[·] = 0; and, the minimum point of the cosk[·] function. In this regard, completeness requires verification that the roots of the quadratic are both real and that the solution of the first order condition of the cosk[·] function is a minimum.

In order to establish the properties of the cosk[·] function, a number of useful corollaries can be developed from the Proposition. Examining the proof of the Proposition reveals that the cosk[·] term is the operative part of the derivative (d skew[W]/dλ). Hence, solving for the roots of the quadratic as solutions to the problem cosk[·] = 0 is the same as solving (d skew[W]/dλ) = 0. It follows that roots of the cosk[·] function correspond to optimum points of skew[W]. Hence, the following corollary can be derived:

Corollary 1: Properties of max skew[W][13]

At the point where skew[W_{t+1}] is maximized, the slopes of EU_{MV} and EU_{MVS} will be equal:

$$\frac{dEU_{MV}}{d\lambda}\bigg|_{\lambda_{msk}} = \frac{dEU_{MVS}}{d\lambda}\bigg|_{\lambda_{msk}}$$

where λ_{msk} is the value of λ when skewness is maximized. Because cosk[λ] = 0 at this point:

$$\lambda_{msk} = -\frac{\sigma_{q^2,R}}{skew[q]} - \frac{\sqrt{\sigma^2_{q^2\,R} - skew[q]\ \sigma_{q,R^2}}}{skew[q]}$$

Except in limiting cases, λ_{msk} is not equal to the optimum values λ_{MV} and λ^*.

Because the slopes of the MV and MVS EU functions are equal at max skew[W], the optimum values for λ_{MV} and λ_{MVS} are either equal or **both** greater than λ_{msk} or both less than λ_{msk}. If $\lambda^* > \lambda_{MV}$, then cosk[λ] > 0 and d{EU_{MV} - EU_{MVS}}/dλ < 0 at λ^*. If $\lambda^* < \lambda_{MV}$, then cosk[λ] < 0 and d{EU_{MV} - EU_{MVS}}/dλ > 0 at λ^*.

To establish that $\lambda^* < \lambda_{MV}$, it is expedient to consider the second derivative d^2skew[W]/dλ^2. The solution of this derivative is λ_{min}, the value of λ which minimizes cosk[λ]. This leads to:

Corollary 2: The minimum value of cosk[λ]

The minimum value of cosk[λ] is given by:

$$\frac{d^2 skew[W_{t+1}]}{d\lambda^2} \quad \rightarrow \quad \frac{dcosk}{d\lambda} = 0 \quad \rightarrow \quad \lambda_{\min} = -\frac{\sigma_{q^2,R}}{skew[q]}$$

At λ_{min}, cosk[λ_{min}] = cov[q,R^2] + λ_{min} cov[q^2, R] $\leq$ 0. It follows that at λ_{min}:

$$\frac{dEU_{MV}}{d\lambda}\Big|_{\lambda_{\min}} - \frac{dEU_{MVS}}{d\lambda}\Big|_{\lambda_{\min}} \geq 0$$

With the aid of two additional theoretical results, it is possible to combine the Corollary 2 result with the slope condition at λ_{msk} from Corollary 1 and show that $\lambda_{msk} \leq \lambda^* \leq \lambda_{MV}$. More precisely, Corollaries 1 and 2 provide information on the relative slopes of the EU_{MV} and EU_{MVS} functions at two separate points. By imposing some relatively weak restrictions on the discrete state space, it is possible to show that $\lambda_{min} = \lambda_{MV}$. This implies that the slope of EU_{MV} = 0 at this point. Observing that the EU functions are at least locally concave in λ, it follows from the relative slope conditions and $\lambda_{msk} \leq \lambda_{min} = \lambda_{MV}$ that $\lambda_{msk} \leq \lambda^* \leq \lambda_{MV}$.

The result that $\lambda^* < \lambda_{MV}$ cannot be proved in the general case.[14] Without some restrictions on the state space, it is possible to construct examples where either $\lambda^* > \lambda_{MV}$ or where $\lambda^* = \lambda_{MV}$, though it does appear that such situations are uncommon. The $\lambda^* < \lambda_{MV}$ result does hold if some relatively weak assumptions are imposed on the state space. In particular, a **sufficient** set of restrictions on the state space is: the state space is discrete; there are more than two possible states; and, there is only one state, with probability less than one-half, where a put option payoff is permitted. If the assumptions about the discrete state space and one payoff state are imposed but the state space only contains only two possible states then the following special case emerges:

Corollary 3: The Two State Space Case

If there are only two possible futures states, with probabilities p_1 and (1 - p_1) respectively, then: $\lambda^* = \lambda_{MV} = \lambda_{msk} = \lambda_{min}$; and, $(cov[q^2,R])^2$ - skew[q] cov[q,R^2] = 0.

In terms of the quadratic function of λ associated with cosk[$\cdot$], the two state case solution has only one root, with the minimum point of the quadratic corresponding to cosk[$\cdot$] = 0.

Inspection of the closed form solution (8) reveals the specific parametric configurations which determine the correspondence between λ^* and λ_{MV}. More precisely, (8) can be rewritten as:

$$\lambda^* = -\frac{\sigma_{q^2R}}{sk[q]} +$$

$$\frac{\sigma_q^2}{2B\ sk[q]} \{1 - \sqrt{1 + \left[\frac{2B}{\sigma_q^2}\right]^2 (\sigma_{q^2R}^2 - sk[q]\sigma_{R^2q}) + \frac{4B(sk[q])}{\sigma_q^2} (\frac{\sigma_{qR}}{\sigma_q^2} - \frac{\sigma_{q^2R}}{sk[q]})}\ \}$$

The proof of Corollary 3 demonstrates that, in the two state case, the terms inside the square root reduce to 1. In addition, for both the two state and greater than two state cases, the following result holds as a consequence of assuming one payoff state:

$$\lambda_{\min} = \frac{\sigma_{q^2R}}{sk[q]} = \frac{\sigma_{qR}}{\sigma_q^2} = \lambda_{MV}$$

When there are more than two states and only one payoff state for the option, then the last term under the square root is zero, but the middle term can only be signed as:

$$cov[q^2R]^2 - sk[q]\ \sigma_{R^2q} = \sigma_{q^2R}^2 - sk[q]\ \sigma_{R^2q} > 0$$

This result requires the probability on the payout state to be less than one-half. That this expression equals zero in the two state case is a very special case, arising from the ability to construct a zero variance, zero skewness portfolio. To see this, observe that the two state case with one payoff state has $R = \{R_1, R_2\}$ and $q = \{q_1, 0\}$. A portfolio of the option and the asset gives $R + \lambda q = \{R_1 + \lambda q_1, R_2\}$. Setting $\lambda = \{R_2 - R_1\}/q_1$ produces a zero variance, zero skewness portfolio. Except in special cases, this is not possible with more than two states.

Allowing the put option to payout in only one state, together with the assumption that there are more than two possible states, produces the result of immediate interest:[15]

Corollary 4: Conditions Associated with $\lambda^* < \lambda_{MV}$

If there are more than two possible future states and the put option is permitted to payout in only one state with probability less than one-half, then the following conditions apply:

$\lambda_{msk} < \lambda^* < \lambda_{MV} = \lambda_{min}$; and, $(cov[q^2,R])^2 - skew[q]\ cov[q,R^2] > 0$.

Of all the parameters listed in Corollary 4, the sign of $cov[q,R^2]$ is the most difficult to identify theoretically, though it is relatively straightforward to provide a heuristic interpretation. From skew[q] > 0, the condition $cov[q,R^2] > 0$ is expected because the greatest variation in the **negatively skewed** R will be associated with the put option pay out. The more negatively skewed is the return distribution, the greater the need for the put option and the more positive is $cov[q,R^2]$. However, because the put option also acts to reduce the negative skewness in terminal wealth, the greatest variability in max[0,RR-R] will occur when (R-E[R]) is negative, making $cov[q^2,R]$ negative. This discussion illustrates the importance of the proof of Corollary 4 which demonstrates that $(cov[q^2,R])^2$ is large enough that the result $\lambda_{MV} > \lambda^*$ is obtained.

It is also possible to motivate the analysis in another way: by exploiting conditions associated with the Taylor series expansion it is possible to demonstrate that $cosk[\lambda] \leq 0$ **at the optimum** for both EU_{MV} and EU_{MVS} will be the case and $\lambda_{MV} \geq \lambda^*$ is the result of assuming the sufficient restrictions on the state space. One immediate implication of differencing the Taylor series representations for EU_{MV} and EU_{MVS} is the result:[16]

$$\frac{d[EU_{MV} - EU_{MVS}]}{d\lambda} = -\frac{U'''}{3!}\frac{d\ skew[W_{t+1}]}{d\lambda} \qquad (9)$$

In (9), the term associated with $dW_{t+1}/d\lambda$ has been set equal to zero and does not appear due to the assumption that the option is priced on an actuarially fair basis. The condition (9) applies over the range of λ and not just at the optimum. Intuitively, it may be expected that because $skew[W_{t+1}]$ is negative, increases in λ will increase $skew[W_{t+1}]$ by making it less negative. However, this in not the case across the range of λ due to the subtle impact that the introduction of the put option has on $skew[W_{t+1}]$ through $cov[q^2,R]$. The intuitive interpretation is incorrect because $skew[W_{t+1}]$ is not monotonic in λ. Prior to reaching the minimum point for var[W], increases in λ will produce reductions in $skew[W_{t+1}]$.

5. A State Dependent Example: Crop Insurance

Crop insurance can be characterized as a put option on crop yield. The representative farmer plants a crop at time t and harvests it at time t+ 1. Both the price at harvest and the quantity harvested are unknown at time t, the date the relevant risk management and planting decisions are initiated. The farmer's optimization problem will not permit the amount of initial wealth to invest in crop production to vary. In practical terms, taking the costs associated with planting the total acreage to be given corresponds to a stylized "wheat farmer" or "apple orchard" where only one crop is fully and systematically planted over the available acreage. The underlying wealth process for the farmer can be specified as in (6), where W_{t+1} is wealth at time t+ 1 and W_t is the known level of initial wealth; A is the number of acres planted; Y_{t+1} is the random yield per acre observed when the crop is harvested at t+ 1; P_{t+1} is the random spot price at t+ 1; C(A) is the given cost function associated with planting the A acres; r is the riskfree interest rate.

To illustrate the practical implementation of (8), consider the following state dependent example:

	STATE 1	2	3	4
	Pr. = .1	Pr. = .4	Pr. = .4	Pr. = .1
Y_{t+1}	0.6	1.3	1.5	1.2
P_{t+1}	1.1	.85	1.0	0.5

These price yield pairings exhibit a slight, negative correlation of -.0492. From these values, calculation of (1+ R) requires specifying planting costs per acre, C(A)/A. Assuming for simplicity that C(A)/A = 1 gives (1+ R) = {.66, 1.105, 1.5, .6}, with skew[R] = -.0169 and var[W] = .1037. Using this return variable, appropriate selection of the "exercise prices" ($\underline{Y}$) permits identification of the precise amount of the insurance payoffs in the relevant states. For example, taking {$\underline{Y}$ = 1.2} yield insurance will payoff in state 1 such that q = max[0, $\underline{RR}$ - R] = {.66, 0, 0, 0}. The statistical parameters required for determining (8) can now be calculated as:

$$\sigma_{Rq} = -.0335 \qquad \sigma_q^2 = .03904$$

$$skew[q] = .0207 \qquad \sigma_{q^2 R} = -.0177 \qquad \sigma_{q R^2} = .0102$$

Using these statistical parameters, it is possible to derive a precise numerical value for the optimal solution (7).

As stated, (7) is composed of two (rhs) terms, the first corresponding to the mean-variance part of the solution and the second associated with the impact on expected utility of the coskewness in terminal wealth. This method of stating the optimality conditions is used for interpretative purposes, when expressed as a closed form, as in (8), the optimality conditions are less revealing. However, it is apparent from both (7) and (8) that the impact of skewness on the optimal solution depends on parameters of the moment preference expected utility function (b,c). Because the mean-variance component depends only on the ratio of covariance to variance, this part of the solution can be immediately determined from the statistical parameters as $\lambda_{MV} = -\{ \sigma_{Rq}/ \sigma_q^2\} =$ .855. The mean-variance-skewness solution, λ^*, is determined by solving the quadratic equation associated with (7) to get (8). The relevant calculations are facilitated by observing that the dimensionless number B = $x\{3c\ W_t/2b\}$ can be taken to be a constant, dependent on specific preference assumptions.

Using this approach, it is possible to evaluate the optimal yield insurance positions as various parameters are altered. As an initial starting point, take $x\{3c\ W_t/2b\} = \{3c\ C(A)/ 2b\} = 1.5$ which gives $\lambda_{MV} = .855$ and $\lambda^* = .688$. This state dependent example is consistent with the result from of Sec. 4 that **the impact of introducing skewness preference into the moment preference approximation to a general expected utility function results in a reduction of the optimal yield insurance position from that indicated by the mean-variance part of the solution**. This seemingly counter-intuitive result can be attributed to the different impact that λ has on cosk[RR], skew[W] and var[W]. To see this, consider the changes in these statistics as λ changes. Using the parameters from the state dependent example:

λ	skew[W]	var[W]	cosk[RR]	
0	-.0169	.1037	.01020	
.2	-.0127	.0919	.00395	
.3659	-.01181	.0845	.00000	λ_{msk}
.4	-.01185	.0832	-.00065	
.6	-.0132	.0776	-.00359	
.688	-.0143	.0762	-.00436	λ^*
.8	-.0158	.0752	-.00487	
.855	-.0167	.0751	-.00494	$\lambda_{MV} = \lambda_{min}$

1.0	-.0188	.0759	-.00450
1.2	-.0209	.0797	-.00247

The skew[W] and var[W] columns represent the relevant portion of the opportunity set for the farmer's optimal choice problem using mean-variance-skewness objective function. The cosk[RR] is the term which determines the difference between the mean-variance and mean-variance-skewness solutions. Insofar as this term is negative, the introduction of positive skewness preference will result in a reduction in the optimal yield insurance position ($\lambda_{MV} > \lambda^*$). Intuitively, this follows because the mean-variance solution produces a lower skew[W] than λ^*.

From the numbers provided, it is apparent that the λ associated with the minimum variance of wealth point does not correspond with λ for maximum (least negative) skewness. In addition to having implications for modelling the optimal crop insurance position, these results also have more general theoretical interest. Contrary to the examples provided by Brockett and Kahane (1992), the underlying opportunity set associated with the state dependent example exhibits a tradeoff between skewness and variance. While it may be possible to construct counter-intuitive examples where the decision maker will choose the prospect which has both higher variance and lower positive skewness, such situations may be of limited practical importance. The subtle interaction between moments will tend to create tradeoffs (between risk and return/ between risk and skewness). The rankings for these types of choices using moment preference expected utility functions may not differ substantively from rankings derived from exact expected utility functions. However, the state dependent example does illustrate that comparison of optimal solutions obtained from the mean-variance and mean-variance-skewness moment preference functions can produce counter-intuitive results.

Compared to the mean-variance solution, a significant disadvantage of the mean-variance-skewness solution given in the Proposition is the presence of parameters, b and c, associated with derivatives in the Taylor series expansion of the exact expected utility function. Changing any of the terms in x{3c W_t/2b} can produce variation in the optimal solution for the mean-variance-skewness objective function. In turn, changes in b and c can also imply changes in the optimal solution obtained from exact expected utility functions, such as the power or negative exponential. Consider the implications of assuming power utility: U = $\{W^{\alpha}/\alpha\}$. Recalling the Taylor series interpretations of c = U'''/3! and b = -U''/2! , it follows that 3c W_t/2b = $(2 - \alpha)/2$ where {U''' W_t/U''} = $(\alpha - 2)$ can be related to the more familiar measure of relative risk aversion, {U'' W_t/U' } = $(\alpha - 1)$. Variation in the optimal solution λ^* as x{3c W_t/2b} changes can now be contrasted with changes in the optimal solution obtained from directly maximizing power utility (λ_{PEU}).

As discussed in Section 2, Brockett and Kahane (1992), among others, have shown that there is not a direct correspondence between the derivatives of the expected utility function and moments of the return distribution. To explore the practical implications of this observation in the present context, various λ^* and λ_{PEU} were simulated using x= 1 giving:

α	λ_{PEU}	x{3c W_t/2b}	λ^*
.9	.761	.55	.787
.6	.757	.70	.769
.3	.732	.85	.753
.001	.706	1.00	.736
-.3	.679	1.15	.721
-.6	.650	1.30	.706
-.9	.621	1.45	.692
-1.0	.611	1.50	.688
-2.0	.512	2.00	.648
-4.0	.341	3.00	.588
-5.0	.277	3.50	.520
-10.0	.107	6.00	.499
-18.0	.020	10.00	.416
-98		50.00	.385
-19998		10,000	.366
$-\infty$	0		

While the presence of the preference parameters b and c permit λ^* to provide a substantially better approximation than λ_{MV} to the exact power utility solution λ_{PEU}, the mean-variance-skewness approximation is lacking for extreme values of α. In addition, it is not possible to make precise inferences about a given value for x{3c W_t/2b} and the associated value of α. For example, the specific solution provided previously (λ^* = .688) corresponds approximately to α = -.3 for λ_{PEU}. However, for x = 1, x{3c W_t/2b} = 1.5 corresponds to α = -1. Despite these problems, the behaviour of λ^* does reflect the qualitative result that: increases in the degree of relative risk aversion (α becoming more negative) produces a smaller fraction of the total acreage covered with yield insurance.

In addition to the form of the utility function, variation in λ^* can be introduced by changing a number of other parameters such as the "exercise price" $\underline{Y}$ or the assumed distributions for the state variable, in this case the probabilities associated with each state for the state dependent example. Consider the impact of changing $\underline{Y}$:

$\underline{Y}$	q	λ_{MV}	λ^*	λ^*q	skew[q]
1.6	{1.1, .255, .1, .2}	.745	.631	{.694, .16, 0.6, .13}	.0547
1.5	{.99, .17, 0, .15}	.801	.737	{.73, .125, 0, .111}	.0503
1.4	{.88, .085, 0, .1}	.822	.724	{.637, 0.6, 0, .072}	.0409
1.3	{.77, 0, 0, .05}	.790	.680	{.525, 0, 0, .034}	.0321
1.2	{.66, 0, 0, 0}	.855	.688	{.454, 0, 0, 0}	.0207
1.1	{.55, 0, 0, 0}	1.026	.825	{.454, 0, 0, 0}	.0120
1.0	{.44, 0, 0, 0}	1.230	1.03	{.454, 0, 0, 0}	.0061

This λ^* behaviour reveals a constancy-of-income/monotonicity of λ^* result when the insurance pays off in only one state. In order to maintain a constant level of income, as $\underline{Y}$ falls the fraction of total acres to insure must increase, confirming the comparative static result that $(d\lambda^*/d\,\text{skew}[q]) < 0$. As $\underline{Y}$ gets increasing smaller, the increase in λ^* continues until 'overinsurance' ($\lambda^* > 1$) is indicated. However, this result does not extend beyond the single payout state case. As $\underline{Y}$ increases, payouts occur in more states, to the point where all states have payoffs and monotonically declining underinsurance emerges.

In addition to variation in $\underline{Y}$, the amount of over or under insurance depends on the probability attached to the disaster state. In general, as this probability increases the optimal fraction of planted acreage covered by yield insurance decreases. For example, set $\underline{Y} = 1.2$ and $x\{3c\,W_t/2b\} = 1.5$ and change the probabilities for the four states from {.1,.4,.4,.1} to:

Prob.	λ_{MV}	λ^*	r_{PY}
{.2,.3,.3,.2}	.707	.465	-.24
{.3,.4,.25,.05}	.833	.697	-.521
{.4,.25,.25,.1}	.909	.744	-.495
{.5,.2,.25,.05}	.896	.711	-.569

{.05,.4,.4,.05}	.891	.724	+.386
{.05,.25,.25,.45}	.469	.205	+.253

where r_{PY} is the correlation between price and yield. These results do not mimic the constancy of income result associated with changing $\underline{Y}$. A reduction in the acreage covered by insurance does not always occur as the probability of the disaster state increases. While this is partially true, the monotonic increase in λ^* depends not only on the disaster state probabilities but also the probabilities in the other states. Contrary to the $\underline{Y}$ case, λ^* does not increase monotonically until overinsurance emerges. In addition, while useful for illustrative purposes, changing the probabilities also has other less desirable implications, such as altering the correlation between P and Y to values which are not consistent with the observed P and Y behaviour for most crops.

Finally, some state dependent examples were constructed to provide numerical illustrations of situations where $\lambda^* > \lambda_{MV}$. In general, such examples require there to be more than one payout state for the option. One such example is the following:

Pr. = {.1, .4, .3, .1, .1}   (1+ R) = {.2, 1.2, 1.5, 2.0, 2.5}   q = {1.1, 0, 0, .6, 0}

As in the previous case, letting B = 1.5 produces $\lambda^* = 1.07 > \lambda_{MV} = .75$. However, closer inspection reveals that this example violates the implicit assumption associated with the restriction of R to be 'negative skewed'. In other words, one of the option payouts in this example is associated with a state which is not a disaster state. A slight reduction in the payout for this state to q = {1.1, 0, 0, .4, 0} required resetting B = 6 in order to obtain $\lambda^* = 1.04 > \lambda_{MV} = .94$. Using simple trial and error, considerable effort did not produce examples of cases where $\lambda^* > \lambda_{MV}$ when payouts were restricted only to disaster states. In practice, situations where put option payouts occur in high return states would be uncommon, with such situations being ruled out completely for revenue insurance. Hence, it would appear that $\lambda^* < \lambda_{MV}$ is the situation which would be encountered in practical applications.

6. Summary

This paper has examined the relationship between the optimal solutions derived from mean-variance and mean-variance-skewness expected utility functions. The specific decision optimization problem examined involves determining the demand for a put option in a portfolio which also contains an asset with a negatively skewed return distribution. The specification permits both asset prices and yields to be uncertain. The manager is permitted to have access to put options which either protect against asset yields falling below a given floor or against asset prices falling below an at-the-money exercise price. Assuming that the put option is actuarially fairly priced, the optimality conditions for a mean-variance-skewness expected utility function are derived and compared with the mean-variance solutions. Both theoretically and using a state dependent example it was demonstrated that, if managers are risk averse and have a preference for positive skewness, then the optimal demand for put options derived using a mean-variance objective function will typically be **greater** than the optimal demand indicated by a mean-variance-skewness objective function. This is seemingly counter-intuitive since the put option is a security which reduces the negative skewness in asset returns. Explicit recognition of skewness in the objective function should, presumably, result in a higher optimum demand than indicated by an objective function defined only over mean and variance.

In a state dependent example which compared the optimal crop insurance (put option on yield) solutions for the mean-variance and mean-variance-skewness cases, a substantive difference in complexity was revealed. As indicated in the theoretical results, the mean-variance solution depends solely on statistical parameters, while the mean-variance-skewness solution is preference dependent. The added complexity of the preference terms does permit the mean-variance-skewness solution to provide a better approximation for various exact expected utility functions. For the specific case of the constant relative risk aversion power utility function, it was demonstrated that as the degree of relative risk aversion increases (the farmer becomes more risk averse) the optimal fraction of total acres to insure diminishes in both the mean-variance-skewness and exact power utility cases. While this is an improvement over the constancy of the mean-variance solution, there was considerable divergence between the exact power utility solution and the mean-variance-skewness solution when the degree of relative risk aversion was high.

In addition to these results, the optimal solutions also permit a number of other inferences. For example, a number of parameters are found to impact the solutions: the fraction of initial wealth invested in crop production (x); the exercise 'price' levels; and, the assumed distribution for crop yields and prices, including the probabilities attached to the disaster states. Each of these factors are found to impact the effect of skewness on the optimal usage of crop insurance. More precisely, as the fraction of initial wealth invested in crop production (x) increases, the impact of coskewness on the optimal solution also increases. Because coskewness is likely to be negative, this implies that the optimal fraction of total acres to insure will fall as x increases. Similarly, as the exercise 'prices' on the insurance fall, the amount of insurance required to protect revenue against disaster states must increase. The immediate implication is the possibility of overinsurance: that the number of acres to insure may be greater than the number of acres planted. Typically, as the probability of the disaster state increases, the optimal fraction of acres to insure will fall in order to maintain the desired income level. However, this result does depend on the probabilities assigned to other states.

Finally, this paper provides considerable opportunities for future research. The paper demonstrates that $\lambda^* \leq \lambda_{MV}$ for a put option with negatively skewed W and R. This leave three other situations to be worked out: put options in combination with positively skewed W; call options with positively skewed W; and, call options with negatively skewed W. While the approach of this paper could be used in the derivation of these results, it does not necessarily follow that the main result of this paper, $\lambda^* \leq \lambda_{MV}$, applies to those cases. By altering the composition of the underlying portfolio, e.g., to permit short positions in the underlying assets, further situations could also be explored. Another potentially interesting problem which can be considered is to explore the restriction that b is the same for both EU_{MV} and EU_{MVS}. This assumption is essential to the results given in this paper. An alternative approach would be to explore the relationship between b and c required for $\lambda_{MV} = \lambda^*$. Put differently, because skewness and variance are functionally related, the addition of skewness to the moment preference expected utility function requires an adjustment to be made to the assumed level of risk aversion in order to reconcile the optimal solutions. Again, the approach used in this paper could be adapted to resolve this problem.

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Appendix

Proof of Proposition:

The objective is to maximize the mean-variance-skewness expected utility of terminal wealth as specified in (6) using (2) to derive (8). Invoking the assumption of "fair pricing" of options, this leads to the following mean, variance and skewness functions, where q corresponds to the max[0, <u>RR</u> - R] random variable associated with the yield insurance case:

$$E[W^y_{t+1}] = W_t \{(1+r) + x\ E[R-r]\}$$

$$var[W^y_{t+1}] = W_t^2\ \{x^2\sigma_R^2 + \lambda^2 x^2 \sigma_q^2 + 2\lambda x^2 \sigma_{qR}\}$$

$$skew[W^y_{t+1}] = W_t^3\ \{x^3\lambda^3\ skew[q] + x^3\ skew[R] + 3\{x^3\ \lambda^2\ \sigma_{q^2R} + x^3\ \lambda\sigma_{R^2q}\}\}$$

Using these functions, evaluating the first order conditions for $\partial EU/\partial\lambda$ using the expected utility function (EU = $E[W_{t+1}]$ - b var$[W_{t+1}]$ + c skew$[W_{t+1}]$) and assuming the risk parameters b, c > 0, gives the following optimality condition:

$$\frac{\partial EU}{\partial \lambda} = -2b\ W_t^2\{x^2\lambda\ \sigma_q^2 + 2x^2\sigma_{Rq}\} + 3cW_t^3\ \{x^3\lambda^2\ skew[q] + x^3\sigma_{R^2q} + 2x^3\lambda\sigma_{q^2R}\} = 0$$

Manipulating and using the definition for cosk[·] provided in the Proposition gives the required result (8) for the yield insurance case.

Proof of Corollary 1:

From inspection of the first order condition from the Proposition, it follows that the λ that solves {d skew[W]}/dλ = 0 corresponds to the λ that solves cosk[λ] = 0. In this case, the first order conditions for EU_{MV} and EU_{MVS} will be equal, which is equivalent to the stated condition that the slopes of the two functions be equal.

Because cosk[λ] is quadratic in λ, there will be two λ which provide solutions to the quadratic equation corresponding to cosk[λ] = 0. Evaluating which is a maximum and minimum value requires some information on the signs of certain values. To establish that λ_{msk} is the maximum value for skewness, observe that the second derivative of skew[W] will have the same sign as the first derivative of cosk[λ]. Substitute the λ_{msk} from the Corollary into the first derivative of cosk[λ] to get:

$$\frac{dcosk[\lambda]}{d\lambda}\ ||_{\lambda_{msk}} = 2\lambda\ skew[q] + 2\ \sigma_{q^2,R}\ ||_{\lambda_{msk}} = -\sqrt{cov_{q^2,R}{}^2 - skew[q]\ cov[q,R^2]} \le 0$$

When the value under the square root is positive, this indicates that the second derivative is negative and the stated value is a maximum for skew[W]. For the other root, the value under the square root would have to be negative for a maximum, indicating that the solution is imaginary. This case is considered in Corollary 4 and ruled out.

Proof of Corollary 2:

To solve for λ_{min}, set the first derivative of cosk[λ] = 0 and solve. Substituting λ_{min} back into the expression for cosk[λ] provides the formula for cosk[λ_{min}]. To show that cosk[λ_{min}] is negative, substitute the value of λ_{min} into the expression and solve. Also observe that from the λ_{msk} solution in Corollary 1 the value inside the square root will be non-negative and less than or equal to λ_{min}, implying that $\lambda_{msk} \leq \lambda_{min}$. Finally, to better describe the result that $\lambda^* < \lambda_{MV}$, observe that EU_{MV} and EU_{MVS} are concave in λ. Because the slopes of these two functions are equal for λ_{msk} and the slope of $EU_{MV} \geq$ the slope of EU_{MVS} at λ_{min}, the maximum point ($dEU_{MVS}/d\lambda$) must occur at a point no lower than the λ for ($dEU_{MV}/d\lambda$). Because λ_{msk} occurs at cosk[λ] = 0 and $\lambda^* \leq \lambda_{MV}$ implies cosk[λ] ≤ 0, it must also be the case that $\lambda_{msk} \leq \lambda^* \leq \lambda_{MV}$.

Proof of Corollary 3:

The restriction of the state space to permit only one state where there is an option payout facilitates the derivation of specific expressions for almost all the parameters of relevance. More precisely, let there be k possible states and let p_i be the probability of state i. If q_i is the option payout in state i, then the one payout state assumption requires $q_1 > 0$ and $q_i = 0$ for i = {2,....k}. Given this, the following results apply:

$$\sigma_{qR} = \sum_{i=1}^{k} p_i\,(q_i - \bar{q})(R_i - \bar{R}) = \sum_{i=1}^{k} p_i q_i\,(R_i - \bar{R}) - \bar{q}\sum_{i=1}^{k} p_i\,(R_i - \bar{R})$$

$$= \sum_{i=1}^{k} p_i q_i\,(R_i - \bar{R}) = p_1 q_1\,(R_1 - \bar{R})$$

Observing that:

$$\bar{q} = \sum_{i=1}^{k} p_i\, q_i = p_1\, q_1$$

It follows that:

$$\sigma_q^2 = \sum_{i=1}^{k} p_i\,(q_i - \bar{q})^2 = \sum_{i=1}^{k} p_i\,(q_i - \bar{q})(q_i - \bar{q}) = p_1 q_1\,(q_1 - \bar{q})$$

From this:

$$\lambda_{MV} = -\frac{\sigma_{qR}}{\sigma_q^2} = -\frac{R_1 - \bar{R}}{q_1 - \bar{q}} = -\frac{R_1 - \bar{R}}{q_1(1 - p_1)}$$

The other relevant parameters needed to show $\lambda_{MV} \geq \lambda_{min}$ are:

$$\sigma_{q^2R} = \sum_{i=1}^{k} p_i\,(q_i - \bar{q})^2(R_i - \bar{R}) = \sum_{i=1}^{k} p_i\,(q_i^2 - 2q_i\bar{q})(R_i - \bar{R})$$

$$= p_1(q_1^2 - 2q_1\bar{q})(R_1 - \bar{R}) = p_1 q_1^2(1 - 2p_1)(R_1 - \bar{R}) \qquad (since\ \bar{q} = p_1 q_1)$$

$$skew[q] = p_1(q_1^2 - 2q_1\bar{q})(q_1 - \bar{q}) = p_1 q_1^3(1 - 2p_1)(1 - p_1)$$

It can now be shown:

$$\lambda_{\min} = -\frac{\sigma_{q^2R}}{skew[q]} = -\frac{p_1 q_1^2(1 - 2p_1)(R_1 - \bar{R})}{p_1 q_1^3(1 - 2p_1)(1 - p_1)} = -\frac{R_1 - \bar{R}}{q_1(1 - p_1)} = \lambda_{MV}$$

The final parameter of interest does not produce the same degree of simplification as the others:

$$\sigma_{R^2 q} = \sum_{i=1}^{k} p_i(q_i - \bar{q})(R_i - \bar{R})^2 = p_1 q_1 (R_1 - \bar{R})^2 - \bar{q}\,\sigma_R^2 = p_1 q_1 \{(R_1 - \bar{R}) - \sigma_R^2\}$$

Using these results, Corollary 3 follows from evaluation of λ_{msk} and λ^*in the special case where there are only two states. In particular, it is possible to show that the term under the square root sign goes to zero when there are only two possible states, only one of which is a payout state. The following result demonstrates that this outcome is not due to both terms under the square root sign being zero.

Sign of cov[q^2,R] < 0

Consider the two-state case, where p_1 is the probability of the disaster state which is associated with the put option payout and p_2 is the probability of the normal state where there is no option payout, $p_1 = 1 - p_2$ and a bar over a variable indicates the mean. It follows that:

$$cov[q^2,R] = \sum_{s=1}^{2} p_s X_s^2 Y_s = p_1 X_1^2 Y_1 + p_2 X_2^2 Y_2 < 0$$

$$where:\ X_1 = (q_1 - \bar{q}) = (1 - p_1)q_1 - p_2 q_2 > 0$$

$$X_2 = (q_2 - \bar{q}) = (1 - p_2)q_2 - p_1 q_1 < 0$$

$$Y_1 = (R_1 - \bar{R}) < 0 \qquad Y_2 = (R_2 - \bar{R}) > 0$$

The last two sign restrictions are true due to the restriction that "R is negatively skewed" which, by assumption, means that the option payout will only occur in the state where the negative return occurs. Because there is no option payout in state 2, $q_2 = 0$. Substituting these results into the summation formula produces:

$$p_1[(1 - p_1)q_1]^2 (R_1 - \bar{R}) < p_2 [-p_1 q_1]^2 (R_2 - \bar{R})$$

Signing of the variables in this expression proves the conjecture. While not needed, it is also possible to show, by examining the two state representation of the formula for skewness that $p_1 < p_2$ is needed for skew[R] < 0 and skew[q] > 0.

Sign of (cov[q$_2$,R])^2 - skew[q] cov[q,R^2] = 0 for the two state case

Again consider the two state case. The condition requires that:

$$\{\sum_{s=1}^{2} p_s X_s^2 Y\}^2 - \{\sum_{s=1}^{2} p_s X_s^3\}\{\sum_{s=1}^{2} p_s Y_s^2 X_s\} = 0$$

Expanding the summation operator, evaluating the powers and cancelling gives:

$$2\, p_1 p_2\, X_1^2 X_2^2 Y_1 Y_2 - \{p_1 p_2\, X_1^3 X_2 Y_2^2 + p_2 p_1\, X_2^3 Y_1^2\, X_1\} > 0$$

$$p_2 p_1 X_1 X_2\, \{X_1^2 X_2^2 + X_2^2 Y_2^2 - 2 X_1 X_2 Y_1 Y_2\} = p_2 p_1 X_1 X_2\, [\{X_1 Y_2 - X_2 Y_1\}^2] = 0$$

The = 0 follows from observing that, in the last expression, $X_1\ Y_2 =\ \{(1 - p_1)q_1\}\{R_1 - \bar{\ }\ ) = (p_1 q_1)(R_2 - R) =\ Y_1\ X_2$. This equality follows immediately from expanding these expressions and cancelling terms.

Proof of Corollary 4:

The proof requires demonstrating that $(\text{cov}[q_2,R])^2 - \text{skew}[q]\ \text{cov}[q,R^2] >\ 0$ under the assumptions of greater than two states, with only one payoff state for the option which has probability less than 1/2. The proof relies on an initial substitution from the results for the parameters stated above to establish:

$$\sigma_{q^2 R}^2 - sk[q]\sigma_{R^2 q} = p_1^2 q_1^4\ (1 - 2p_1)\{(1 - p_1)\sigma_R^2 - p_1(R_1 - \bar{R})^2\}$$

$$= p_1^2 q_1^4\ (1 - 2p_1)\ \frac{1}{2} \sum_{i=2}^{k} \sum_{j=2}^{k}\ p_i\ p_j\ (R_i - R_j)^2 > 0 \qquad \textit{for } p_1 < \frac{1}{2} \quad \textit{and } k > 2$$

The assumption that $p_1 <\ 1/2$ ensures that the sign of $\{(\text{cov}[q_2,R])^2 - \text{skew}[q]\ \text{cov}[q,R^2]\}$ depends on the sign of the expression $\{(1 - p_1)\sigma_R^2 - p_1\ (R_1 - \bar{R})^2\}$. The sign of this expression can be evaluated by evaluating the relevant expression for an increasing number of states. For three states, the expression gives:

$$(1 - p_1)\sigma_R^2 - p_1(R_1 - \bar{R})^2 \Rightarrow p_2 p_3 \{R_2 - R_3\}^2 > 0$$

For four states:

$$(1 - p_1)\sigma_R^2 - p_1(R_1 - \bar{R})^2 \Rightarrow p_2 p_3\{R_2 - R_3\}^2 + p_2 p_4\{R_2 - R_4\}^2 + p_4 p_3\{R_3 - R_4\}^2 > 0$$

And so on, for increasingly larger number of states.

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Skewness Preference, Mean-Variance and the Demand for Put Options

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ABSTRACT

This paper compares the mean-variance and mean-variance-skewness approaches to modelling expected utility. Attention focuses on a problem in risk management: determining the optimal demand for a put option hedging the return on an asset with a negatively skewed return distribution. It is demonstrated theoretically that incorporating positive skewness preference into the decision maker's objective function typically produces a reduction in the demand for put options when compared to the mean-variance solution. A state dependent example is provided to illustrate how a mean-variance-skewness objective can result in a significant reduction in the optimal amount of crop insurance demanded when compared with the mean-variance solution.

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NOTES

1. Further discussion of issues related to the general properties of a Taylor series expansion for approximating a general expected utility function can be found in Loistl (1976). Hassett et al. (1985) examine specific types of problems with the Taylor series which arise where skewness is involved. Brockett and Kahane (1992) discuss the connection between preference for moments and expected utility rankings of risky prospects, arguing that " $U'' < 0$ and $U''' > 0$ are not related to variance avoidance or skewness preference".

2. In addition to the practical situations already listed, there are numerous other situations where the size of the risky asset position is fixed. Insurance decisions provide many cases, such as those involving fire or earthquake insurance on a house or how much crop insurance to purchase for an apple orchard. Other examples include the purchase of currency put options to protect against changes in exchange rates by a company bidding on a contract denominated in a foreign currency or a metals refinery concerned about declining prices for scrap already in inventory. In most practical situations, the decision about how many put options to purchase is unbundled from the real asset decision. In other words, the hedging decision is separated from the production decision, e.g., Feder et al. (1980).

3. In the domestic asset investment problem, it is typical to assume that $AP_t = AP_{t+1}$ is the initial value of asset units, e.g., shares of stock in the initial investment, making the problem somewhat simpler.

4. Extending the analysis to situations where $x < 0$ does changes the underlying conditions of the decision problem somewhat. For example, optimal solutions would involve the sale of put options. In many practical situations, e.g., crop insurance, this would not be possible. In some situations, the purchase of call options could be a feasible alternative. In addition, when the shape of the return distribution is negatively skewed and $x > 0$, this leads immediately to a negatively skewed distribution for terminal wealth. This situation changes when $x < 0$.

5. It is also possible to specify the put option using futures prices. However, because this involves the introduction of basis considerations, this complicates the analysis. Because exchange traded options are often written using futures prices, construction using futures prices is in some cases potentially more realistic. The assumption that the option is at-the-money is not restrictive and is used only for notational convenience.

6. It is simple to extend the profit function for the yield insurance case to cover revenue insurance. For revenue insurance, instead of two random variables associated with price and yield interacting to determine revenue (PY), there is only one random variable for revenue (R). The put option decision problem involves determining (Q_R/A), the fraction of A covered or insured with the revenue put option. Substitution of $\underline{R} = \{\underline{R}\ A\}/C(A)$ into (6) motivates the relevant profit function.

7. This assumption involves somewhat more than is stated. More precisely, this assumption requires that the put option payouts will occur in states where the returns are low. Cases where put option payouts occur and revenue is high are excluded to avoid having to consider pathological cases.

8. If the option is not fairly priced, in practice it will probably be underpriced, due to government subsidies. In the crop insurance context, underpricing will produce an increase in the usage of insurance.

9. **The optimal demand for the put option depends fundamentally on parameters in cosk[λ] which may be unfamiliar, such as cov[q^2,**R]. Interpretation of these terms by

numerical example is considered by example in Section 5.

10. While almost identical, the optimal demand for a yield put option does differ from the price put option in that $\gamma^*/x = [Q_z P_t]/C(A)$ and $\lambda^* = Q_y/A$. These decision variables have a somewhat different interpretation. For the put option based on prices, it is the fraction of the initial dollar value of investment in the risky asset which is of interest, while for the yield put option it is the fraction of the physical size of the asset. The optimal demand for put option on revenue is the same as that for yield insurance, with the proviso that the actual value of the various parameters, i.e., variances, covariances and the like, will have different values.

11. The underlying moment preference objective function requires that dEU/d skew[W] > 0. Using (8) it is now possible to do comparative static analysis on specific parameters, e.g., to determine that $\{d\lambda^*/d\ \mathrm{skew}[q]\} < 0$. This point is illustrated in section 5.

12. This inherent advantages of using (7) over (8) is apparent from the simple operation of verifying that the $\lambda_{MVS} = \lambda_{MV}$ when B = 0. To show this in (7) is immediate. However, (8) requires the application of l'Hospital's rule, i.e., evaluating the limit as B goes to zero.

13. **Examining cosk[·], it is apparent that {d skew[W]}/dλ is quadratic in λ, so there will be both a minimum skew[W] and a maximum skew[W] solution for λ. This point is considered in detail in the Appendix.**

14. In the following, trivial cases are ignored. For example, if the put option finishes out of the money in all future states of the world then $\lambda_{MV} = \lambda^*$. Similarly, if the distribution of R is symmetric and the put option finishes in the money in all futures states of the world then $\lambda_{MV} = \lambda^*$ is assured by linearity of the put option payout.

15. The proof for this Corollary also requires that the probability of the option payout state be less than one-half.

16. In general, this derivative takes the form:

$$\frac{d[EU_{MV} - EU_{MVS}}{d\lambda} = -\frac{U^{///}}{3!}\frac{dskew[W_{t+1}]}{d\lambda} - \frac{U^{///}}{3!}\frac{dW_{t+1}}{d\lambda}skew[W_{t+1}]$$

However, this expression reduces to (9) due to the assumption that the option is actuarially, fairly priced which produces the result that $dW/d\lambda = 0$.