The Hemicycle of Notre-Dame of Paris: Gothic Design and Geometrical Knowledge in the Twelfth Century

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Figure 1 Notre-Dame, Paris, hemicycle plinths in the ambulatory (author's photo). See *JSAH* online for a zoomable image in color

The Hemicycle of Notre-Dame of Paris

Gothic Design and Geometrical Knowledge in the Twelfth Century

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In the cathedral of Notre-Dame of Paris, the side aisles and ambulatories are remarkable spaces, hallways with large columns meticulously placed at even intervals.[1] The extraordinary regularity of these passages of circulation betrays the application of strict rules during their construction. It required specific skills to create this architecture. However, more than eight hundred years after the construction of the cathedral, little is known of these skills and the expertise of the builders of the cathedral of Notre-Dame remains unclear. The layout of four large plinths in the inner ambulatory reflects the knowledge of the builders and offers insight into the mechanisms by which this remarkable early Gothic structure was achieved (Figure 1). The four plinths that support the hemicycle columns were placed on a polygonal plan, a layout that was a component of the overall scheme that the builders prepared for this part of the church (Figure 2). The cathedral demonstrates the success of their architectural enterprise.

The regularity of Gothic buildings in general has inspired scholars to develop various theories about the knowledge of medieval builders.[2] For example, in the nineteenth century, Eugène-Emmanuel Viollet-le-Duc insisted that geometry was essential to Gothic design.[3] For him, the proportions of a Gothic cathedral were based on a system of isosceles triangles.[4] This type of triangle stood for the concept of stability, and its geometry defined the proportions of the Gothic churches that Viollet-le-Duc found the most appealing, including Notre-Dame of Paris and Amiens Cathedral.[5] However, Viollet-le-Duc's theory derived from his desire to discover this ideal shape in the proportions of the buildings he admired. He neither considered the intellectual context nor questioned whether medieval builders had any interest in the geometry of this type of triangle.

Conversely, about a century later, Erwin Panofsky argued that the intellectual context, more precisely medieval scholastic philosophy, inspired the Gothic cathedral builders.[6] However, Panofsky's theory was based on formal similarities between Gothic churches and the organization of scholastic treatises, and his approach remained superficial.[7] It did not, for example, consider issues involved in the translation of theory into architecture. Furthermore, neither Viollet-le-Duc nor Panofsky considered the sources of the builders' geometrical knowledge or critically assessed the nature of the social and organizational conduits through which such knowledge might have circulated.

Later scholars investigated these questions, searching in the surviving written sources and treatises for evidence of the knowledge of medieval architects. Lon R. Shelby studied not only geometry familiar to medieval masons but also its linkages to the mathematical treatises of the Middle Ages, and Peter Kidson analyzed the plan of Saint-Denis in relation to

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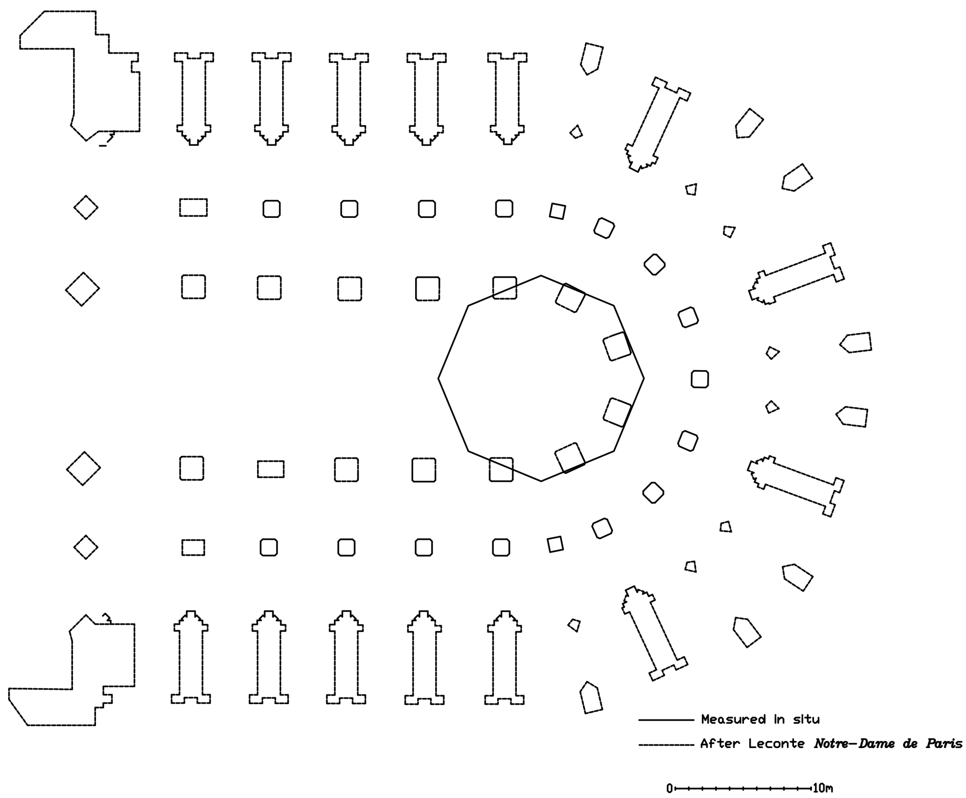

Figure 2 Notre-Dame, plan of the main supports of the chevet as in the fourteenth century, showing the geometry of the hemicycle (author's drawing)

a treatise from Antiquity about regular polygons.[8] Both concluded that there was little evidence in the twelfth century for a linkage between architecture and mathematical theory.[9] Shelby indicated that at that time builders and theoreticians belonged to different social circles—a separation of class that made knowledge exchange unlikely. Moreover, both Shelby's and Kidson's research have demonstrated that there is little chance to discover a mathematical treatise specifically composed to guide the architects of the twelfth century.

A different approach is possible. Instead of a specific theory, a combination of mathematics from different treatises that were circulating in Paris at the moment of the construction of the cathedral can be shown to have been useful to plan the four plinths of the hemicycle of Notre-Dame. The builders of Notre-Dame could have used methods similar to those studied by scholars to convert a geometric plan into a problem involving arithmetic, round figures, and linear measurements.[10] The correspondence between dimensions in the building and values in mathematical treatises suggests that when the cathedral was built either the boundaries between theory and practice were fluid or, more likely, theoretical mathematics and construction practice drew from the same scientific lore.[11] It might be concluded that after the twelfth century mathematics evolved toward a specialized theoretical discipline while building practice developed its own geometrical procedures.[12] But in twelfth-century Paris, a city that was in full development and poised to become capital of royal France, scholars and builders adopted similar methods, applying scientific knowledge to the interpretation of Scripture and resolving architectural challenges with inventiveness and pragmatism.

The Geometry of the Scholars

During this time, in Paris, intense building activity coincided with the vibrant intellectual life of the schools.[13] One after another, monastic foundations dismantled portions of their outdated buildings and replaced them with new Gothic structures. Saint-Denis might have called the tune, but the community there was not alone in its verve for building.[14] At Saint-Martin-des-Champs and Saint-Germain-des-Prés architects also experimented with innovative ideas, and around 1160, in this atmosphere of creativity, the chapter and bishop of Paris undertook the construction of a church reflecting the status of their institution, a cathedral intended to be the most prestigious church in Western Europe.[15]

While prelates ordered new churches, Parisian scholars aspired to knowledge of encyclopedic breadth and demonstrated great inventiveness in their investigations.[16] For example, Hugh of Saint-Victor composed a geometrical treatise and explored the possibilities that mathematics offered to understand the ark of Noah.[17] His works provide precious information about the mathematical knowledge of a twelfth-century intellectual. However, if Hugh's science was remarkable and innovative, it was rooted in a mathematical tradition.[18] His example illustrates that much of the mathematical knowledge that circulated at the time of the construction of Notre-Dame came down from late antiquity.

Already early in the Middle Ages fragments circulated of Euclid's *Elements*, the work that today is recognized as the basic reference text on geometry.[19] A first version, a sixth-century translation attributed to Boethius, is now known as the *Geometry I*. It was a partial list of the theorems of Euclid, each illustrated with simple freehand drawings, but no proofs of Euclid's theories were presented. In the eleventh century a second version, known as the *Geometry II*, surfaced.[20] While this offered some of the proofs, the crudeness of the drawings—even in the copies of the *Geometry II* from around the middle of the twelfth century—indicate that Euclid was misunderstood.[21] This ignorance of Euclid would last at least until the second half of the twelfth century, when scholars started to assimilate Euclid's principles in their translation of Arabic texts.[22] The number of extant copies of *Geometry I* and *Geometry II* is thus more an index of a desire to place works with a special aura on monastic library shelves than a sign that Euclidean geometry was understood.[23] Consequently, chances that Euclid had any direct influence on the practical work of medieval architects are close to none.[24]

However, other works circulating in the Middle Ages contained applications of geometry that were useful for building practice. From the ninth century on, copies of Vitruvius's treatise on architecture became available in the Latin West, and while it is, of course, unlikely that medieval builders had any interest in the construction of classical temples, Vitruvius contained some basic mathematics useful for any type of architecture.[25] For example, he explained a simple technique to orient a colonial town according to the points of the compass. The method consisted of tracking the shadow of the sun cast by a stick fixed in the center of a circle.[26] Next, an exercise with a compass used the points of intersection of the shadow and the circle to establish a line oriented in the north-south direction. Vitruvius also discussed Plato's theorem for doubling the area of a square by making the side of the second square equal to the diagonal of the first (Figure 3).[27] Finally, Vitruvius offered a technique that he ascribed to Pythagoras, explaining how to obtain a

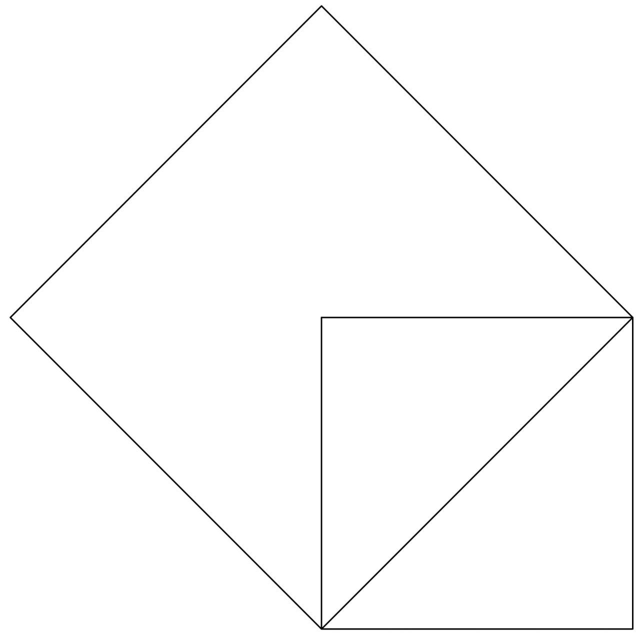

Figure 3 Plato's method for doubling the area of a square (author's drawing)

right angle by simply connecting lines of length 3, 4, and 5 feet.[28] These three mathematical methods in Vitruvius provided simple, useful techniques for orienting buildings, harmonizing proportions between square elements, or placing structures at right angles.

Another work that contained techniques useful for building was the collection of treatises bundled under the name of *Corpus agrimensorum* or *Writings of the Roman Land Surveyors*.[29] This compendium, of which the oldest surviving copy probably dates from the sixth century AD, repeated some of the techniques described by Vitruvius but included additional information.[30] It explained, for example, how to design a city on a grid plan.[31] The basic unit of this grid plan was the *actus quadratus*, or square acre, an area of 120 by 120 feet. This unit could be subdivided into areas of 10 by 10 feet.[32] To establish such a grid, the surveyor used an instrument called *groma*.[33] This technique was basic though efficient, enabling a designer to lay out the streets of a city or fix the positions of the columns of a building.[34]

In addition to these writings on architecture and surveying, another influential text that circulated in the Middle Ages was the *Gerbert Geometry*, a geometry treatise from the end of the tenth century.[35] It emphasized the value of right triangles. From the hand of Gerbert of Aurillac, Pope Sylvester II, it began by presenting material from land surveying manuals, such as the *actus* of 120 by 120 feet, or 12 by 12 perches.[36] After mentioning this square module, the treatise

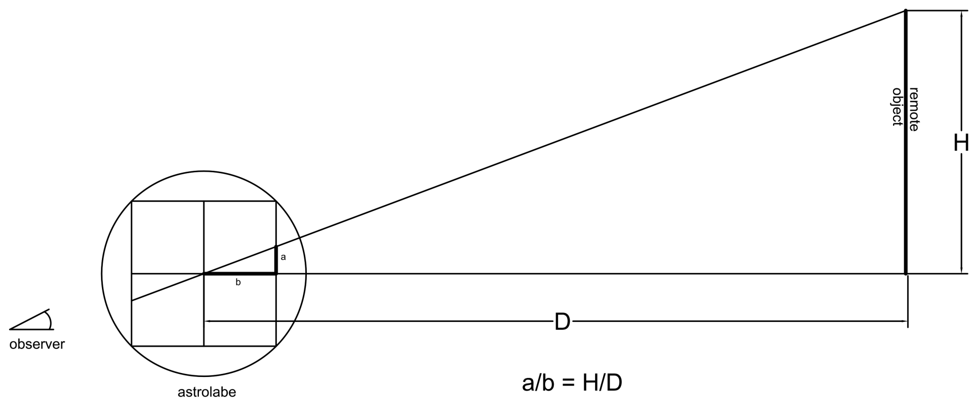

Figure 4 Diagram illustrating the functioning of the astrolabe (author's drawing)

continued with an innovative discussion of right triangles, to which Gerbert dedicated more than half of his treatise, and with the *Gerbert Geometry*, this figure forcefully entered medieval science.[37]

From the theoretical point of view, this sudden interest in right triangles can be explained by the peculiar relationship among the sides of that shape. Vitruvius mentioned this property, but only for a triangle with sides of 3, 4, and 5 feet.[38] Gerbert generalized this discussion to include triangles of different sizes. He multiplied and divided the dimensions by a constant, showing that scaling did not alter the properties, and he also described the relationship among the sides of all right triangles, the Pythagorean Theorem, which shows how one can always compute the length of a side given the two other sides of a right triangle.[39] Gerbert overwhelmed the reader with a plethora of examples, many with elaborate fractions.

If the theoretical properties of the right triangle were seductive, its practical applications proved even more attractive. This is indicated by the so-called *Anonymous Geometry* often attached to the *Gerbert Geometry* in medieval manuscripts.[40] The *Anonymous Geometry* presented a variety of applications involving right triangles.[41] It explained, for example, how to use an astrolabe to determine the height of a distant object by triangulation.[42] The technique consisted of using the instrument to create a miniature right triangle having the same proportions as the triangle formed between the observer and the measured object (Figure 4). Because two sides of the miniature triangle had the same ratio as large one, it was possible to compute the height of the object. Other measuring techniques that relied on the ratio between the orthogonal sides of a small and large right triangle were also described.[43]

The *Anonymous Geometry* also presented a few exercises with a compass; one created a hexagon and a second created a regular octagon inside a square.[44] More than just clever constructions, these exercises established the relationships among various geometrical shapes. The second construction exposed, for example, how an octagon resulted from clipping triangles from corners of a square (Figure 5). But, more importantly, in this construction one could see that the diagonal of the square—or more precisely half the diagonal—determined the length of each side of the inscribed octagon. This exercise offered theoretical insight into the relationship between an octagon and a square.

Finally, the treatise contained a discussion of the value for the diagonal of a square, a topic also debated in an eleventh-century letter exchange between two schoolmasters, Master Ragimboldus of Cologne and Master Radulphus of Liège.[45] Writing to his colleague Radulphus claimed that the diagonal measured $\frac{7}{5}$ of the side of the square. In his answer, Ragimboldus conceded that this was an acceptable approximation,

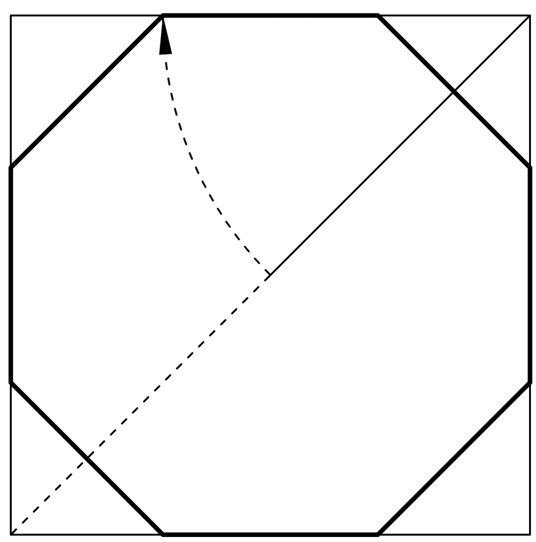

Figure 5 Drawing an octagon inside a square (author's drawing)

but demonstrated that $\frac{17}{12}$ was a more accurate ratio.[46] That this subject appears both in the *Anonymous Geometry* and in the letter exchange indicates scholarly curiosity for the value of the diagonal of a square.

At the beginning of the twelfth century, the properties of right triangles still inspired admiration, and they provided the basis for a geometrical treatise by the prominent Parisian theologian Hugh of Saint-Victor. Hugh's *Practica geometriae*, or *Practical Geometry*, drew on earlier mathematics, but it was better organized than earlier treatises.[47] Like prior works it introduced techniques of measurement based on right triangles, but presented these in a logical and systematic way. The text began by explaining the difference between "practical" and "theoretical" geometry.[48] For Hugh, theoretical geometry was a purely mental exercise while practical geometry established proportions and dimensions with instruments. In the *Practical Geometry*, the instrument par excellence was the astrolabe already introduced in the *Anonymous Geometry*.[49] However, that treatise only described how to measure with the instrument, while Hugh, who used the *Anonymous Geometry*, added a discussion of its workings.[50] In a lengthy theoretical introduction, he explained that the astrolabe operated with right triangles.[51] Step-by-step, he demonstrated how, when measuring with the astrolabe, two right-angled triangles were formed by a revolving needle on a graduated square on the side of the device.[52] Hugh then explained in detail what the *Anonymous Geometry* simply mentioned, and he clearly presented how the ratio of the sides of the miniature triangle on the instrument allowed the viewer to calculate the height of a remote object at a known distance. After discussing the theoretical principles of the astrolabe, Hugh introduced a series of applications, all in a logical sequence of height, planar, and celestial measurements. The *Practical Geometry* reveals this twelfth-century theologian's interest in the possibilities offered by right triangles and his admiration for the efficiency of the astrolabe.

Although the interest of a theologian in a mechanical device might be surprising, it was probably rooted in Hugh's peculiar conception of biblical studies. He considered that theological investigation required universal knowledge, a view expressed in his manual on the curriculum for aspiring theologians, the *Didascalicon*. This text reflected Hugh's double interest in worldly and divine knowledge; its first section introduced both the liberal and the mechanical arts. The liberal arts consisted of the sciences of the mind and included mathematics, while the mechanical arts included physical activities such as construction.[53] For Hugh, knowledge of the liberal arts formed the essential basis for theology, the science that he described in the second section of his treatise.[54] It was thus a desire to understand the mysteries of Scripture that nourished Hugh's interest in mathematics.

Hugh himself translated his method into practice in another treatise in which he analyzed a built structure described in Scripture, the ark of Noah.[55] Composed around 1125–30, Hugh's analysis demonstrates the use of geometry for biblical exegesis, and it also shows how a twelfth-century scholar understood the value of geometry for the study of a man-made structure.[56] The long text was generally devoted to the ark's symbolism, but in a short passage Hugh expounded on his understanding of the physical aspects of the biblical ship. This passage consisted of two separate fields of inquiry: analysis of shape and dimensions.[57]

The church father Origen had imagined that the ark was a pyramid with a rectangular base and an opening of one cubit at its summit, and, over the years, this had been accepted as its canonical form.[58] Hugh reanalyzed Origen's conception. A different reading of Scripture indicated the existence of doors in a wall below the pyramid, establishing the existence of a structure underneath the pyramid conceived by Origen. For Hugh, the ship Noah built consisted of a pyramidal upper section on an oblong hull. However, while Hugh transformed the ark into something closer to an actual ship or a barge, his conception remained an abstract, angular structure that could be analyzed in its geometrical constituents. In Hugh's version, the ark was composed of the four triangular faces of the pyramid placed on the six rectangular faces of the oblong. It only consisted of rectangles and triangles.

In the second section, Hugh analyzed the dimensions of the ark and its geometrical components.[59] Following a brief recalculation of its size, he computed the surface area of the deck and then introduced a series of computations involving its components. He first divided the rectangular deck along its diagonal. Then, he used the Pythagorean Theorem to compute the length of this diagonal. Half this length corresponded to the distance from a corner to the center of the deck, where Hugh imagined a mast. This mast, the half diagonal, and a rib of the pyramidal roof formed, together, a right triangle (Figure 6). Because the Pythagorean Theorem geometrically connected these elements, it unified the ark. Hugh thus did not simply add a hull to the biblical vessel, he added a hull that formed a harmonious whole with the pyramidal superstructure. The relationship between sides in a right triangle permitted Hugh to understand and interconnect all the dimensions of the ark.

Hugh's exegesis reveals a scholar applying mathematics to analyze a three-dimensional structure in order to bolster his argument, and it also demonstrates a method, a scientific framework. Hugh probably had a rhetorical agenda

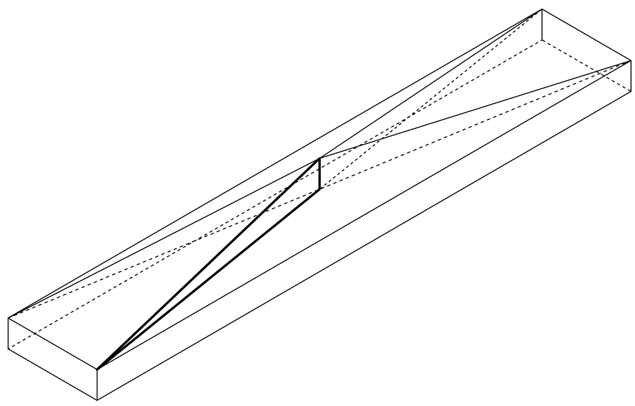

Figure 6 The right triangle formed between the deck and the pyramidal roof of the ark (author's drawing)

when he drew on his mathematical knowledge.[60] To convince his audience of the Bible's literal truth, he desired to create a model of an Ark that could be rationally explained by mathematics. As an intellectual, he judged that the ship of Noah was an abstract geometric assemblage of rectangles and triangles, all shapes that he mastered. Then he applied mathematical principles to combine these basic shapes into a whole.

The Geometry of the Builders

In the same years that Hugh assembled triangles and rectangles to solve questions about Scripture, Paris turned into a city full of building activity, a movement that culminated with the construction of the great church of Notre-Dame. Only a few years after Hugh composed his *Practical Geometry* and the *De archa*, in the 1130s, builders started a new chevet for the abbey church of Saint-Martin-des-Champs in the Parisian suburbs.[61] Despite its somewhat clumsy layout, this building was more spacious and better illuminated than any structure ever built in Paris. These properties must have pleased, as other buildings soon followed its example. Early in the 1140s, Suger, the abbot of Saint-Denis, rebuilt the chevet of his abbey church, and only a few years later, around 1145, Hugh, abbot of Saint-Germain-des-Prés, ordered the construction of yet another Gothic choir.[62] These buildings possessed more and more of the characteristics associated with Gothic, such as spaciousness and intricacy, and to these the cathedral of Notre-Dame added height.[63] Begun early in the 1160s, its vaults were the first to reach an elevation equal to that of such prestigious older buildings as the great Romanesque abbey church of Cluny and Old Saint Peter's in Rome.[64] Because of its size, creating the plan of Notre-Dame was challenging, and its layout required a systematic approach to obtain the desired geometry. The planning of its double ambulatory, with two wide passages of circulation, demanded a system that could locate piers over a wide area. Moreover, because the builders apparently started with the outer envelope of the chevet, their planning had to determine the dimensions of the future inner choir, even while its location was inaccessible at the start of the construction.[65] Indeed, while there must have been place for some improvisation, it is inconceivable that the builders did not establish the interior layout of this complex chevet at the same time they placed the foundations for the outer walls. In this building, a contraction of the outer envelope resulting from a misalignment of, for example, the outer responds would have had immediate repercussions on the proportions of the inner choir.

Furthermore, the overall proportions of the projected building must have been subject to the constraints of the urban building site. There remain many questions about the structures present on the site before the construction of the cathedral, but this part of the Île de la Cité was a built-up area. For about six centuries it had been the location of the institution of Notre-Dame, and at the time of the construction this area was occupied by canons' houses, churches, and an episcopal palace.[66] Unfortunately, because the location of some of these structures is unclear, it is difficult to determine with exactitude the constraints that their presence placed on the layout of the projected cathedral.[67] However, although this topographical information is lacking, it is possible to investigate other factors that influenced the overall dimensions of the building.

The interior width of the cathedral east of the transept measures 36.37 meters from wall to wall.[68] In the French capital, chances are that the builders used the royal foot as the standard length unit.[69] This dimension corresponds almost exactly to 112 royal feet (36.40 meters). This is only four feet less on either side than the module of 120 by 120 feet that is discussed in the *Writings of the Roman Land Surveyors* and in the *Gerbert Geometry*, hinting that there might be some connection between the size of the chevet and this recommended unit of area (Figure 7). Possibly, as the interior dimension of the chevet is less than the sides of the 120 by 120 square module, the builders made an allowance for the thickness of the foundations, walls, or buttresses.[70]

The plan of the choir of Notre-Dame consists of two roughly distinct components: the rectangular bays and the hemicycle (Figure 8). While their distinct designs indicate that they were individually planned, their combination to form a whole must have taken place within an overall framework. Those involved in planning the building might have judged an orthogonal grid to be the best system in which to compute the positions of the supports of the chevet.

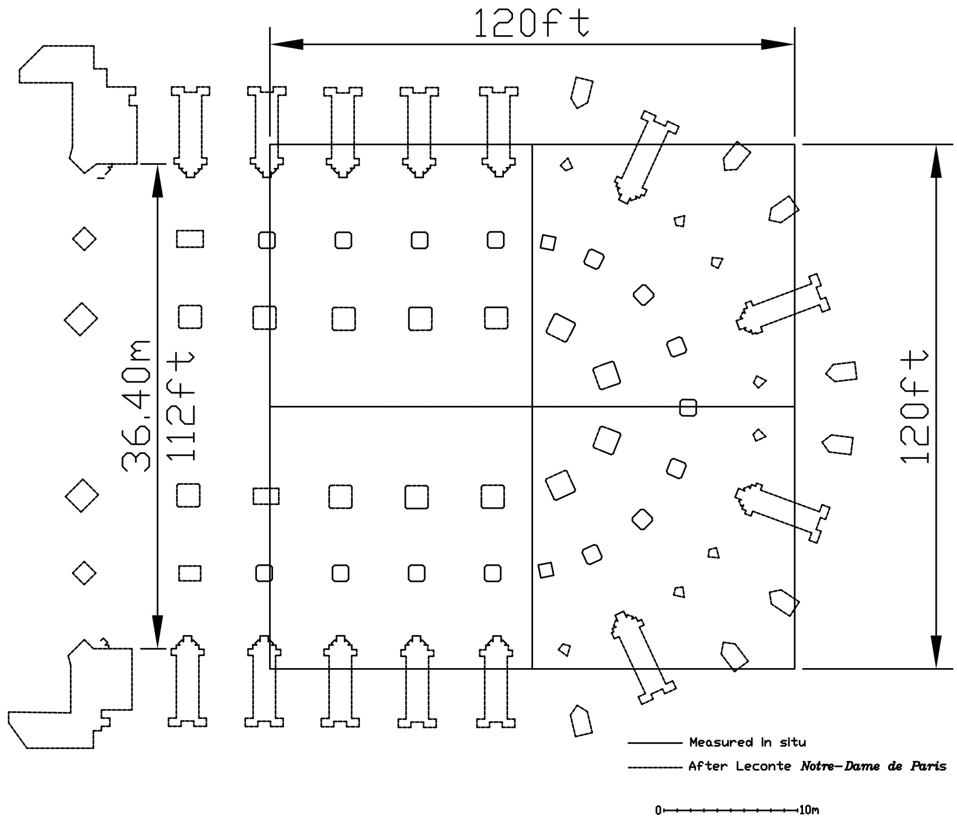

Figure 7 Notre-Dame, overall dimensions of the chevet (author's drawing)

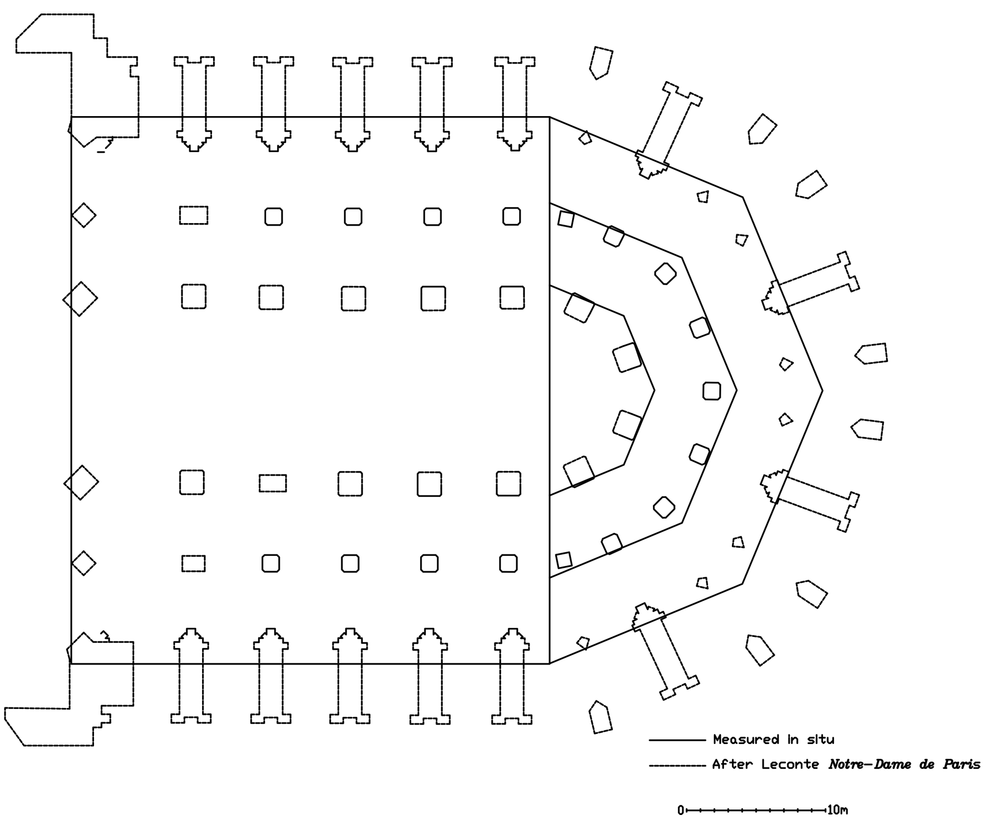

Figure 8 Notre-Dame, geometry of the hemicycle vs. the rectangular bays (author's drawing)

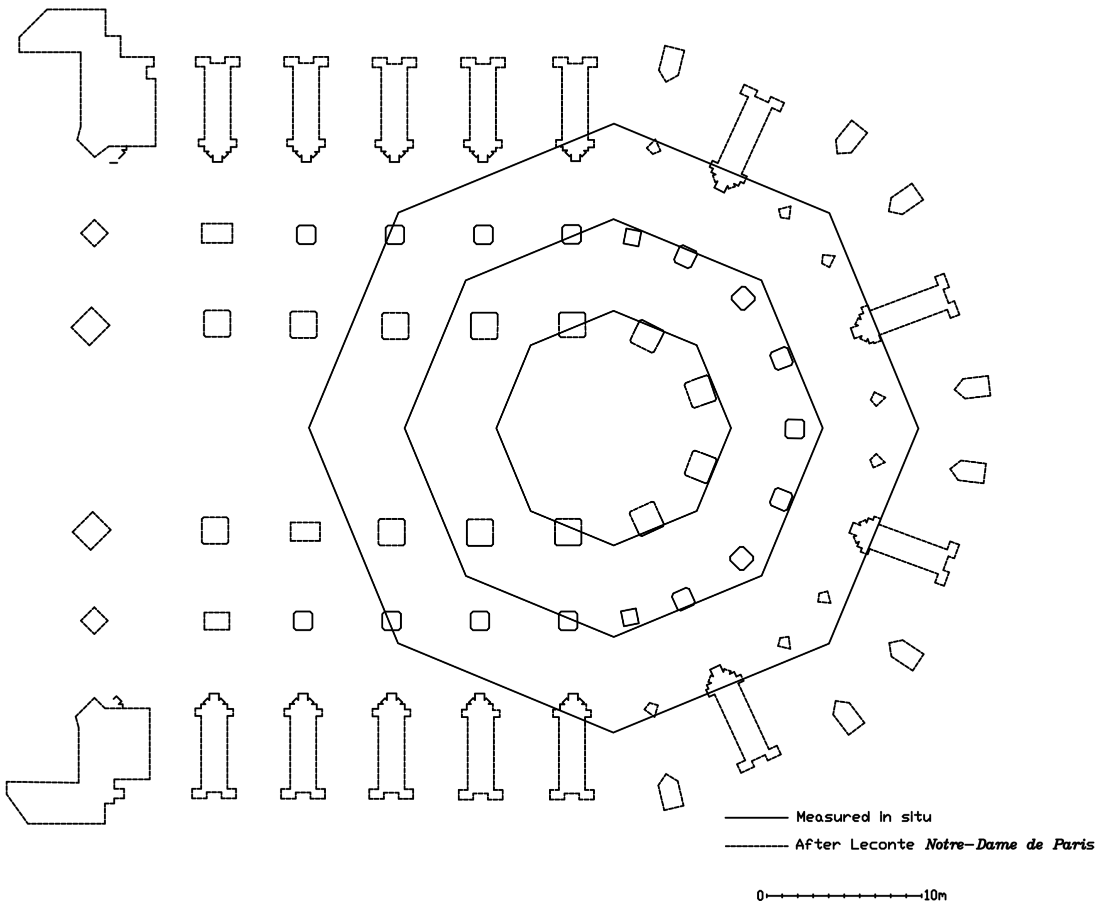

Figure 9 Notre-Dame, octagonal layout of the plinths (author's drawing)

In the cathedral, the layout of all the supports east of the choir is based on concentric octagons (Figure 9). The careful arrangement of the four large central plinths ending the choir at the east betrays an efficient and well-coordinated method.[71] Unfortunately, unlike Hugh, the builders of the cathedral left no discussion of their methods. But studying the geometrical configuration of these four supports in light of the mathematics discussed above suggests how the octagonal hemicycle could have been fit into a rectangular grid.

Indeed, it appears that, like Hugh's understanding of the ark, the plan conceived by the builders of the choir of Notre-Dame was a system of rectangles and triangles, and that the relationship between these components was used to correlate the two distinct sections. The octagon that defines the plan of the hemicycle could be considered as consisting of eight right triangles placed around four squares (Figure 10). These triangular components could fit into the square layout of the rectangular bays because the Pythagorean Theorem defines the arithmetical relationship between oblique triangle sides and orthogonal rectangle sides. Each element could thus fit inside an overall framework, in which the hemicycle and the rectangular bays are integrated by the theorem.

However, while this integration resembles Hugh's discussion of the ark, the problems that the builders solved were

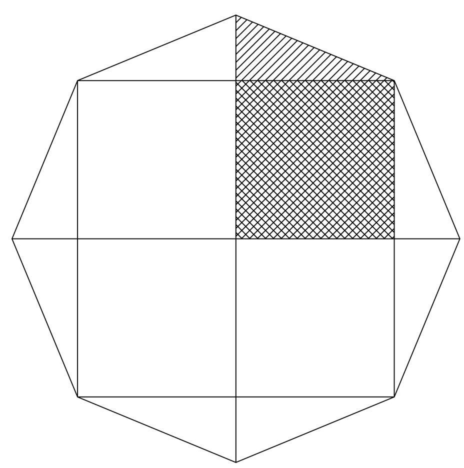

Figure 10 Diagram illustrating an octagon as a composition of squares and right triangles (author's drawing)

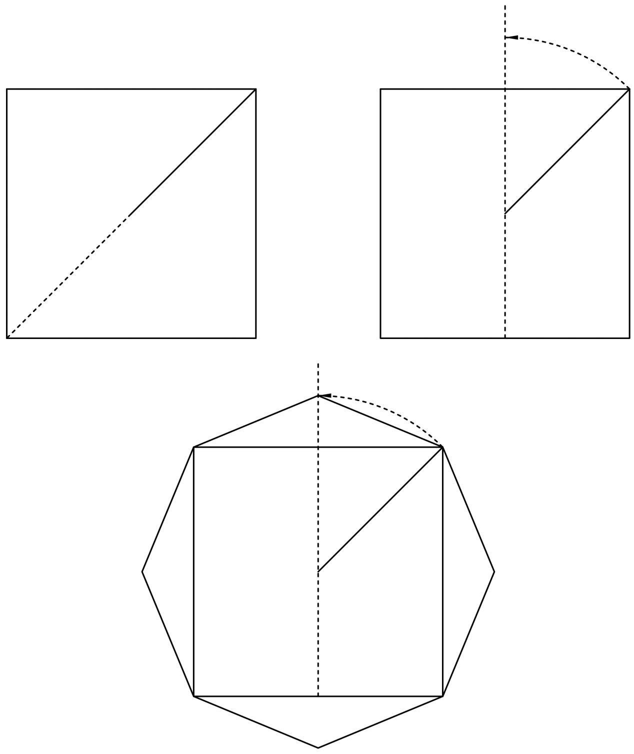

Figure 11 Diagram illustrating a method to draw an octagon circumscribing a square (author's drawing)

different. Hugh separated the ark into components and analyzed them individually. Those who created the cathedral did not analyze a plan but produced one, and, furthermore, for practical reasons in the field, linear measurements and round figures would prove useful. They needed geometrical and arithmetical knowledge to devise a scheme that, first, integrated rectangular bays and the octagonal hemicycle, second, established the layout and the dimensions of the inner choir, and, third, generated uncomplicated dimensions to guide the workmen.[72] The treatises show that this knowledge was available.

First, a geometric construction could show the relationship between the dimensions of an octagon and the square it circumscribed. The eleventh-century *Anonymous Geometry* contained a method for drawing such an octagon *inside* a square. A straightforward variation was drawing an octagon circumscribing a square: instead of using half the square's diagonal to establish the points at which the square was chamfered, the diagonal was swung to establish new vertices of the octagon (Figure 11). In this construction, four vertices of the octagon coincide with the corners of the original square; the four new vertices are located at a distance equal to half the diagonal from the center on perpendicular lines bisecting sides of the square. This method offered a geometric solution for integrating the octagonal hemicycle and the

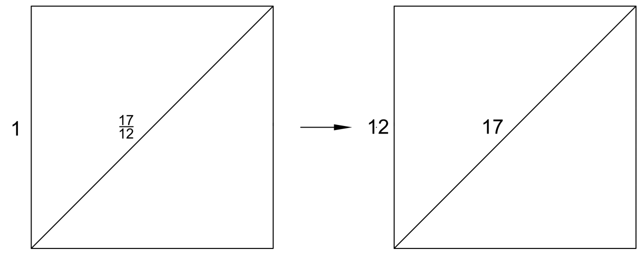

Figure 12 Diagram illustrating how to convert the diagonal of a square expressed as a fraction into a whole figure (author's drawing)

rectangular bays. However, while this offered a solution for the integration of the two principal geometric components of the plan, the chevet still needed dimensions. Again, the arithmetic needed to translate the octagon into a plan with actual dimensions was available.[73]

In the construction of an octagon inscribed in a square, the size of a circumscribing octagon was defined by half the diagonal of the square, and the diagonal of a square was discussed, for example, in the letter exchange between Ragimboldus and Radulphus. In those documents, its value is expressed as a fraction, $\frac{17}{12}$. Simple multiplication could turn this fraction into a round number: if a square with a side that was one unit long had a diagonal equal to $\frac{17}{12}$, a square of side 12 had a diagonal of 17 (Figure 12).[74] From these numbers, other values for the diagonal could easily be derived. Simple manipulations generated other squares having sides and diagonals with whole number measurements. The easiest operation was to continue to multiply the sides of the square by a whole number. A more sophisticated solution was to apply Plato's theorem for doubling the area of a square. This method, described in Vitruvius, could generate a square of side 17 and diagonal 24 (Figure 13). In Notre-Dame, it

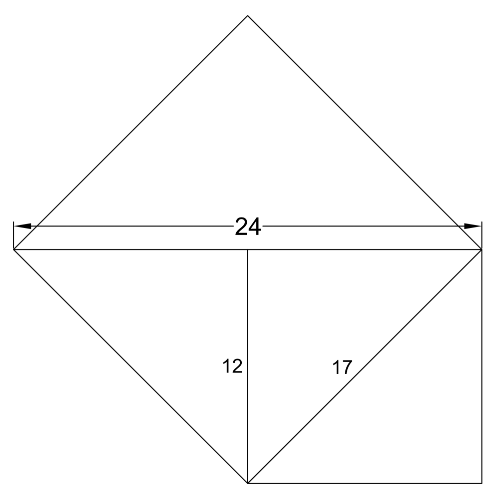

Figure 13 Diagram illustrating Plato's theorem to find an alternative value for the side and the diagonal of a square (author's drawing)

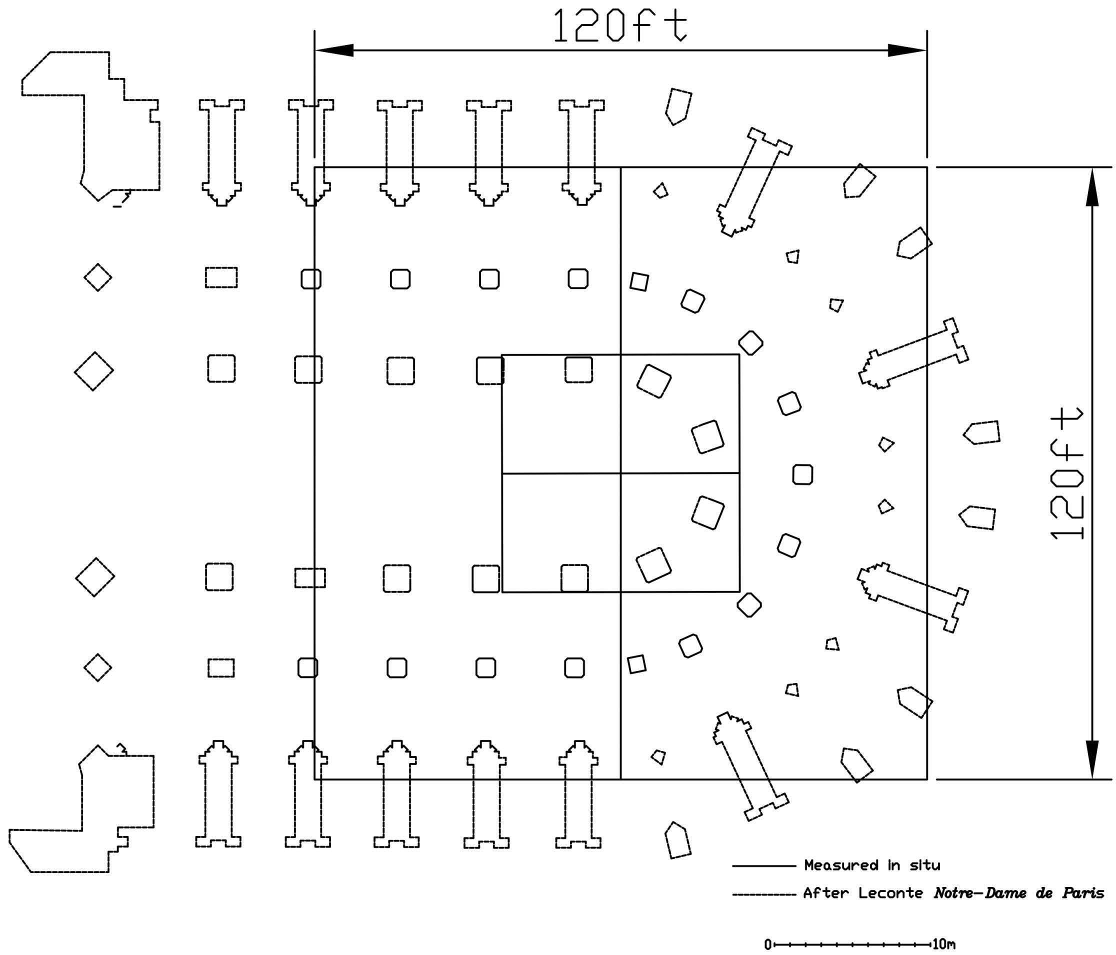

Figure 14 Notre-Dame, plan showing four quadrants in the center of the hemicycle (author's drawing)

appears that such a square was at the basis of the proportions of the hemicycle.

In the field, theoretical schemes and small-scale drawings were enlarged by exchanging the draftsman's instruments for the architect's measuring rod or perch.[75] While it is impossible to trace exactly how the builders laid out the hemicycle, the mathematics available at the time of the construction of Notre-Dame make possible the following method.

Once the site was cleared of older structures, the location of the center of the hemicycle is fixed.[76] A straightforward choice for the location of this center point is to make it coincide with the middle of the square module (see Figure 7). Around this point the builders conceive of a square working area of four quadrants (Figure 14). This area will form the core of the hemicycle. It consists of a large square subdivided by two main axes into four smaller squares, each with a diagonal equal to D (Figure 15). From the central point, c, the builders measure along one of the main axes a distance

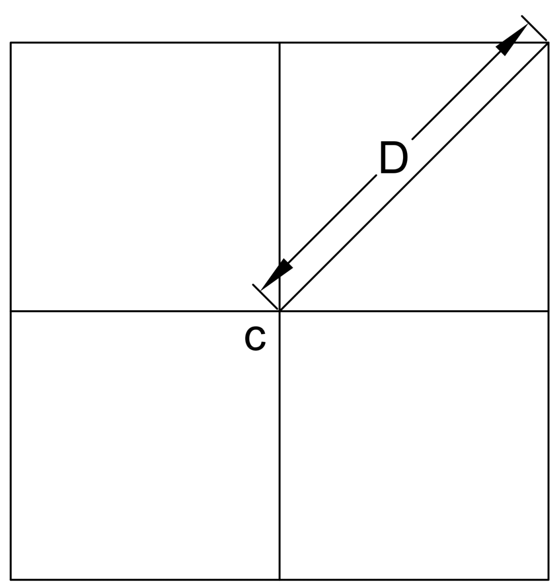

Figure 15 Diagram illustrating four quadrants around the center "c," and having a diagonal equal to "D" (author's drawing)

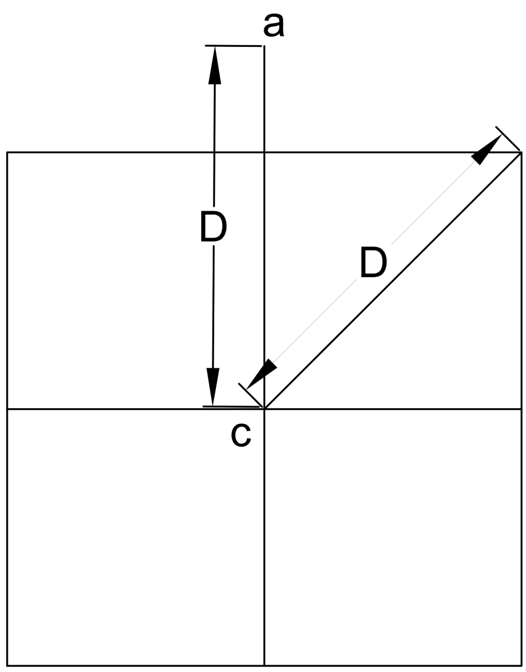

Figure 16 Diagram illustrating how at a distance "D" from the center point "a" is marked (author's drawing)

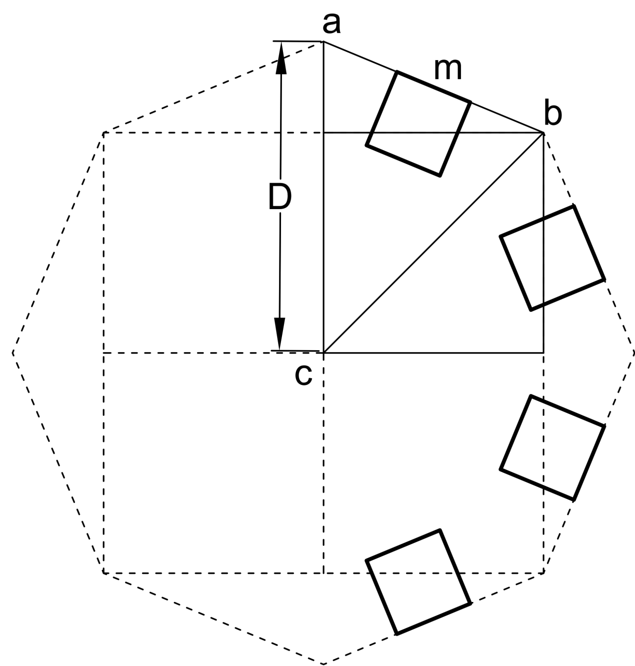

Figure 18 Diagram illustrating how the plinths can be placed in the middle of each side (author's drawing)

equal to the diagonal of the quadrant, and mark the endpoint, a (Figure 16). Connecting this endpoint with the adjacent corner, b, of the square establishes one side of the octagon (Figure 17). In the middle of this side, m, a plinth is placed

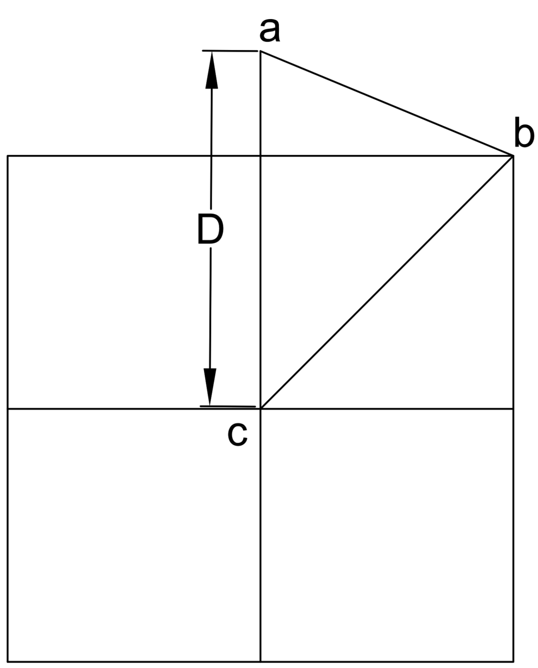

Figure 17 Diagram illustrating that connecting "a" and "b" produces an octagon side (author's drawing)

(Figure 18). This process is repeated three more times to position the remaining plinths.

On the building site, those responsible for the construction of the chevet might well have been informed about the geometrical figures that underlay such a plan. A survey of the chevet reveals interesting similarities between the numbers that appeared in the treatises and the dimensions of the hemicycle. As mentioned, applying Plato's theorem on a square of side 12 and diagonal 17 generates a square of side 17 and diagonal 24 (see Figure 13). In the field an octagon surrounding a square of these dimensions would have vertices at 24 royal feet, or 7.8 meters, from the center. The overlay of an octagon of this ideal size on a surveyed plan of the actual four plinths of Notre-Dame shows that the octagon sides almost coincide with the outer edges of the plinths (Figure 19).[77] This correspondence is an indication that the builders used similar numbers.[78] While the layout of the two easternmost plinths closely corresponds to this figure, the position of the two other plinths is slightly off, and the hemicycle appears to widen out toward the west. This is possibly due to errors made during the layout, but it can also be attributed to adjustments between the hemicycle and the easternmost plinths of the main arcade of the rectangular bays, for which the builders also tried to use whole number measurements.[79] Despite these discrepancies, this analysis suggests the use of a rigorous method. It suggests that figures from the

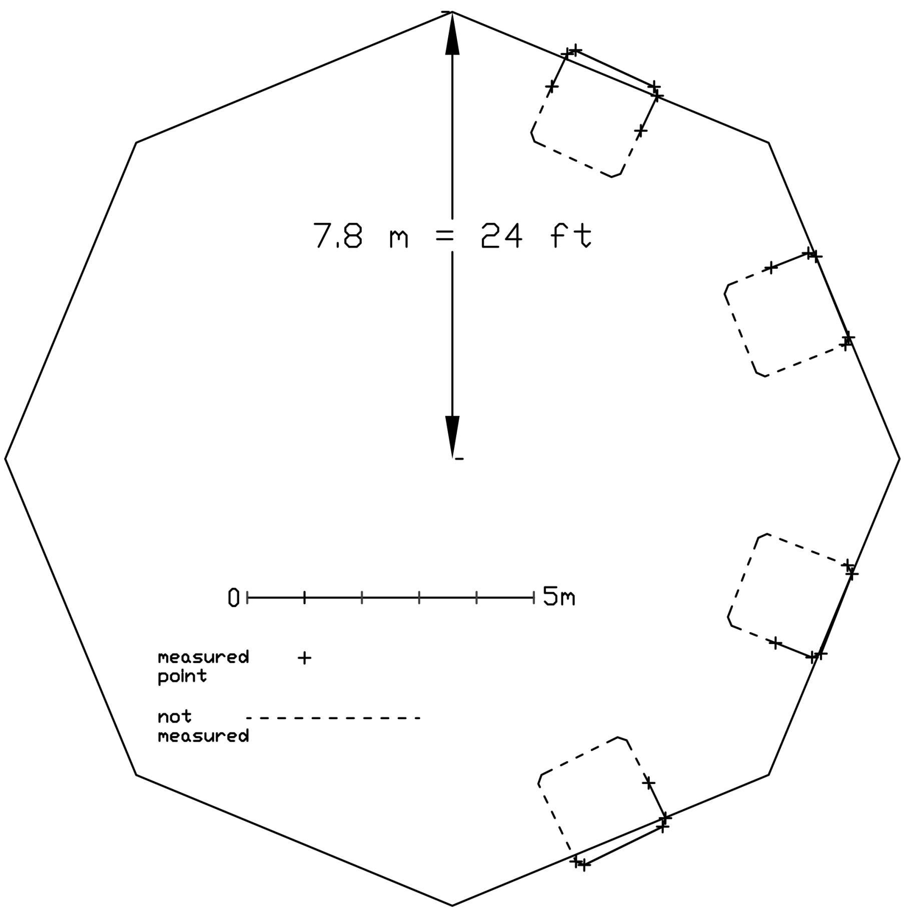

Figure 19 Notre-Dame, overlay of an octagon with a radius of 24 royal feet and a surveyed plan of the plinths of the hemicycle (author's drawing)

Western mathematical lore were adduced to design the octagonal plan of the hemicycle of Notre-Dame.

It is thus possible that the intellectual dynamics observed in Paris in the twelfth century had their counterpart in building practice. The series of new buildings that were built in Paris in the years preceding the construction of the cathedral evince the general characteristics of Gothic—diminished number of supports, increasing size of buildings, widespread use of rib vaults, and standardized working methods—that may signal an improvement in the computational systems used for building. It is possible that building science followed a development similar to the intellectual renewal undertaken by Hugh of Saint-Victor and Thierry of Chartres. Hugh's endeavors did not introduce a new type of mathematics, but his writings reveal a sophisticated, systematic, and organized use of existing knowledge, and his computations went back to the tradition of western mathematics. Twelfth-century builders may have experimented similarly with known methods and figures to achieve new architectural possibilities. With each experiment, builders would have increased their proficiency and contributed to the expertise of the following generation.[80]

This analysis of the layout of the four hemicycle plinths of Notre-Dame suggests that around the middle of the twelfth-century builders applied methods to architecture that were not disconnected from theoretical mathematical knowledge. The dichotomy that is usually said to exist between intellectual and practical work might derive from overestimating the mathematical knowledge of the twelfth-century scholar and underestimating the mathematical skills of the contemporary builder. The study of more monuments may confirm or invalidate the results presented here, and new research on the exchange of knowledge, a better understanding of the origins of the expertise of medieval builders, and more knowledge of twelfth-century society may also provide further answers about the relationship between architects and scholars in this most creative period.

Notes

1. This article issues from my dissertation, and I am particularly thankful to my thesis director, Stephen Murray, for making this research possible and for his precious comments and advice on this essay. I am also grateful to Michael T. Davis and Meredith Cohen for their thoughtful input and

suggestions during the preparation of this article, as well as to the anonymous reviewers for the *JSAH* who offered very useful comments on this research. I am also thankful for the opportunity that I received to present this research at the Robert Branner Forum, Columbia University. Finally, I want to acknowledge the financial support of the Willson Center for the Humanities at the University of Georgia for this publication. Translations are mine unless otherwise indicated.

2. On the education of builders, see John Harvey, *The Mediaeval Architect* (London: Wayland, 1972), 87–100; Lon R. Shelby, "The Education of Medieval English Master Masons," *Mediaeval Studies* 32 (1970), 1–26. See also, Werner Müller, *Grundlagen gotischer Bautechnik: ars sine scientia nihil est* (Munich: Deutscher Kunstverlag, 1990), 105–110. An example of collaboration between builders and a mathematician in the late fourteenth century is one of the projects for Milan cathedral. See James S. Ackerman, "'Ars Sine Scientia Nihil Est': Gothic Theory of Architecture at the Cathedral of Milan," *Art Bulletin* 31, no. 1 (June 1949), 84–111; Paul Frankl and Erwin Panofsky, "The Secret of Medieval Masons with an Explanation of Stornaloco's Formula," *Art Bulletin* 27, no. 1 (March 1945), 46–64. On the education of medieval patrons, Günther Binding, *Der Früh-und Hochmittelalterliche Bauherr als sapiens architectus*, 2nd ed. (Darmstadt: Wisseschaftliche Buchgesellschaft, 1998).

3. Under the entry "Proportion," in Eugène-Emmanuel Viollet-le-Duc, *Dictionnaire raisonné de l'architecture française du XI^e^ au XVI^e^ siècle* (Paris: A. Morel, 1854–68), 7: 534, 543–55. And in the "Neuvième entretien," in Eugène-Emmanuel Viollet-le-Duc, *Entretiens sur l'architecture* (Ridgewood, New Jersey: Gregg Press; reprint, 1965), 1: 385–448. For a critical discussion on proportions in Gothic, Konrad Hecht, *Maß und Zahl in der gotischen Baukunst*, 2nd ed. (Hildesheim and New York: Georg Olms, 1979). See also Müller, *Grundlagen gotischer Bautechnik*, 35–105.

4. In particular equilateral triangles and also an isosceles triangle having base 4 and height 2½ that Viollet-le-Duc called the "Egyptian" triangle. See Viollet-le-Duc, *Dictionnaire*, 7: 534–35, 543; Viollet-le-Duc, *Entretiens*, 1: 402–5.

5. Viollet-le-Duc, *Dictionnaire*, 7: 543; Viollet-le-Duc, *Entretiens*, 1: 405–8.

6. E. Panofsky, *Gothic Architecture and Scholasticism* (Latrobe, Penn.: The Archabbey Press, 1951). In another work Panofsky proposed the influence of light metaphysics on Gothic, E. Panofsky, *Abbot Suger on the Abbey Church of St.-Denis and its Art Treasures* (Princeton: Princeton University Press, 1946). For a critical discussion, Christopher Markschies, *Gibt es eine 'Theologie der gotischen Kathedrale'? Nochmals: Suger von Saint-Denis und Sankt Dionys vom Aeropag* (Heidelberg: C. Winter, 1995). More recently, Nigel Hiscock has proposed that Platonic geometry lay at the basis of medieval church design, see Nigel Hiscock, *The Wise Master Builder: Platonic Geometry in Plans of Medieval Abbeys and Cathedrals* (Aldershot: Ashgate, 2000); Nigel Hiscock, *The Symbol at Your Door: Number and Geometry in Religious Architecture of the Greek and Latin Middle Ages* (Aldershot: Ashgate, 2007).

7. On the issues about connecting scholasticism to Gothic, see Erik Inglis, "Gothic Architecture and a Scholastic: Jean de Jandun's 'Tractatus de laudibus Parisius' (1323)," *Gesta* 42, no. 1 (2003), 73–78.

8. Peter Kidson, "Panofsky, Suger and St Denis," *Journal of the Warburg and Courtauld Institutes* 50 (1987), 1–17; Lon R. Shelby, "The Geometrical Knowledge of Medieval Master Masons," *Speculum* 47, no. 3 (July 1972), 395–421. In another article Kidson compared the polygonal layout of Sens and Canterbury cathedrals, "Peter Kidson, Gervase, Becket, and William of Sens," *Speculum* 68, no. 4 (Oct. 1993), 969–91. There exist many studies discussing the mathematical underpinnings of medieval building practice, but few investigate mathematical treatises that circulated in the Middle Ages. Nigel Hiscock discusses at length sources related to religious architectural design, Hiscock, *The Symbol*. For a compilation of studies on geometry and medieval architecture, see Nancy Y. Wu, ed., *Ad Quadratum: The Practical Application of Geometry in Medieval Architecture*, AVISTA Studies in the History of Medieval Technology, Science, and Art (Aldershot: Ashgate, 2002). A full list of publications on this subject is too long but some examples from after 1990 are: Michael T. Davis and Linda E. Neagley, "Mechanics and Meaning: Plan Design at Saint-Urbain, Troyes and Saint-Ouen, Rouen," *Gesta* 39, no. 2 (2000), 161–82; Eric Fernie, "A Beginner's Guide to the Study of Architectural Proportions and Systems of Length," in *Medieval Architecture and its Intellectual Context: Studies in Honour of Peter Kidson*, ed. Eric Fernie and Paul Crossley (Southampton: Southampton Book Company, 1990), 229–37; Hiscock, *The Wise Master Builder*; Peter Kidson, "A Metrological Investigation," *Journal of the Warburg and Courtauld Institutes* 53 (1990), 71–97; Stephen Murray, *Notre-Dame, Cathedral of Amiens: The Power of Change in Gothic* (Cambridge and New York: Cambridge University Press, 1995), 39–43. Sabine Lepsky and Norbert Nußbaum, *Gotische Konstruktion und Baupraxis an der Zisterzienserkirche Altenberg: Band 1 Die Choranlage*, Veröffentlichungen des Altenberger Dom-Vereins (Bergisch Gladbach: Heider Druck, 2005), 33–93.

9. In his essay, Shelby was particularly interested in short notes in Villard de Honnecourt's thirteenth-century portfolio that mentioned that some of its drawings were copied from treatises on geometry. Shelby, "The Geometrical Knowledge," 395, 398–99. Shelby investigated this claim. However, failing to find any serious linkage between medieval geometrical treatises and the sketchbook, Shelby concluded that the drawings in Villard's manuscript were not copied from any treatise but that they displayed knowledge that had been orally transmitted inside the mason lodge: Shelby, "The Geometrical Knowledge," 408. The short notes mentioning geometry are on fols. 1v, 19v, and 20r. For a color edition of the manuscript, Carl. F. Barnes, *The Portfolio of Villard de Honnecourt (Paris, Bibliothèque Nationale de France, MS Fr 19093)* (Farnham, UK; Burlington, Vt.: Ashgate, 2009). For a discussion on the denomination of this technical manuscript, see Barnes, *The Portfolio of Villard de Honnecourt*, 1–2. Barnes has presented a series of arguments supporting that Villard was himself not an architect: Carl. F. Barnes, "Le 'problème' Villard de Honnecourt," in *Les bâtisseurs des cathédrales gothiques*, ed. Roland Recht (Strasbourg: Éditions des Musées de la Ville de Strasbourg, 1989), 209–23; Barnes, *The Portfolio of Villard de Honnecourt*, 217–19. On the content of this manuscript see also Roland Bechmann, *Villard de Honnecourt: La pensée technique au XIII^e^ siècle et sa communication* (Paris: Picard, 1991); Hans R. Hahnloser, *Villard de Honnecourt: Kritische Gesamtausgabe des Bauhüttesbuches ms. fr. 19093 der Pariser Nationalbibliothek* (Vienna: Anton Schroll, 1935). Kidson indicated that Heron of Alexandria's treatise on regular polygons could offer solutions for the design of the hemicycle Saint-Denis, see Kidson, "Panofsky," 15. However, Kidson observed that this work was unknown in the West at the time of the construction of the new choir of Saint-Denis. He mentioned a manuscript, CLM13084, containing some rough geometric construction possibly derived from Heron, but concluded that these were of little use for architectural design. The geometric constructions from the CLM13084 are edited in Paul Tannery, "Un nouveau texte des traités d'arpentage et de géométrie d'Epaphroditus et de Vitruvius Rufus," in *Sciences Exactes au Moyen-Age*, ed. J.-L. Heiberg, *Mémoires scientifiques* (1896; Toulouse: Édouard Privat; Paris: Gautier-Villars, 1922), 29–78. Özdural points, nonetheless, to Heron as the eventual source for the design of a crusader church. Alpay Özdural, "The Church of St. George of the Latins in Famagusta: A Case Study of Medieval Metrology and Design Techniques," in *Ad Quadratum: The Practical Application of Geometry in Medieval Architecture*, ed. Nancy Y. Wu, *Avista Studies in the History of Medieval Technology, Science, and Art* (Aldershot: Ashgate, 2002), 222, 226.

10. The use of linear measurements has also been observed in the plan of the thirteenth-century abbey church in Altenberg, see Lepsky and Nußbaum, *Gotische Konstruktion*, 1: 55, 58–59.

11. The measurements in this essay are from a 2008 survey of Notre-Dame of Paris with a Total Station.

12. This evolution would explain the obvious difference between building and theoretical mathematics in later periods. Shelby talks about "constructive geometry" for architectural design, see Shelby, "The Geometrical Knowledge," 420–21.

13. On the development of Paris, see Robert-Henri Bautier, "Quand et comment Paris devint une capitale," *Bulletin de la société de l'histoire de Paris et de l'Île-de-France* (1978), 17–46. Bautier speaks of an "explosion" of the city, see Robert-Henri Bautier, "Paris au temps d'Abélard," in *Abélard en son temps: Actes du colloque international organisé à l'occasion du 9e centenaire de la naissance de Pierre Abélard (14–19 mai 1979)*, ed. Jean Jolivet (Paris: Les Belles Lettres, 1981), 23. On urbanism in Paris, see Jacques Boussard, *Nouvelle histoire de Paris: De la fin du siège de 885–886 à la mort de Philippe Auguste* (Paris: Hachette, 1976), 227–61. On buildings erected in Paris in the first half of the twelfth century, William W. Clark, "Context, Continuity, and the Creation of a National Memory in Paris, 1130–1160: A Critical Commentary," *Gesta* 45, no. 2 (2006), 161–75; Stephen Gardner, "The Theory of Centripetal Implosion and the Birth of Gothic Architecture," in *World Art: Themes of Unity in Diversity*, ed. Irving Lavin (University Park and London: Penn State University Press, 1989), 111–16. On intellectual life in Paris at the turn of the eleventh century, Boussard, *Nouvelle histoire de Paris*, 108–27. In the first half of the twelfth century, the important intellectual life appears through the number of scholars then active in Paris; see Richard W. Southern, "The School of Paris and the School of Chartres," in *Renaissance and Renewal in the Twelfth Century* (Conference 26–29 Nov. 1977), ed. R. L. Benson and G. Constable (Cambridge: Harvard University Press, 1982), 113–37. For a general overview of architecture and thought, Charles M. Radding and William W. Clark, *Medieval Architecture, Medieval Learning: Builders and Masters in the Age of Romanesque and Gothic* (New Haven and London: Yale University Press, 1992).

14. Gardner, "The Theory," 111. On Saint-Denis, William W. Clark, "Suger's Church at Saint-Denis: The State of Research," in *Abbot Suger and Saint-Denis: A Symposium*, ed. Paula L. Gerson (New York: Metropolitan Museum of Art, 1986), 105–30; William W. Clark, "'The Recollection of the Past is the Promise of the Future': Continuity and Contextuality: Saint-Denis, Merovingians, and Paris" in *Artistic Integration in Gothic Buildings*, ed. Virginia Chieffo Raguin, Kathryn Brush, and Peter Draper (Toronto: University of Toronto Press, 1995), 92–113; Sumner McKnight Crosby, *The Royal Abbey of Saint-Denis: From Its Beginnings to the Death of Suger, 475–1151* (edited and completed by Pamela Z. Blum), ed. George L. Hersey, Yale Publications in the History of Art (New Haven: Yale University Press, 1987).

15. On Saint-Martin-des-Champs, see M. E. Lefèvre-Pontalis, "Eglise de Saint-Martin des Champs à Paris," *Congrès Archéologique de France (1919)* 82 (1920), 106–26; Yves Plagnieux, "Le Chevet de Saint-Martin-des-Champs à Paris: Incunable de l'architecture gothique et temple de l'oraison clunisienne," *Bulletin monumental* 167, no. 1 (March 2009), 3–39. On Saint-Germain-des-Prés, Philippe Plagnieux, "l'abbatiale de Saint-Germain-des-Prés et les débuts de l'architecture gothique," *Bulletin Monumental* 158, no. 1 (2000), 7–86. Aubert's monograph remains an important work on Notre-Dame, Marcel Aubert, *Notre-Dame de Paris, sa place dans l'histoire de l'architecture du XIIe au XIVième siècle* (Paris: H. Laurens, 1920). Aubert's construction sequence has been reviewed in Caroline Bruzelius, "The Construction of Notre-Dame in Paris," *Art Bulletin* 69, no. 4 (Dec. 1987), 540–69. For the date of the start of the construction of Notre-Dame, see Bruzelius, "The Construction of Notre-Dame in Paris," 555. David Stanley sees in the scale of this building the expression of ancient Merovingian authority and of the Cult of the Virgin, David J. Stanley, "The Plan of the Cathedral of Notre-Dame: History and Ideology in Twelfth Century Paris," *Arte Medievale* 5, no. 1 (2006), 83–97. On the Merovingian revival, Clark, "The Recollection," 92–113; Clark, "Context," 161–75. The bibliography on Notre-Dame is too long to list here. For the principal works, see Alain Erlande-Brandenburg, *Notre-Dame de Paris* (Paris: Editions de la Martinière, 1997), 240.

16. In the *Didascalicon*, a manual describing the ideal curriculum for aspiring theologians, Hugh of Saint-Victor expresses the encyclopedic scope of his research. See in particular Hugh of Saint-Victor, *Didascalicon* 6.3. For an edition and translation in German, Thilo Offergeld, *Hugo von Sankt Viktor: Didascalicon de studio legendi*, Fontes Christiani, vol. 27 (Freiburg: Herder, 1997). For an English translation, Jerome Taylor, *The Didascalicon of Hugh of St. Victor: A Medieval Guide to the Arts* (New York and London: Columbia University Press, 1961; reprint, 1991). On Hugh of Saint-Victor, Dominque Poirel, *Hugues de Saint-Victor* (Paris: Cerf, 1998); Patrice Sicard, *Hugues de Saint-Victor et son Ecole*, ed. Pascale Bourgain, Témoins de notre histoire (Turnhout: Brepols, 1991). Another major scholar active in Paris in the first half of the twelfth century is Thierry of Chartres. He composed the *Heptateuchon*, an encyclopedia containing all what was known about the liberal arts at that time (Chartres, Bibliothèque de la Ville 498). The *Heptateuchon* was, fortunately, microfilmed before its destruction. On Thierry's presence in Paris, Southern, "The School of Paris and the School of Chartres," 130. Katzenellenbogen linked the sculpture of the west portals in Chartres to Thierry, see Adolf Katzenellenbogen, *The Sculptural Programs of Chartres Cathedral* (Baltimore: Johns Hopkins Press, 1959), 18–19.

17. Roger Baron, *Hugonis de Sancto Victore opera propedeutica: Practica geometriae, De grammatica, Epitome Dindimi in philosophiam*, ed. Philip S. Moore, Publications in Medieval Studies from the University of Notre Dame, vol. 20 (Notre Dame, Ind.: University of Notre Dame, 1966), 15–64. For a translation, Frederick A. Homann, S.J., *Practical Geometry*, ed. S. J. Roland Teske, Mediaeval Philosophical Texts in Translation, vol. 29 (Milwaukee: Marquette University Press, 1991), 33–70. For an edition of Hugh's treatise on the ark, Patrice Sicard, *Hugonis de Sancto Victore: de Archa Noe. Libellus de formatione Arche*, Corpus Christianorum Continuatio Mediaeualis, vol. 176 (Turnhout: Brepols, 2001), 1–117.

18. For Hugh's mathematical sources, see Baron, *Hugonis de Sancto Victore*, 5–6. Homann, *Practical Geometry*, 16–17.

19. On Euclid in the West, Marshall Clagett, "The Medieval Translations from the Arabic of the *Elements* of Euclid, with Special Emphasis on the Versions of Adelard of Bath," *Isis* 44, no. 1/2 (Jun. 1953), 16–42; Gillian R. Evans, "The 'Sub-Euclidean' Geometry of the Earlier Middle Ages, up to the Mid-Twelfth Century "*Archive for History of Exact Sciences* 16, no. 2 (1976), 105–18. For a list of manuscripts containing the *Elements*, see Menso Folkerts, "Euclid in Medieval Europe," *Questio* (Winnipeg, Canada: The Benjamin Catalogue for History of Science, 1989), http://www.math.ubc.ca/~cass/Euclid/folkerts/folkerts.html (accessed May 2010). And the revised edition, Menso Folkerts, "Euclid in Medieval Europe," in *The Development of Mathematics in Medieval Europe: The Arabs, Euclid, Regiomontanus*, *Variorum* (Aldershot: Ashgate, 2006), sec. 3, 1–64. These fragments of Euclid often circulated together with the treatises of the Roman land surveyors discussed below. On combining land surveying with theoretical geometry, see Evgeny A. Zaitsev, "The Meaning of Early Medieval Geometry. From Euclid and Surveyor's Manuals to Christian Philosophy," *Isis* 90, no. 3 (Sept. 1999), 527–28.

20. Menso Folkerts, *"Boethius" Geometrie II: ein mathematischen Lehrbuch des Mittelalters* (Wiesbaden: Franz Steiner, 1970), 105–6.

21. A manuscript from a little before the middle of the twelfth century that contained both the *Geometry II* and a version of Euclid translated from

Arabic was the second volume of Thierry of Chartres' *Heptateuchon*, ibid. On the origin of the Euclid translation in this treatise, see Hubert L. L. Busard and Menso Folkerts, *Robert of Chester's (?) Redaction of Euclid's Elements: The So-called Adelard II Version*, ed. Erwin Hiebert and Hans Wussing, vols. 8–9, Science Networks Historical Studies (Basel: Birkhäuser, 1992), 22–26.

22. For an overview of translations, see Marie-Thérèse d' Alverny, "Translations and Translators," in *Renaissance and Renewal in the Twelfth Century* (Conference 26–29 Nov. 1977), ed. R. L. Benson and G. Constable (Cambridge: Harvard University Press: 1982), 421–62. On the slow assimilation of theoretical geometry, see Guy Beaujouan, "The Transformation of the Quadrivium," in *Renaissance and Renewal in the Twelfth Century* (Conference 26–29 Nov. 1977), ed. R. L. Benson and G. Constable (Cambridge: Harvard University Press: 1982), 470–71; Evans, "The 'Sub-Euclidean' Geometry," 117–18; John E. Murdoch, "The Medieval Euclid: Salient Aspects of the Translations of the *Elements* by Adelard of Bath and Campanus of Novara," *Revue de Synthèse* 89 ser. 3, nos. 49–52 (1968), 67–94.

23. Zaitsev's speaks of the "exaltation of geometry." See Zaitsev, "The Meaning," 530–31.

24. As also observed by Shelby. See, Shelby, "The Geometrical Knowledge," 397.

25. Charlemagne's biographer Einhard who supervised the construction of Aachen mentions Vitruvius in one of his letters, edited and translated in Caecilia Davis-Weyer, *Early Medieval Art 300–1150*, vol. 17, Medieval Academy Reprints for Teaching (Toronto, Buffalo, and London: Toronto University Press, 1986), 107–108. For a list of manuscripts, Carol Herselle Krinsky, "Seventy-Eight Vitruvius Manuscripts," *Journal of the Warburg and Courtauld Institutes* 30 (1967), 36–70. See also, Eugene Dwyer, Peter Kidson, and Pier Nicola Pagliara, "Vitruvius," *Grove Art Online. Oxford Art Online* http://www.oxfordartonline.com/subscriber/article/grove/art/T089908 (accessed Sept. 2008). Harvey, *The Mediaeval Architect*, 20–21. Briefly on its reception in the Middle Ages, Robert Suckale, "La Théorie de l'architecture au temps des cathédrales," in *Les Bâtisseurs des cathédrales gothiques*, ed. Roland Recht (Strasbourg: Editions des Musées de la Ville de Strasbourg, 1989), 44–45. For an illustrated translation of Vitruvius, Ingrid D. Rowland, *Vitruvius: Ten Books of Architecture* (Cambridge: Cambridge University Press, 1999).

26. Vitruvius 1.6.6–7.

27. Plato, *Meno*, 82b–85b; Vitruvius 9.0.4.

28. Vitruvius 9.0.6.

29. On surveying in Roman times, see Oswald A. W. Dilke, *The Roman Land Surveyors* (Newton Abbot: David & Charles, 1971). For editions of the writings of the Roman surveyors, see A. Blume, F. Lachmann, and K. Rudorff, *Schriften der Römischen Feldmesser* (Berlin: Georg Reimer, 1848–52), 2 vols.; Brian Campbell, *The Writings of the Roman Land Surveyors: Introduction, Text, Translation and Commentary* (London: Society for Promotion of Roman Studies, 2000); Carl Thulin, *Opuscula Agrimensorum Veterum*, Corpus Agrimensorum Romanorum (Leipzig: Teubner, 1913), vol. 1. On manuscripts and transmission, see Nikolaus M. Bubnov, *Gerberti opera mathematica (972–1003)* (Berlin: R. Friedländer & Sohn, 1899; reprint, Hildesheim: Georg Olms, 1963), 394–553; Theodor Mommsen, "Die Interpolationen des gromatischen Corpus," *Bonner Jahrbücher. Jahrbücher des Vereins von Altertumsfreunden im Rheinlande* 96–97 (1895), 272–92; Tannery, "Un Nouveau texte des traités d'arpentage," 29–78; Carl Thulin, "Die Handschriften des Corpus Agrimensorum Romanorum," *Abhandlungen der königlich preussischen Akademie der Wissenschaften, philosophisch-historische Klasse* 2 (1911), 1–102; Carl Thulin, "Zur Überlieferungsgeschichte des Corpus Agrimensorum: Exzerptenhandschriften und Kompendien," *Göteborgs Kungl. Vetenskaps- och Vitterhets-Samhälles Handlingar* 14–15 (1914), 3–69. On the notion that these texts were considered as geometries, Zaitsev, "The Meaning," 527. See also, Homann, *Practical Geometry*, 8–12.

30. For the date of the earliest copy, Campbell, *The Writings of the Roman Land Surveyors*, xxi.

31. For example in Julius Frontinus and Hyginus 2. Ibid., 2–5, 134–63.

32. Ibid., 151 and diagram 2. See also Rowland, *Vitruvius: Ten Books of Architecture*, 167.

33. This instrument consisted of a stick with a metal point at the base, and, at the top, two perpendicularly intersecting cross arms with four plumb lines hanging from each end. The surveyor stuck the metal point of his *groma* in the ground, leveled the device, and obtained two perpendicular lines in the field by placing markers at regular intervals along the lines of sight extending from each cross arm. Campbell, *The Writings of the Roman Land Surveyors*, 330 note 35. On the *groma*, Dilke, *The Roman Land Surveyors*, 66–70. For a detailed description of this instrument, M. J. T. Lewis, *Surveying Instruments of Greece and Rome* (Cambridge: Cambridge University Press, 2001), 120–33. For an illustration of the generation of an orthogonal grid, see Rowland, *Vitruvius: Ten Books of Architecture*, 170.

34. Though by Carolingian times the Roman surveying techniques based on square modules were outmoded for establishing cities, the orthogonal layout of the plan of Saint Gall is similar to drawings in copies of the surveyor's manuals originating from the Carolingian court scriptorium at Aachen, see Charles B. McClendon, *The Origins of Medieval Architecture: Building in Europe, A.D. 600–900* (New Haven and London: Yale University Press, 2005), 170; Homann, *Practical Geometry*, 12–15; Müller, *Grundlagen gotischer Bautechnik*, 25. See also Zaitsev, "The Meaning," 527 note 10.

35. This work generally named *Geometria Gerberti* or *Gerbert Geometry* is edited in Bubnov, *Gerberti*, 48–97; Gerbertus Auriliacensis, "*Geometria*," *Patrologia Latina*, ed. Jacques-Paul Migne (1844–55 and 1862–65), http://pld.chadwyck.com (accessed Jan. 2009); A. Olleris, *Oeuvres de Gerbert. Pape sous le nom de Sylvestre II* (Clermont-Ferrand: F. Thibaud; Paris: Dumoulin, 1867).

36. Bubnov, *Gerberti*, 60–61. In this treatise, the perch corresponded to 10 feet.

37. Ibid., 71–97.

38. See note 28.

39. Bubnov, *Gerberti*, 82–3. In any right triangle, the sum of the squares of the right sides is equal to the square of the hypotenuse. Gerbert mentions that for other types of triangles there exists no exact solution.

40. Ibid., 317–64. Following Bubnov the earliest copies date from the eleventh century, Bubnov, *Gerberti*, 313–15.

41. This treatise was strongly inspired by the Roman land surveyors, and, for example, discussed the computation of areas.

42. Bubnov, *Gerberti*, 317–22. This instrument worked as the dioptra of the Greeks and was different from the sophisticated plane astrolabe. On the dioptra and the plane astrolabe in Antiquity, Lewis, *Surveying Instruments of Greece and Rome*, 51–108.

43. The treatise described for example, how a stick, a mirror, a basin filled with water, or, simply a wooden template of a right triangle could serve to measure large structures. See, Bubnov, *Gerberti*, 323–28.

44. Some manuscripts contained an additional, similar though different, passage. It presented a third method illustrating the drawing of an octagon starting from one side. Ibid., 552–53.

45. Ibid., 352. This letter exchange is edited in Paul Tannery and Abbé A. Clerval, "Une correspondence d'écolâtres du onzième siècle," in *Sciences Exactes au Moyen-Âge*, ed. J.-L. Heiberg, *Mémoires scientifiques* (1901; Toulouse: Edouard Privat; Paris: Gautier-Villars, 1922), 229–303. Apparently, as different editions give different values, the value varied from manuscript to manuscript. See Tannery and Clerval, "Une correspondence," 236–37 and 237 note 1.

46. Tannery and Clerval, "Une correspondence," 267 and 270–71.

47. See note 17. Hugh composed it before 1125. See Damien van den Eynde, *Essai sur la succession et la date des écrits de Hugues de Saint-Victor*, Spicilegium Pontificii Athenaei Antoniani (Rome: Antonianum, 1960), 51–52 and table, 214–15.

48. Baron, *Hugonis de Sancto Victore*, 16.

49. See note 42.

50. On the sources of Hugh, Baron, *Hugonis de Sancto Victore*, 5–6.

51. Ibid., 22.

52. Ibid., 26–27.

53. *Didascalicon* 2.3, 2.6–17. Hugh classified construction under armament because he considered a building as a shelter protecting from the elements. *Didascalicon* 2.22. On the mechanical arts in the Middle Ages, see Elspeth Whitney, *Paradise Restored: The Mechanical Arts from Antiquity through the Thirteenth Century*, Transactions of the American Philosophical Society 80, no. 1 (Philadelphia: American Philosophical Society, 1990).

54. *Didascalicon* 4–6. See esp. *Didascalicon* 6.3.

55. For an edition of this passage, Sicard, *Hugonis de Sancto Victore: de Archa Noe*, 18–23.

56. For the date of this treatise, see van den Eynde, *Essai sur la succession*, 69–73 and table, 214–15. Later Hugh composed a second treatise on the ark dating probably from 1128 or 1129. See van den Eynde, *Essai sur la succession*, 80–83 and table, 214–15.

57. "videlicet formam et quantitatem," Sicard, *Hugonis de Sancto Victore: de Archa Noe*, 18.

58. On Origen's and Hugh's interpretation of the ark, Grover A. Zinn, "Hugh of St. Victor and the Ark of Noah: A New Look," *Church History* 40, no. 3 (1971), 262–65.

59. For a translation of this passage, Homann, *Practical Geometry*, 85–86.

60. Similar rhetorical motivations explain Richard of Saint-Victor's (d. 1173) use of the Pythagorean Theorem to solve problems with dimensions of the Temple in his *In visionem Ezechielis*. For a recent edition, Jochen Schröder, *Gervasius von Canterbury, Richard von Saint-Victor und die Methodik der Bauerfassung im 12. Jahrhundert*, ed. Günther Binding, 2 vols., Veröffentlichung der Abteilung Architekturgeschichte des Kunsthistorischen Instituts der Universität zu Köln, vol. 71 (Cologne: Kleikamp, 2000), 494–97.

61. Plagnieux, "Le Chevet de Saint-Martin-des-Champs," 10.

62. Crosby, *Royal Abbey*, 215–22; Plagnieux, "L'abbatiale de Saint-Germain-des-Prés," 7–9, 21.

63. Bruzelius, "The Construction of Notre-Dame in Paris," 540.

64. On the original flying buttresses of this building with vaults at ca. 32 meters from the ground, see Stephen Murray, "Notre-Dame of Paris and the Anticipation of Gothic," *Art Bulletin* 80, no. 2 (June 1998), 229–53.

65. For the building sequence, Caroline Bruzelius, "The Construction of Notre-Dame in Paris," 543; William W. Clark, "The Early Capitals of Notre-Dame de Paris," in *Tribute to Lotte Brand Philip*, ed. William W. Clark, et al. (New York: Abaris Books, 1985), 39. Based on archaeological findings, some scholars advanced the existence of an earlier church under the choir of Notre-Dame. If this was the case, this structure might have been kept during the earlier phases of construction and its presence would explain why construction started from the outside. Starting from the outside around an existing building was not uncommon. This was, for example, the case in Reims, see Jean-Pierre Ravaux, "Les Campagnes de construction de la cathédrale de Reims au XIII[e] siècle," *Bulletin Monumental* 137, no. 1 (1979), 15–29. But the nature of the structures observed in the foundations of the cathedral of Paris seems highly questionable, see Noël Duval, Patrick Perin, and Jean-Charles Picard, "Paris," in *Province ecclésiastique de Sens*, ed. Jean-Charles Picard, *Topographie chrétienne des cités de la Gaule des origines au milieu du VIII[e] siècle* (Paris: De Boccard, 1992), 112–13. This skepticism is maintained by Didier Busson's in his detailed discussion on the subject, see Didier Busson, *Paris*, Carte archéologique de la Gaule (1998), 447.

66. While buildings are mentioned in documents, the location of, for example, the first episcopal palace and of earlier churches remains unclear. See Busson, *Paris*, 447 ; Marcel Aubert, "Les anciennes églises épiscopales de Paris, Saint-Etienne et Notre-Dame, au XI[e] siècle et au début du XII[e] siècle," *Académie des Inscriptions et Belles-Lettres. Comptes rendus* (1939), 319–27; Alain Erlande-Brandenburg, *Notre-Dame de Paris* (Paris: Éditions de la Martinière, 1997), 43–48; Michel Fleury, "La Cathédrale mérovingienne Saint-Etienne de Paris. Plan et datation," in *Landschaft und Geschichte. Festschrift für Franz Petri zu seinem 65. Geburtstag am 22. Februar 1968*, ed. Georg Droege, et al. (Bonn: Ludwig Röhrscheid, 1970), 211–221; Jean Hubert, "Les origines de Notre-Dame de Paris," in *Huitième centenaire de Notre-Dame de Paris, Congrès des 30 Mai–3 Juin 1964. Recueil de travaux sur l'histoire de la cathédrale et de l'église de Paris.*, *Bibliothèque de la société d'histoire ecclésiastique de la France* (Paris: J. Vrin, 1967), 1–22; Philippe Lorentz and Danny Sandron, *Atlas de Paris au Moyen Âge: espace urbain, habitat, société, religion, lieux de pouvoir* (Paris: Parigramme, 2006), 116–25; Victor Mortet, *Étude historique et archéologique sur la cathédrale et le palais épiscopal de Paris du VI[e] au XII[e] siècle* (Paris: Alphonse Picard, 1888); Francis Salet, "Notre-Dame de Paris: État présent de la recherche," *La sauvegarde de l'art français* 2 (1982), 89–113. Specifically on the difficulties to locate the first episcopal palace, T. Crépin-Leblond, "Recherche sur les palais épiscopaux en France au Moyen Age (XIIe–XIIIe siècle)" (thesis, École Nationale des Chartes, 1987), 229–30.

67. Recently, Didier Busson discovered in the notes of the nineteenth-century archaeologist Théodore Vacquer the existence of the foundations of a forgotten religious building right next to the north side of the cathedral, Didier Busson, "Un édifice religieux inconnu du groupe épiscopal," *Archéologia* 475 (Mar. 2010), 6–7.

68. This is different from the distance between the outer walls of the chapels which have been inserted between the buttresses in the thirteenth and fourteenth centuries. On these chapels, see Michael T. Davis, "Splendor and Peril: The Cathedral of Paris, 1290–1350," *Art Bulletin* 80, no. 1 (March 1998), 34–66.

69. This is also apparent from the many measurements in the chevet that converted in royal foot correspond to a whole number of these units. For example, the diagonal distance between the faces of the plinths in the side aisles is 20 royal feet. Possibly, this distance was taken a whole figure because it corresponds to the distance that a diagonal rib subtends. This would facilitate the production of the wooden forms. On the reuse of forms for different bays at Saint-Denis, see Crosby, *Royal Abbey*, 257–58. A royal foot is defined as 0.324893 m in Angelo Martini, "Manuale di metrologia." (Turin: Biblioteca Nazionale Braidense, 1883), http://www.braidense.it/dire/martini/indice.htm (accessed Dec. 2008). I used 0.325 m.

70. The uprights of the buttresses partially fall outside of this module. On the extension of the uprights, Stephen Murray, "Notre-Dame of Paris," 236. See also under the entry "*Cathédrale*" in Viollet-le-Duc, *Dictionnaire*, 2: 293.

71. In the nineteenth century Viollet-le-Duc restored this area. Based on the limited documentation—by contemporary standards—that he left of his restoration, this intervention does not seem to influence the results of the present research. In his restoration journal, Viollet-le-Duc does not indicate any underpinning of the four supports of the hemicycle, while he mentions such intervention on third arcade pier on the north, and also details insertions of large chunks of masonry in the plinths of the hemicycle. In the journal that he kept 1844–64, Viollet-le-Duc mentions on April 8, 1859: "On a terminé l'incrustement de gros morceaux de base au piliers de la partie circulaire ainsi que la reprise en sous-oeuvre du troisième pilier du choeur nord." ("We have finished the insertion of large chunks of plinths under the supports of the hemicycle as well as the underpinning of the third column of the north choir."), see "Journal rédigé par l'inspecteur en chef des travaux de restauration de la Métropole," (80/14/10 Archives de la Médiathèque de l'Architecture et du Patrimoine, Paris, 1844–64).

72. This does not exclude that they used different geometric constructions with, for example, a compass for other purposes such as the design of plinths, but for generating this octagon of relatively large dimensions, methods based on linear measurements and whole figures were likely more efficient. In a passage from the Didascalicon Hugh recounts how he drew figures on the ground and walked along their sides to compare their areas. *Didascalicon* 6.3.

73. Suger mentions the use of arithmetical and geometrical instruments for adjusting the new choir to the old structures at Saint-Denis. However, it is unclear what these instruments were. See Panofsky, *Abbot Suger*, 100–101. Crosby discussed this at different occasions. See his conclusion in Crosby, *Royal Abbey*, 286–87.

74. Rowland mentions that the Roman surveyors knew these values. Rowland, *Vitruvius: Ten Books of Architecture*, 167 note 87. Moreover, it is also possible that builders knew that surrounding a square of these dimensions with eight triangles of sides 5, 12, and 13 generated an octagon. This specific triangle is mentioned in the *Gerbert Geometry* and in the eleventh-century CLM 13084. See, Bubnov, *Gerberti*, 97 and table II, fig. 41; Tannery, "Un nouveau texte des traités d'arpentage," 55.

75. The use of the perch in the field is mentioned in sources, see, for example, Victor Mortet and Paul Deschamps, *Receuil de textes relatifs à l'architecture et la condition des architectes en France, au Moyen Âge XIIe et XIIIe siècles* (Paris: Auguste Picard, 1929), 51, 190. On the perch see also, Günther Binding and Susanne Linscheid-Burdich, *Planen und Bauen im frühen und hohen Mittelalter nach den Schriftquellen bis 1250* (Darmstadt: Wissenschaftliche Buchgesellschaft, 2002), 104–5, 126–33; Kidson, "A Metrological Investigation," 74–81.

76. In the schema presented, the computations of the dimensions of the hemicycle could, however, be made ahead even if this central point was not accessible due to older structures on the site. Before the construction started, the positions of each of the projected structures could be recorded as coordinates in a large grid. For the actual layout, these coordinates, numbers, could be translated into linear distances once the site cleared.

77. This does not exclude that also the position of the center points of the supports played a crucial role. It is possible that during construction other computations were made to locate the center points of the plinths. The plinths of Notre-Dame are large pieces of masonry which were possibly easier to position by fixing their edges. In other buildings, or even at other places in Notre-Dame, distances between supports could be planned between center points. An example of the use of the support axes as reference points is Altenberg, see Lepsky and Nußbaum, *Gotische Konstruktion*, 1: 55.

78. Kidson observed that a short text included in an early-medieval codex with writings of the Roman land surveyors mentioned four perches of sizes equal to 10, 12, 15, or 17, and linked this to the approximation for the square root of two. If perches in these sizes were still used at the time of the construction of Notre-Dame, the perch of 17 and 12 could be used for creating the octagon. See Kidson, "A Metrological Investigation," 75, 77.

79. See note 69.

80. Kidson suggested that with Saint-Denis geometry entered into church design, Kidson, "Panofsky," 17.