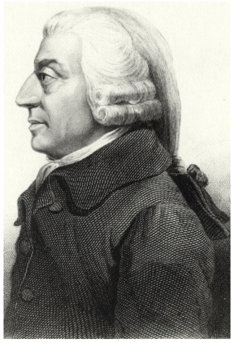
PART II:

Theories of Equity Security Valuation



"The value of a share in a joint stock is always the price which it will bring in the market; and this may be either greater or less, in any proportion, than the sum which its owner stands credited for in the stock of the company."

Adam Smith (1723-1790),
Wealth of Nations (1776, p.232)

"Academics ... like to define investment 'risk' differently, averring that it is the relative volatility of a stock or portfolio of stocks – that is, their volatility as compared to a large universe of stocks. Employing data bases and statistical skills, these academics compute with precision the 'beta' of a stock – its relative volatility in the past – and then build arcane investment and capital-allocation theories around this calculation. In their hunger for a single statistic to measure risk, however, they forget a fundamental principle: It is better to be approximately right than precisely wrong."

Warren Buffett (1993), as quoted in Cunningham (2002, p.82)

"When you enter the stock market, you are going into a competitive field in which your evaluations and opinions will be matched against some of the sharpest and toughest minds in the business. You are in a highly specialized industry in which there are many different sectors, all of which are under intense study by men whose economic survival depends upon their best judgment. You will certainly be exposed to advice, suggestions, offers of help from all sides. Unless you are able to develop some market philosophy of your own, you will not be able to tell the good from the bad, the sound from the unsound."

John Magee, as quoted in Edwards and Magee (1992, p.viii)

Chapter 4 Discounted Cash Flow Models

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Discounted Cash Flow Prior to J.B. Williams

Macaulay (1938, p.130-2) makes the following observation:

> Because the good that the common stock offers to its purchaser is an expectation of future money payments, the relation of its present-money price to its future-money payments is as ***unmistakably an interest phenomenon*** as is the relation of the present-money price of a bond to its future-money payments. In the fullness of time the stock will have a 'realized' or 'actual' yield just as will the bond. And, though the stock makes no 'promise', as does the bond, and therefore has no 'promised' or 'hypothetical' yield, its price ***discounts estimated future payments*** as truly as does the price of a bond. (emphasis added)

Despite this recognition of using discounted cash flow methods to value common stocks, Macaulay objects quite strongly to the practicality of using such methods:

> The 'assumption of payment', which must be made before the promised or 'hypothetical' yield of a bond can be calculated ... may, as we have seen, be a mere mathematical fiction for all except the highest grade of bonds. But, for common stocks it is not only a mathematical fiction but also ***an economic absurdity***.

For Macaulay, the difficulty of estimating the cash flows generated by common stock prevented the practical application of discounting methods to value such securities. This skepticism is a useful backdrop to the following discussion on the use of discounted cash flow techniques to value equity securities.

4.1 History of Equity Valuation Models

A. Equity Value in Partnerships

The problem of determining the value of an equity position in a business venture goes back to antiquity (Heckshер 1955, vol.1). In particular, because transport of goods over long distance was expensive, arduous and risky, a number of merchants would often combine resources in a business venture to transport goods to and from a foreign trading location. Such ventures were common in the seaborne carry trade and usually were dissolved when the voyage was completed. The valuation problem involved ***the division of the terminal equity value into the shares*** determined by the contribution of each partner to the venture. For the seaborne trade, these shares were usually determined when the business venture was initiated. However, for more permanent ventures, such as a farm, wholesale business or a manufacturing facility, equity contributions might occur at different times. Equity valuation was required, not only when a share in a venture was sold or wound-up, but also in estate probate where more than one individual has a claim to the equity share of the deceased.

A range of equity valuation problems are posed and solved in the commercial arithmetics that were used in training merchants in the 15th to 17th century reckoning schools (Swetz 1987; Poitras 2000, ch.4) The ***Treviso Arithmetic*** (1478) is not a particularly memorable book from the standpoint of high theory. The primary relevance of the book is that it was the first printed book on commercial arithmetic and, more generally, on mathematics. As such, the book almost certainly provides a snapshot of the teachings that an unnamed *maestri d'abbaco* gave to the students of his reckoning school. The book is untitled and draws its name from being an arithmetic published in Treviso, a town in the Venetian republic, some 26 kilometres northwest of Venice. Treviso was a town of some economic importance, being located about one day's journey on the main trade route linking Venice with northern and central European centres such as Vienna and the German cities. Economic activity in Treviso was sufficiently robust that, as early as 1372, the town was capable of supporting a *maestri d'abbaco* (Swetz 1987, ch. 1).

In the *Treviso*, three problems are given where simple interest rate methods are used to calculate the returns from partnership (Swetz 1987, pp.138-9). No other attention is given to any situations involving interest payments. The first of these problems is an elementary application of the rule of three:

> Three merchants have invested their money in a partnership ... Piero put in 112 ducats, Polo 200 ducats and Zuanne 142 ducats. At the end of a certain period they found that they had gained 563 ducats. Required is to know how much falls to each man so that no one shall be cheated.

The solution to the problem is uninteresting from a mathematical viewpoint.[1] However, the problem is of interest in illustrating the general framework for practical partnership problems. In addition, the author is careful to implicitly observe that if all partners have funds invested for the same length of time, the solution to the problem is independent of the endpoint of the partnership.

The second of the *Treviso* problems is more complicated in that the partners are permitted to be involved in the partnership for different time periods (Swetz 1987, p.143):

> Two merchants, Sebastino and Jacomo, have invested their money for gain in a partnership. Sebastino put in 350 ducats on the first day of January, 1472, and Jacomo 500 ducats, 14 grossi on the first day of July, 1472; and on the first day of January, 1474 they found that they had gained 622 ducats. Required is the share of each (man so that no one shall be cheated).

The proposed solution to this problem follows as an extension of applying the rule of three given in the first problem. As such, this is also a simple interest method of solution. Observing that the stated solution does not admit the possibility of compound interest provides considerable insight into the methods of calculation used in mercantile practice during this period.

Considering the proposed solution in more detail requires knowing that 1 ducat = 24 grossi and 1 grossi = 32 pizoli. The solution to the problem proceeds by applying the rule of three which, in this case, involves expressing the two contributions in grossi, 8400 grossi for Sebastino and 12014 grossi for Jacomo with the addendum that 'since Sebastino has had his share in 6 months longer than Jacomo, we must multiply each share by the length of its time'. Multiplying by 24 months gives Sebastino's share as 201,600 and by 18 months gives Jacomo's share as 216,252. Taking the sum of these two shares (417,852) for a divisor and applying the 'rule of three' gives the solution of 300 ducats, 2 grossi, 8 pizoli and a remainder for Sebastino and 321 ducats, 21 grossi, 13 pizoli and a remainder for Jacomo.

The <u>*Treviso*</u> solution to the partnership problem does not involve ***the use of compound interest.*** Using semi-annual compounding, the inclusion of compound interest would involve solving:

$$850\frac{14}{24} + 622 = 350\left(1 + \frac{r}{2}\right)^4 + 500\frac{14}{24}\left(1 + \frac{r}{2}\right)^3$$

The solution of r = 34.694% requires the evaluation of a quartic equation. The associated shares would be 308.4 ducats (308 ducats, 9 grossi, 19 pizoli and remainder) for Sebastino and 313.6 ducats (313 ducats, 14 grossi, 12 pizoli and remainder) for Jacomo, a decidedly different result than the 'just' result proposed in the *Treviso*.[2]

Evidence that knowledge of compound interest calculations was common in the commercial practice of the 15th century, at least in the important financial centres such as Lyons, can be found in Chuquet's **<u>Triparty</u>**. On the subject of compound interest, the <u>Triparty</u> makes explicit reference to the incongruity between the theoretically correct mathematical calculation and recommended commercial practice for calculating shares in partnerships reflected in the basic commercial arithmetics. The manuscripts contained in the <u>Triparty</u> are actually three main sections concerned with algebraic theory, and three other parts containing problems, a geometry and a commercial arithmetic. The latter is generally similar in content to the <u>*Treviso*</u>, reflecting the similarity in the study of commercial arithmetic throughout Europe. However, unlike the <u>*Treviso*</u>, the handling of compound interest is recognized directly (pp.306-7):

> Three merchants formed a company, one of whom put in 10 ecus which remained there for

> the space of three years. The second put in 6 ecus which remained there for 7 years, and the third put in 8 ecus which remained there for four years. At the end of a period, 20 livres of profit was found. One asks how much comes to each, considering the money and the time that each has used it.

The answer proceeds with the usual application of the rule of three as in the *Treviso*. After presenting this method and the solution Chuquet states (p.307):

> And the calculation is done, according to the style and opinion of some. And in order for such reckoning to be of value, it is necessary to presuppose that the principal or the capital alone has made a profit, and not the profit (itself). And inasmuch as it is not thus, for the profit and the profit on the profit made in merchandise can earn profit and profit on profit in proportion to the principal, from day to day, from month to month and from year to year, whereby a larger profit may ensue. Thus such calculations are null, and I believe that among merchants no such companies are formed.

Though the compound interest solution is not provided, Chuquet appears to hold that calculation of compound interest is the fair practice in calculating the returns from partnerships of unequal duration.[3]

B. The Emergence of Discounted Cash Flow Models

The use of discounted cash flow (*DCF*) methods to value investments and real assets goes back centuries, e.g., Poitras (2000, ch.4). The application of these techniques to the earliest issues of equity securities involved the use of valuation methods developed for fixed income securities to determine the 'intrinsick' value of the stock. Well into the 20th century, ***the dividend yield*** was considered to be the most important measure of equity security value, especially for investment grade stocks, e.g., Rutterford (2004). There were various reasons for this preference, including the absence of sufficiently accurate accounting information prior to reforms of the securities markets begun by passage in the US of the Securities Act (1933) and the Securities and Exchange Act (1934) and the Companies Act (1948) in the UK.[4] Dividends paid were the most visible and reliable source of information about firm performance. In addition, in the UK and Europe corporations tended to have high dividend payout ratios and a preference for raising capital for expansion from new share issues (Graham and Dodd 1934, p.331), instead of having a lower dividend payout and reinvesting retained earnings to expand the business.

The shortcoming with using fixed income methods to value common stock is that the dividend is not fixed over time. The historical transition from traditional fixed income valuation to discounted cash flow valuation can be traced to ***a number of disparate contributions by actuaries, real estate appraisers and engineers made in the mid-19th century***. These more advanced valuation methodologies permitted the future cash flows associated with an equity security to vary over time. In particular, a UK mining engineer, William Armstrong, used discounted cash flow methods to value mining company issues and mining leases (Pitts 2001). Around this time actuaries also became interested in valuation for variable annuities. The appearance of the first Institute of

Actuaries textbook (Sutton 1882) contained a section on variable annuities. Todhunter (1901) modelled the common stock price using a perpetuity with a constant growth rate, determining the pricing solution with an infinite number of dividend payments. This approach to equity valuation continued to attract attention until the 1960's, e.g., Clendenin and van Greave (1954), Durand (1957a), Solodofsky (1966).

While not directly concerned with *DCF* valuation, Smith (1925) (Common Stocks as Long-Term Investments) arguably marks the beginning of ***the transition from equity valuation based on dividend yields to equity valuation based on earnings***. Providing detailed empirical evidence, Smith (1925, 1927) changed the perception of the importance of earnings relative to dividends. Prior to Smith, the convention was to measure the value of common stock relative to bonds: being the most junior of all securities with the greatest variability of cash flow common stock required the highest yield among the securities on offer from a given corporation. Smith demonstrated that the *ex post* returns on equities outperformed bonds due to the compounding effect on future dividends and future share prices of reinvesting 'this increasing surplus in productive operation' (Smith 1925, p.77). This view was well suited to US companies which had benefited from strong economic growth over the 1866-1922 sample period that Smith examined. The superior presentation of accounting information by US as compared to UK and European companies, e.g., in the reporting of consolidated versus unconsolidated earnings, also facilitated the use of valuations based on earnings as opposed to dividends.

INSERT US Dividend versus AAA Bond Yield (Figure 4.1.a)
INSERT Figure 4.1.b Ma and Kanas (2004) 1871-1996 P/D ratio

The connection between equity valuation based on earnings and *DCF* valuation is that once reinvestment of retained earnings is recognized then the dividend is expected to rise over time with the share price. This represents a significant change in the character of corporate finance which favoured distributing the bulk of earnings in dividend payments, leaving undistributed earnings in reserves designed to maintain the dividend payment in adverse conditions. Faced with favourable growth prospects, US companies tended to have lower dividend payout ratios than firms in the UK and Europe that were often more mature and did not have the same type of need for capital to fund growth opportunities as those operating in the tariff-protected US market. With high dividend payout and lower growth prospects, unconsolidated reported earnings were primarily used to measure the 'dividend cover' – similar to the use of interest coverage ratios to value corporate bonds. Recognition that the dividend payment will increase over time led to theoretical contributions providing appropriate pricing models. For example, Guild (1931) developed a model of the share price using the sum of a finite period of constant growth in cash flow plus a terminal share value measured using the discounted value of a price/earnings multiple.

While Guild (1931) was written for the investment trade, similar contributions were appearing in academic publications. Preinreich (1932) presented a model with a capital base that expands as earnings grow over a finite period. Significantly, Preinreich concludes: "only discounted cash flow techniques could value such growth firms correctly" (Rutterford 2004, p.139). Despite the ability to model the price of equity claims using a discounted stream of growing dividends, the widespread recognition and acceptance of these methods by academics and some practitioners to value common stocks begins with the theoretical and empirical applications in ***John Burr Williams*** The Theory of Investment Value (1938). The classic Graham and Dodd (1934) gives no explicit discussion or recognition of discounted cash flow valuation for equities, though in the final edition of Security Analysis, Graham, Dodd and Cottle (1962), *DCF* methods are recommended to estimate intrinsic value without actually executing such a calculation.

The basic notion advanced in Williams (1938) was that the present value for an equity security, such as a common stock, can be determined by discounting the future stream of expected cash inflows minus expected cash outflows at the appropriate rate of interest. The notion that "every stock price represents a discounted value of future dividends and earnings of that stock" (Fisher 1930, xxii) was well known by the time Williams (1938) appeared. What is significant about Williams (1938) is that the common stock valuation takes centre stage in the discussion. There is detailed discussion of how the distribution of company earnings into dividends and retained earnings impacted the future growth of both earnings and the balance sheet. While only simplistic growth assumptions are used, Williams (1938) devoted pages to actual valuations of companies such as General Motors. Recognizing that "investment value" for Williams corresponded to "intrinsic value" in Graham and Dodd (1934), Williams (1938) provided a promising and well-structured method of determining a numerical 'intrinsic value'.

The theoretical development of the basic discounted dividend model advanced in Williams (1938) reaches a climax with Durand (1957a) where a connection is made between the constant dividend growth version of ***the discounted dividend model and the St. Peterburg paradox***. Though the constant dividend growth variant of the discounted dividend model is often attributed to Gordon (1962), as Durand (1957a, p.351) observes this version of the model is given in Williams (1938). Durand (1957a) considers a number of theoretical nuances arising from the dividend discount model with growing dividends. Despite this long history, the eponym for the simplified *DDM* belong to Myron Gordon, currently Professor Emeritus at the University of Toronto. Because Gordon (1962) was concerned with the important practical problem of valuing companies in regulated industries, the formula provided a 'killer application' for the emerging capital asset pricing model to determine the unobservable discount rate for the stock. In recognition of the significance that the model played at that time, the constant dividend growth version of the discounted cash flow model is referred to as the 'Gordon growth model', e.g., Damodaran (1994, p.99).

C. Rational Bubbles and Intrinsic Bubbles

To review the derivation of ***the basic DCF model***, assume for the moment that the future is known with certainty and that perfect market assumptions apply. In the present context, this means there are no taxes and the term structure of discount rates is flat. Given this, consider the problem of determining the current price of a common stock by discounting the future cash flows. Assume that the stock to be valued is purchased at price *P(0)* and held for one period and then sold. This current price can be modeled as the discounted value for the sum of the dividend to be received in the next period (*Div(1)*) and the price *P(1)* received from selling the stock. Assuming the dividend is paid at the point the stock is sold:

$$P(0) = \frac{Div(1) + P(1)}{1 + k} \qquad \rightarrow \qquad k = \frac{P(1) - P(0)}{P(0)} + \frac{Div(1)}{P(0)}$$

This is the 'basic valuation equation', sometimes incorrectly referred to as the 'absence of arbitrage' condition. In effect, the (expected) return on the stock can be decomposed into two parts: the (expected) capital gain and the (expected) dividend yield. Dropping the assumption that future cash flows are known with certainty, leads to the result that $k = E[R_S]$, the expected return on the stock.

Accounting for randomness in the future cash flows by taking expectations conditional on information available at $t=0$, the ***infinite horizon discounted dividend model*** is derived by making a progressive substitution for prices:

$$E[P(1)] = \frac{E[P(2)] + E[Div(2)]}{1 + k} \quad \rightarrow \quad P(0) = \frac{E[Div(1)]}{(1 + k)} + \frac{E[P(2) + Div(2)]}{(1 + k)^2}$$

$$P(0) = \sum_{t=1}^{T} \frac{E[Div(t)]}{(1 + k)^t} + \frac{E[P(T)]}{(1 + k)^T} \quad \rightarrow \quad P(0) = \sum_{t=1}^{\infty} \frac{E[Div(t)]}{(1 + k)^t}$$

The relevance of other perfect markets assumptions follows appropriately. For example, introducing taxes requires a distinction to be made between the stream of expected dividends and the expected capital gain, which will be taxed at different rates. Combining this with differences in the relative riskiness for these two types of cash flows leads to the possibility that different discount rates might be required for dividends and capital gains. In addition, relaxing the assumption of a flat term structure of discount rates requires different k's to be used to discount cash flows occurring at different points in time.

Careful analysis of the substitution process involved in deriving the infinite horizon discounted dividend model reveals a problem: what if there are other variables that affect prices than just discounted stream of future dividend payments? Then this component of the price will be ignored in the progressive substitution process and the proposed infinite horizon solution would be faulty. This theoretical difficulty is compounded by the empirical observation, initially advanced by Flood and Garber (1980), LeRoy and Porter (1981) and Shiller (1981) using the assumption of a constant discount rate, that the observed variation in stock prices was too volatile to be consistent with the infinite horizon discounted dividend model. The ***"excess volatility hypothesis"*** has subsequently generated an impressive collection of empirical articles that have explored a range of possible failings in the initial statement of the hypothesis, including Campbell and Shiller (1987), Evans (1991), Campbell and Kyle (1993) and McMillan (2007). Despite these considerable efforts, the basic empirical result still remains: the variability of common stock prices cannot be sufficiently explained by the variation in dividends.

The appearance of the empirical 'excess volatility hypothesis' generated a demand for theoretical contributions that could explain this stylized fact. An initial step in this direction was the rational speculative bubble or ***'rational bubble'*** proposed by Blanchard and Watson (1982). To motivate this approach, consider the discounted dividend model in continuous time:

$$p(t) = V(t,T)\ p(T) + \int_t^T V(t,u)\ d(u)\ du \tag{1}$$

where $p(t)$ is the real (price index deflated) stock price observed at time t, $p(T)$ is the anticipated real stock price at time T, $d(u)$ is the expected continuously paid real dividend over $u \, \varepsilon \, (t,T]$, and T is the terminal or maturity date for the valuation problem ($T \geq t \geq 0$). In this formulation, the set of valuation operators $\{V(t,T)\}$ involves both discounting and expected value operations. The same set of valuation operators is applied to both prices and dividends.

As illustrated above in discrete time, progressive substitution for $p(T)$ in (1) produces the infinite horizon, discounted-dividend model:

$$p(t) = \int_t^{\infty} V(t,u) \; d(u) \; du = p_F(t) \qquad (2)$$

Convergence of the operator as $T \rightarrow \infty$ is required for the satisfaction of a ***transversality condition***:

$$\lim_{T \rightarrow \infty} V(t,T) \; p(T) = 0 \qquad (3)$$

This condition is needed to ensure that the stochastic difference equation identified by (1) will not have an infinite number of solutions, e.g., Craine (1993). In other words, (3) is a technical condition required for the solution to the pricing problem to be unique.

The importance of the transversality condition can be seen by observing that it is possible the stock price $p(t)$ has two components: $p(t) = p_F(t) + B(t)$, the "market fundamentals" component $p_F(t)$ associated with the infinite stream of discounted future dividend payments and a rational bubbles component $B(t)$, where $p_F(t)$ is given in (2) and $B(t)$ can be any random variable that satisfies $B(t) = V(t,T) \; B(T)$. Because in this case $p(t)$ incorporates both fundamental and bubble information, progressive substitution for $p(T)$ produces restrictions on the bubble component:

$$p(t) = p_F(t) + B(t) = V(t,T)\Big(p_F(T) + B(T) \Big) + \int_t^T V(t,u) \; d(u) \; du$$

$$= \int_t^{\infty} V(t,u) \; d(u) \; du + V(t,T) \; B(T)$$

In order for the transversality condition (3) to be satisfied, then an explosive bubble is ruled out. However, it is not enough to have $B(T)$ vanish in the limit as T goes to infinity. As initially demonstrated by Blanchard (1979), in order for $B(t) = 0$ and the current price to be determined solely by $p_F(t)$, it is necessary to rule out a 'speculative, periodically collapsing bubble'. Following Evans (1991, p.924), for stock price models: "Bubbles do not appear to be empirically plausible unless there is a significant chance that they will collapse after reaching high levels."[5]

Considerable initial effort in the empirical analysis of rational bubbles was concerned with arriving at a correct formulation of the testable hypothesis. For example, Evans (1991) showed that the dynamics of a bubble could fool a testing strategy that involved comparing the stationarity properties of stock prices and dividends, as suggested by Diba and Grossman (1988), Hamilton and Whiteman (1985) and others. Accumulating empirical evidence against the 'rational bubble' model led to the introduction of ***'intrinsic bubbles'*** by Froot and Obstfeld (1991). Because the intrinsic bubble component of the price also depends on dividends, the resulting relationship between the log of prices and dividends is non-linear. This non-linearity hypothesis has received much more favourable empirical support, e.g., Ma and Kanas (2004). Non-linearity is now the norm in econometric studies of the relationship between stock prices and dividends, e.g., McMillan (2007), Balke and Wohar (2006; 2009). However, it is well known that non-linearity in the relationship between stock prices and dividends can be generated by a variety of observationally equivalent models, e.g., stochastic regime switching, temporal variation in the parameters of the dividend process and intrinsic bubbles all produce non-linearity.

> Balke and Wohar (2006, p.55) provide the following summary of the empirical relationship between stock prices and dividends:
>
> "the data have difficulty distinguishing a stock price decomposition in which expectations of future real dividend growth is a primary determinant of stock price movements from one in which expectations of future excess returns are a primary determinant. The data cannot distinguish between these very different decompositions because movements in the price-dividend ratio are very persistent whereas neither real dividend growth nor excess returns are; most of the information about low-frequency movements in dividend growth and excess returns is contained in stock prices and not the series themselves ... this inability to identify the source of the stock price movements is not solely due to poor power and size properties of our statistical procedure, nor does it appear to be due to the presence of a rational bubble."

A quarter century after the excess volatility hypothesis first appeared, econometric explanations are still a work in progress. As indicated by the Balke and Wolhar (2006, p. 55) quote, there is an impasse in the line of empirical research that aimed to exploit properties of covariance stationary processes to decompose stock price movements: "The data cannot distinguish between ... very different decompositions [of the stock price] because movements in the price-dividend ratio are very persistent whereas neither real dividend growth nor excess returns are". Based on closer inspection of the 'clean surplus' equation, e.g., Lundholm (1995), it appears that ***cash dividends are not the most appropriate variable to use for cash flows*** in a given period if stock valuation is the objective. Rather, substituting earnings and book value for dividends has a stronger accounting foundation. It is also apparent that the 'narrow dividends' variable almost always employed in empirical bubble studies does not capture all cash flows contained in the 'broad dividends' variable defined by the clean surplus relationship, especially share repurchases (see Figure 4.1.c). So even if dividends are to be used as the cash flow, the dividend variable that has been used is too 'narrow' to be consistent with clean surplus accounting. While the implications of the clean surplus relationship are well known to accountants, this point is only gradually being recognized by financial economists, e.g., De Angelo and De Angelo (2006); Handley (2008).

INSERT Fig.4.1.c
Prices, Narrow and Broad Dividends

from Jiang and Lee (2005, p.1481)

4.2 A Variety of *DCF* Models

A. The Gordon Growth Model

Despite impressions to the contrary, e.g., Cunningham (2002, p.93), Williams (1938) did not originate the discounted cash flow model for security analysis. Rather Williams popularized acceptance of the approach.[6] Prior to Williams, informed opinion was generally against the validity of the model, e.g., Macaulay (1938). What disturbed previous writers about this approach to common stock valuation was not the basic formulation but, rather, the difficulties of determining the future cash flows for stock. The application of discounted cash flow techniques to assess the value of stocks, where value is measured using a model price vs. market price, depends fundamentally on the forecasting of future payments. In cases where the cash flows from the equity security are relatively predictable, then discounted cash flow techniques can provide a reasonable estimate for the intrinsic value of the security. Given this, where in the present common stock universe can such situations be commonly found?

The motivation for the Gordon growth model (Gordon 1962) is to provide a simplification of the general form of the discounted dividend model (*DDM*), where determining the price involves evaluating the infinite sum of discounted dividends, a clearly impractical exercise. The ***Gordon growth model*** permits the dividend to change over time according to the assumption: $D(t+1) = D(t)(1 + g)$, where g is the assumed constant growth rate in dividends. Dropping the expectation for ease of notation and substituting this result into the general form of the discounted dividend model produces the simplified *DDM* model or, other words, the Gordon growth model:

$$P(0) = \sum_{t=1}^{\infty} \frac{Div(t)}{(1 + k)^t} = \sum_{t=1}^{\infty} \frac{D(0)(1 + g)^t}{(1 + k)^t}$$

$$= \frac{D(0)(1 + g)}{(1 + k)} \left[1 + \frac{1 + g}{1 + k} + \frac{(1 + g)^2}{(1 + k)^2} + \frac{(1 + g)^3}{(1 + k)^3} + \ldots \right]$$

$$= \frac{D(0)(1 + g)}{(1 + k)} \left[\frac{1}{1 - \dfrac{1 + g}{1 + k}} \right] = \frac{D(1)}{k - g} \qquad for\ k > g$$

By assuming that the dividend grows at a constant rate over time the Gordon growth model is able to provide a simple common stock valuation model: $P(0) = D(1) / (k - g)$. The example that Macaulay provides refers to the situation where $g \to k$ which gives "a very high price level for stocks".

In the context of the Gordon growth model with $g \to k$, Macaulay (1938) develops an interesting implication:

> If the dividends were \$4(1.03), \$4(1.03)^2, \$4(1.03)^3, etc. ... and if these dividends were discounted at 3 per cent per annum, the price of a share of the stock that was to pay the dividends, should be just four times the *number of payments* that were to be made; in other words, four times the *number of years* that the succession of dividends was to continue.

Though this result is relatively obvious from inspection of the original sum, this result is not so obvious from inspection of the $D(1) / (k - g)$ formulation of the model where $k > g$ is required for convergence of the perpetual sum. Over time, a number of developments of the basic Gordon growth model have appeared that introduce more complicated patterns for future dividend payments. For example, Malkiel (1963) has a two stage model where dividends grow at a constant rate for a finite number of years and then grows at a rate typical of other firms in the economy thereafter. Molodovsky et al. (1965) has a three stage model where dividends initially grow at a constant rate, then decline over a second period to be followed by a constant steady state dividend payment thereafter. While theoretically appealing, the two, three and multi-stage growth models lack practical applications in all but the most specialized situations. Due to the proliferation of parameters and terms, the formulas quickly get unwieldy. Useful simplifications of multi-stage growth models can be obtained for purposes of doing relative value analysis using P/E ratios, e.g., Leibowitz (1999, 2000).

Clean Surplus Accounting

The Gordon growth model makes the precise statement that common stock pricing depends on three variables: the dividend to be received next period, $Div(1) = D(1)$; the expected return on the common stock, k; and the long term growth rate of dividends, g. In terms of expected returns the model maintains that: $E[R_S] = k = (D(1)/P(0)) + g$ and $(D(1) / P(0)) = (k - g)$. Returning to the 'basic valuation equation', this implies that the expected capital gain is equal to the expected growth rate of dividends. While this might seem to be quite unrealistic, it does have a reasonable interpretation. If the *P/E* ratio does not change over time, the growth rate in earnings will equal the growth rate in dividends if the dividend payout ratio does not change over time. Under these assumptions, g will be translated into the capital gain (see sec. 4.3). Yet, if these assumptions are adopted, then the model can be manipulated to produce other interesting results that can be used to interpret widely used valuation measures.

More precisely, assuming for the moment that the Gordon growth model is correct, it is possible to manipulate the model to provide precise statements for two important valuation measures, the price-earnings (*P/E*) ratio and the price-to-book value (*P/BV*) ratio. In turn, these values can be used to provide an interpretation for g. To convert the Gordon model to *P/E* form requires the ***'clean surplus' equation*** for earnings: $E(t) = Div(t) + RE(t) = Div(t) + (BV(t) - BV(t\text{-}1))$, where $E(t)$ is earnings available to common stockholders, $RE(t)$ is the retained earnings and $BV(t)$ is the book value of equity, all observed at time t and expressed on a per share basis. In keeping with currently accepted accounting practice, e.g., Bernstein (1989, p.747), the clean surplus equation requires that all items involving gain or loss in income are accounted for in the period in which these items occur.

In effect, *E(t)* is either paid out in dividends or is retained earnings and accounted for by changing the book value of equity. Though there are some accounting qualifications to this condition, the clean surplus equation is sufficient detail for purposes of equity security valuation.

Letting *b* represent the dividend payout ratio (*Div(t) / E(t)*), i.e., *b E(t) = Div(t)*, substituting this result into the Gordon growth model produces the ***simplified DDM P/E ratio***:

$$P(0) = \frac{b\ E(1)}{k - g} = \frac{b\ E(0)(1 + g)}{k - g} \qquad \rightarrow \qquad \frac{P(0)}{E(0)} = \frac{b\ (1 + g)}{k - g}$$

The common practice of using the 'forward' P/E ratio (*P(0)/E(1)*) to measure the earnings multiple eliminates the (1+*g*) term in the numerator. Taking the dividend payout ratio to be fixed over time produces: *D(t+1) = bE(t+1) = b E(t) (1 + g)* → *E(t+1) = E(t)(1 + g)*. In words, with a constant dividend payout ratio, the constant growth in dividends assumption translates into an assumption about the constant growth in earnings. To derive the price to book ratio involves observing that *(E(t) / BV(t-1)) = ROE(t)*, where *ROE(t)* is the return on equity at time *t*. (Though it is more conventional to use *BV(t)* in defining *ROE(t)*, this definition will not be used here.) Making the appropriate substitution produces the ***simplified DDM P/BV ratio***:

$$\frac{P(0)}{BV(0)} = \frac{b\ \dfrac{E(1)}{BV(0)}}{k - g} = \frac{b\ ROE(1)}{k - g}$$

It follows that the price to book value ratio will depend on the dividend payout ratio, the *ROE*, the expected return on the stock and the growth rate in earnings.

B. Abnormal Earnings and Other Models

The failure of discounted dividend models, such as the Gordon model, to value most common stocks was not lost on accounting researchers (Lee 1999, Jiang and Lee 2005). From an accounting perspective, the dividend discount model is relatively naive as it ignores the variety of inter-relationships between accounting numbers that follow from the clean surplus relationship. As such, dividends in the clean surplus equation include not only conventional cash dividends but also other forms of cash payouts to shareholders (e.g., share repurchases, acquisitions). It is conventional for studies of the dividend discount model to use actual cash dividends paid, ignoring other forms of payments from earnings to shareholders. A number of different variations on the *DCF* model can be derived.[7] Ohlson (1991, 1995) and Feltham and Ohlson (1995) popularized the application of the clean surplus relationship, *BV(t) = BV(t-1) + E(t) - D(t)* into the ***abnormal earnings form of the DCF model*** (English 2001, p.334-5):[8]

$$P(0) = \sum_{t=1}^{\infty} \frac{D(t)}{(1 + k)^t} = \sum_{t=1}^{\infty} \frac{E(t) - \Delta BV(t)}{(1 + k)^t}$$

$$= BV(0) + \sum_{t=1}^{\infty} \frac{(ROE(t) - k)\ BV(t-1)}{(1 + k)^t} = BV(0) + \sum_{t=1}^{\infty} \frac{AE(t)}{(1 + k)^t}$$

where $AE(t) = (ROE(t) - k)\ BV(t-1)$ is the "abnormal earnings" attributable to equity in period *t* and *BV* is expressed on a per share basis.

A number of different titles have been given to the abnormal earnings model. Ritter and Warr (2002, p.36) refer to this form the *DCF* model as the "***residual income model***" while Penman (2001, chap.6) uses "***residual earnings model***".[9] Ritter and Warr make numerous adjustments to the model to account for the impact of inflation and the use of accounting accruals. Dechow et al. (1999) and Callen and Segal (2004) provide a detailed examination of this model while Penman and Sougiannis (1998) compares this *DCF* model with the free cash flow and dividend discount variants. This formula has intuitive appeal because it relates the current price to the initial value of capital raised, reflected in *BV(0)* adjusted for the ability of the firm to earn more (or less) on invested capital (*ROE)* than the cost of maintaining the capital stock, as reflected in *k*. This formulation captures the idea that securities with superior investment potential create wealth (*ROE* > *k*) as opposed to destroying wealth (*ROE* < *k*).

In practical applications it is often useful work with the ***simplified residual income model*** which corresponds to the Gordon growth model for the discounted dividend valuation. One version of this model follows from manipulation of the *DDM* following substitution the clean surplus equation:

$$P(0) = \frac{P(1) + D(1)}{1 + k} = \frac{P(1) + E(1) - \Delta\ BV(1)}{1 + k} = \frac{P(1) + E(1) - g_B(1)\ BV(0)}{1 + k}$$

$$\rightarrow \quad P(0) = BV(0) \left(\frac{ROE(1) - g_B(1)}{k - g} \right)$$

where g_B corresponds to the growth in book value: $BV(t+1) = BV(t)\ (1+ g_B(t+1))$. This formula captures the theoretical connection between the growth in earnings and the growth in book value. More precisely, except in special cases, both *ROE* and g_B will be different and will vary over time.

To see the connection between *ROE* and g_B consider the following example:

Initial values at *t*=0: $BV(0) = 100 \quad g = 10\% \quad b = .6$

At *t*=1: $E(1) = 10 \rightarrow D(1) = 6 \quad RE(1) = 4 = \Delta BV = BV(1) - BV(0) \rightarrow BV(1) = 104$
$ROE(1) = E(1)/BV(0) = 10/100 = 10\% \quad g_B(1) = \Delta BV(1) / BV(0) = 4/100 = 4\%$

At *t*=2: $E(2) = E(1)(1 + g) = 10\ (1+g) = 11 \quad D(2) = 6.6 \quad RE(2) = 4.4 \rightarrow BV(2) = 108.4$
$ROE(2) = E(2)/BV(1) = 11/104 = 10.58\% \quad g_B(2) = \Delta BV(2) / BV(1) = 4.4/104 = 4.23\%$

At t=3: $E(3) = E(2)(1+g) = 11(1+g) = 12.1 \quad D(3) = 7.26 \quad RE(3) = 4.84 \rightarrow BV(3) = 113.24$
$ROE(3) = E(3)/BV(2) = 12.1/108.4 = 11.16\% \quad g_B(3) = \Delta BV(3) / BV(2) = 4.84/108.4 = 4.47\%$

When the defined growth rate for earnings, *g*, is fixed and *b* > *0*, this imposes a time path where earnings growth will exceed the growth in book value. It follows that the $ROE(t) = \{E(t)/BV(t-1)\}$ has to increase over time. The theoretical relationship for the growth in book values can be specified as:

$$RE(1) = (1 - b)E(1) = \Delta BV(1) = g_B(1)\ BV(0)$$

$$\rightarrow \quad g_B(1) = (1 - b)ROE(1) = (1 - b)\frac{E(1)}{BV(0)}$$

$$g_B(2) = \frac{(1 - b)E(2)}{BV(1)} = \frac{(1-b)(1+g)\ E(1)}{(1 + g_B(1))\ BV(0)} \quad \rightarrow \quad g_B(2) = \frac{1 + g}{1 + g_B(1)}\ g_B(1)$$

Using the numbers from the example above 4.23% = {(1.1)(.04)}/(1.04), which illustrates the evolution of the growth rate of book value. The simplified residual income model follows:

$$P(0) = \frac{E(1)}{1 + k} + \frac{E(2)}{(1 + k)^2} + \ldots - \frac{g_B(1)\ BV(0)}{1 + k} - \frac{g_B(2)\ BV(1)}{(1 + k)^2} - \ldots$$

$$= \frac{E(1)}{k - g} - \left(\frac{g_B(1)}{1 + k} + \frac{\frac{1 + g}{1 + g_B(1)} g_B(1)\ (1 + g_B(1))}{(1 + k)^2} + \ldots \right) BV(0)$$

Cancelling the powers of $(1 + g_B)$ gives the simplified residual income model:

$$P(0) = BV(0) \left(\frac{ROE(1) - g_B(1)}{k - g} \right) = BV(0) \left(\frac{b\ ROE(1)}{k - g} \right)$$

For complete dividend payout, $b = 1$, this equals the Gordon growth model. Just as in the Gordon model, the result depends on a constant dividend payout.

A useful manipulation of the abnormal earnings model follows from making a substitution for the dividend payout ratio, b, using the clean surplus relationship: $D(t) = b\,E(t) = (E(t) - (BV(t) - BV(t-1))$. Observing that constant growth in dividends with a constant dividend payout gives $BV(t) = (1 + g)BV(t-1)$, it follows (English 2001, p.353-4):

$$b = \frac{E(t) - gBV(t-1)}{E(t)} = 1 - \frac{g}{k} \left(\frac{k\ BV(t-1)}{E(t)} \right)$$

Recalling that the definition for *AE(t)* requires, *k BV(t-1) = E(t) - AE(t),* the expression for *b* can be manipulated to get:

$$b = \frac{1}{k}\left[k - g + g\left(\frac{AE(t)}{E(t)}\right)\right]$$

Substituting this result into the *P/E* ratio expression associated with the Gordon growth model produces the ***abnormal earnings form of the P/E ratio***:

$$\frac{P(0)}{E(0)} = \frac{1 + g}{k}\left\{1 + \left(\frac{AE(1)}{E(1)}\right)\left[\frac{g}{k - g}\right]\right\}$$

Compared to the simple Gordon growth model formulation of the *P/E* ratio, by making the connection between *P/E* and the ability to generate 'abnormal earnings' this formulation is more revealing.

Following Jiang and Lee (2005), the ***abnormal earnings model has a number of advantages over the DDM*** in estimating stock values. In particular, while the dividend process is too smooth to explain volatile stock prices, earnings and book value are considerably more volatile than dividends and, as such, have greater potential for explaining stock prices. Another more practical rationale for the abnormal earnings model is that, for many stocks, companies do not pay dividends significantly complicating application of the *DDM*. The earnings and book value inputs to the abnormal earnings model are both available even though dividends are not. Similarly, the abnormal earnings model makes direct use of the clean surplus accounting relationship while the 'dividends' used in the *DDM* are usually too narrow to be consistent. This lack of conceptual consistency extends to the interpretation of 'dividend irrelvance': "dividend policy irrelvance integrates naturally into the [abnormal earnings model]. The key is that dividends are paid out of book value but not out of current earnings. That is, residual income is invariant to changes in dividend policy" (Jiang and Lee 2005, p.1468).

Dividends and Share Repurchases

Recognition of the clean surplus relationship led to empirical studies on the properties of 'broad dividends', effectively cash dividends and share repurchases. To this end, there are numerous studies of 'narrow dividends' that document an overall decline in aggregate dividend payout ratios since the 1950's, marked by periods of persistence where the dividend payout ratio is relatively constant. This ***declining importance of dividends*** has been parallelled by an ***increasing importance of share repurchases***. For example, using a sample of all industrial firms listed on the Compustat database, Jagannathan et al. (2000) find that from 1985 to 1996 the number of open market repurchase program announcements by U.S. industrial firms increased 650% from $115 to $755, with an announced value increase of 750% from $15.4 billion to $113 billion. Over the same period, dividends increased by a factor of just over two during with aggregate dividends rising from $67.6 billion to $141.7 billion. Grullon and Michaely (2002), report an increase in the value of share repurchases from $1.5 billion to $194.2 billion over a 1972 and 2000 sample while dividends only rose from \$17.6 billion to \$171.7 billion.

The study of share repurchases has progressed considerably as more and more firms have adopted this method of returning cash to shareholders, e.g., Stephens and Weisbach (1998), Jagannathan et al. (2000), Kahle (2002), Lee and Rui (2007). At the time of the seminal work on corporate dividend policy by Lintner (1956), it was acceptable for corporate income to be effectively distributed among dividends, retained earnings, and taxes. Stock repurchases were not a practical part of the mix. But times have changed and, from the significant number of recent studies, ***a number of stylized facts*** have emerged. In particular, share repurchases and dividends are used at different times and by different types of firms. While dividends are typically paid by firms with a higher level of "permanent" operating cash flows, firms using share repurchases tend to have higher "temporary", non-operating cash flows. Both the cash flows and distributions from repurchasing firms tend to be substantially more volatile.

Against this backdrop of increased usage of share repurchases, Ferris et al. (2009) examine the number of firms paying dividends in 1994 and 2007 for a sample of 25 countries and examine the global decrease in the percentage of firms that are dividend payers. Combined with the more detailed results in Fama and French (2001) on US dividend payers for a 1926-1999 sample (see Figure 4.2.a), some key developments are apparent. Recognizing that there has been a significant increase in the total number of firms over the period, the percentage of firms paying dividends has fallen over 30% in common law countries from 74% to 43% between 1994 and 2007, with a more modest reduction of 8% from 70% to 62% for civil law countries.[10] The aggregate common law result brings these countries closer in line with the US which saw dividend payers fall from 36% to 25%. Following a period in the 1970's when the payers exceeded non-payers, since 1980 the number of dividend non-payers in the US has increasingly exceeded payers. The civil law countries, which include European firms other than UK and Ireland and the important emerging markets of China and Brazil, even saw some increases in the percentage of dividend payers in smaller European countries such as Spain and Finland. The percentage of dividend paying firms in Germany fell from 71% to 46%, while the important sample of Japanese firms fell only from 88% to 86%.

INSERT Figure 4.2.a
Dividend Payers from Fama and French (2001, Fig. 1)

Given the secular decline in the importance of cash dividend payout relative share repurchase programs, details on share repurchase activity increasing assumes importance. For example, various studies observe that the increases in stock repurchases have been pro-cyclical. Examining the ongoing process of corporations substituting share repurchases for dividends, Grullon and Michaely (2002) provide evidence that firms are not simply cutting dividends and replacing them with repurchases. Rather, ***large dividend-paying firms are repurchasing stocks rather than increasing dividends***, and that much of the growth in popularity of share repurchases is due to large dividend-paying firms. They report that although the dividend payout ratio of U,S. companies has been declining since the mid-1980s, the total payout ratio has remained more or less constant, which suggests that corporations have been substituting repurchases for dividends. They also show the average dividend payout ratio fell from 22.3% in 1974 to 13.8% in 1998, while the average repurchase payout ratio increased from 3.7% to 13.6% during the same period.

The connection of dividend payments with permanent cash flow and share repurchases with temporary, non-operating cash flows fits well with the traditional theory of dividends that originates with Lintner (1956). Based on a survey of corporate managers about the most important factors influencing dividend policy, Lintner found that the most important determinant of a change in the dividend payment is a change in company earnings that results in a dividend that is "out of line" with the firm's target payout ratio. Firms seek to make only partial adjustments in the actual payout ratio toward the target payout ratio and are strongly averse to cutting the regular cash dividend payment. Hence, managers smooth dividends in the short run to prevent fluctuations in the dividend cash flow to shareholders. ***Lintner's partial adjustment model for dividends*** can be formalized as: $D(t) = D(t-1) + \lambda\,(D^*(t) - D(t-1))$ where $\lambda\ \varepsilon\ [0,1]$ is the speed of adjustment coefficient and $D^*(t)$ is the target dividend payment.

The Lintner model of dividend payment behaviour has ***a number of testable hypotheses***. One testable implication is that dividend decreases will be relatively rare and only be associated with historically poor performance. Similarly, following dividend decreases (increases), the bad (good) operating performance of the firm will continue. In more recent studies, this hypothesis has been formulated along the lines that dividend increases will be related to the "permanent" component of cash flow and not to the "temporary" component of cash flows. Another implication of the Lintner model is that, due to the need for higher and more stable cash flows, dividend-paying firms will be larger than non-dividend paying firms. Strong empirical support for the Lintner model, in one form or another, has continued from the early cross-sectional work by Fama and Babiak (1968) to the recent survey work of Benartzi et al. (1997) and Baker et al. (2001).

The aversion of firms to cutting dividends is understandable given the significantly negative stock market reaction to dividend cuts, e.g., Ghosh and Woolridge (1988) and Denis et al. (1994) both report an average stock price decline of about 6% on the three days surrounding the announcement of a dividend cut. This punitive market response to dividend decreases has been identified as an argument in favour of share repurchase programs where there is no commitment to initiate a new repurchase program when the old program expires. As such, stock repurchases are a sensible way for firms to pay out "temporary" cash flows that have a high likelihood of not being sustainable. The typically favourable investor tax treatment for capital gains over dividend income is another argument in favour of share repurchases. The arguments in favour of share repurchases are so compelling that Black (1976) coined ***"the dividend puzzle"***, e.g., Crockett and Friend (1988); Christie (1990); Frankfurter (1999): why do firms use dividends to distribute corporate income and investors prefer this form of distribution when dividends are subject to double taxation?

In the US, the dividend puzzle has been largely resolved by the evolution of firm dividend policy. For example, Fama and French (2001) find that the percentage of firms paying cash dividends fell from 66.5% in 1978 to 20.7% in 1998 What has emerged more then three decades after Black introduced the dividend puzzle is that there is ***considerable heterogeneity in the dividend policy decision***. this is particularly true if the scope of discussion includes international firms Truong and Heaney (2007, p.684, Table 3) demonstrate the importance across firms and countries of corporate ownership composition, particularly the presence of a large shareholder or shareholder group, to the determination of dividend policy. While not as significant an issue with US stocks, in the Truong and Heaney sample of over 8000 firms form 37 countries, at least 50% of the firms had one shareholder or a shareholder group owning at least 25% of the equity in the firm, with the largest shareholder owning, on average, more than 30% of total voting shares.

The PEG Ratio and Other Models

The relationship between ***the P/E ratio and growth of the firm*** is a subject that receives attention in almost every introductory investments textbook, e.g., Bodie et al. (1999). The conventional starting point is the 'present value of growth opportunities' (*PVGO*) formulation of the *P/E* ratio. The simplifying, if somewhat confusing, assumption is made that there is benchmark 'firm' that is able to generate a constant stream of earnings into perpetuity that are fully paid out to common shareholders, i.e., $b = 1$. The value of this firm would be $P(0)^* = E(1) / k$. The definition of the *PVGO* reflected in the stock price for any given firm follows appropriately as: $P(0) = P(0)^* + PVGO$. Expressed as a *P/E* ratio, this formulation is: $(P(0) / E(1)) = (1 / k)\ [\ 1 + (PVGO / P(0)^*)]$. Hence, the *P/E* ratio will be higher for firms with higher growth opportunities. Within this framework, it can be shown that $g = ROE\,(1 - b)$ and, substituting into the Gordon growth model *P/E* ratio gives: $(P(0)/E(1)) = (b\ /\ \{\ k - (1 - b)(ROE)\})$. In this case, firms with higher *ROE*, which reflects growth opportunities, will have a higher *P/E* ratio. If accurate, these types of formulas would provide precise information about the relationship between *P/E* and growth opportunities.

One application of the different formulas for the *P/E* ratio is to illustrate the behavior of the ***"PEG" ratio***, or *P/E* to growth rate ratio. This ratio is sometimes used as a crude rule of thumb to determine under/over valuation for a common stock. For example, a PEG rule could be formulated as: if the PEG ratio is less than one then the stock is undervalued because the 'cost of growth' as measured by the *P/E* is less than the actual growth. The *AE* form of the *P/E* can be used to show that this rule is difficult to apply in practice, even when simplifying assumptions are made, i.e.:

$$\frac{P(0)}{100g\ E(0)} = \frac{PEG}{100} = \frac{1}{100}\left[\frac{1+g}{kg} + \frac{1+g}{k}\,\frac{AE(1)}{E(1)}\,\frac{1}{k-g}\right]$$

Scaling by 100 follows from recognizing that the PEG rule assumes the growth rate is expressed as a percentage whole number. It follows that if $AE(1) = 0$ because, say, the firm is in 'competitive equilibrium', then if $k = g = .1$ the PEG rule will be approximately correct. However, if $k = g = .05$, then the PEG will equal 4. Even without examining cases where $AE(1) \neq 0$, the PEG ratio rule can be seen to have significant limitations.

There is not complete agreement about: ***what is to be considered a DCF model?*** To the capital budgeting purists, a *DCF* model discounts the net cash flows. In this case, the variable being discounted is cash flow and is to be interpreted in a cash accounting sense. Assuming that the only cash payments received by the buy-and-hold common stock investor are dividends, an assumption which ignores share repurchases, leads to the discounted dividend model and its variants, such as the Gordon model. As illustrated, it is possible to expand the *DCF* universe to include cash flows that involve accrual accounting numbers, as in the residual income models. At some point, a boundary is crossed and traditional manipulations of GAAP numbers no longer can capture the type of cash flow required. For example, use of unadjusted earnings is problematic in the valuation of REITs as GAAP accounting involves a number of non-cash expenses, such as depreciation and amortization, that make little sense where the value of the assets is usually appreciating. As a consequence, REIT associations, such as REALPAC in Canada and NAREIT in the US, recommend the use of "funds from operations" as the appropriate cash flow measure.

Instead of starting with available accounting numbers to determine relevant cash flow numbers, it is possible to start with the characteristics of the desired cash flow and work backward to assemble a useful proxy from the accounting numbers. This is the basis of the 'economic free cash flow' model proposed by Warren Buffett. In its purest form,the appropriate variable to discount is ***economic profit*** – the net payment to invested capital after allowance for the economic cost of factors expended in the production process, including the opportunity cost of investment capital. In accrual form, this is the basis of the abnormal earnings model. However, there is a heterogeneity in the amount and type of distortion in the relationship between 'true economic value' and the reported accrual value. Recognizing that economic profit cannot be observed directly, the analytical problem is to fashion a proxy from available accounting numbers, using a combination of accrual and cash flow accounting.

The most widely used proxy for economic profit is ***'free cash flow'***. Being a non-GAAP number, there is no standardized definition currently in use. Examining a sample of US firms that voluntarily report free cash flow (*FCF*) information in 10-Q and 10-K filings, Adhikari and Duru (2006) find CFO-based – as opposed to net income-based – definitions of *FCF* are most commonly used. Within the group of firms reporting CFO-based *FCF*, Adhikari and Duru were able to identify seven different types of adjustments made to arrive at the reported *FCF* number. Compared to a matched sample of non-disclosing firms; firms disclosing *FCF* estimates tended to be less profitable; more leveraged; have lower credit ratings and higher dividend payout. Even for firms that view *FCF* as an important enough non-GAAP number to warrant reporting in the annual SEC filings, *FCF* definitions vary widely. While this does limit the immediate comparability of *FCF* disclosures across firms, it also reflects the difficulties in arriving at a specific measure of economic profit.

For purposes of valuing common stock using free cash flow, the relevant pricing model is the ***free cash flow to equity model*** While seeking to provide a more representative cash flow measure, there is conflicting academic evidence on whether these techniques are successful at achieving that objective. The free cash flow to equity model (*FCFE*) aims to measure the return to equity above the amount required to: maintain existing production levels; or, alternatively, to keep the firm on a particular growth path. To derive the *FCFE DCF* model, observe that the cash flow in this case is free cash flow to the firm, adjusted for net debt payments. Discounting of these cash flows leads to the general and simplified forms of ***the FCFE DCF model***:

$$P(0) = \sum_{t=1}^{\infty} \frac{FCFE(t)}{(1 + k)^t} \quad \rightarrow \quad P(0) = \frac{FCFE(1)}{k - g_f}$$

where $FCFE(t)$ is expressed on a per share basis and g_f is the expected growth rate in free cash flow. Recognizing that constant growth in dividends with a constant dividend payout does not ensure that *FCFE* will also grow at the same rate, it is possible to assume that *FCFE* grows at a constant rate g_f such that $FCFE(t) = FCFE(t\text{-}1)(1 + g_f)$, this produces the simplified free cash flow valuation model: $P(0) = \{FCFE^*(0)(1 + g_f)\} / \{k - g_f\}$. While it may be possible to assume $g = g_f$ under certain assumptions, given the variability in free cash flow, this is an unlikely situation.

There is a fundamental difficulty with using *FCF* to measure the 'true cash flow' or economic profit. There is no assurance that the cash amount spent on capital expenditures in a given period and the amount required to keep the firm on the assumed growth path will be the same. It is not even clear which items to include in the capital expenditure item. Room for improvement is indicated. The search for a measure of economic profit is not restricted to its uses in the valuation of equity securities. In particular, the management consulting industry is fundamentally concerned with corporate performance measurement for purposes such as setting executive compensation levels. To this end, Stern Stewart Management Services introduced the ***economic value added*** (EVA) measure (Stewart 1991). The EVA approach is conceptually the same as measures proposed by a number of other major management companies, such as the 'Economic Profit Model' of McKinsey & Company (Copeland et al. 1996). EVA-type models require a large number of possible adjustments; in the case of EVA more than 250 adjustments to accounting information are possible, though no more than 15 adjustments are usually of material significance (Bacidore et al. 1997).

As described by Copeland et al. (1996, p.149-50), the economic profit model and ***EVA-type models are an advance over DCF valuation using FCF*** because these measures can be used for understanding a company's performance in any single year, while free cash flow fluctuates too much to be useful. For example, tracking a company's progress by comparing actual and projected free cash flow, is too difficult because free cash flow in any year is determined by highly discretionary investments in fixed assets and working capital. Management delays or accelerates investments for various reasons. To target free cash flow in a given year at the expense of long term value creation is problematic. The basic idea behind the EVA-type models is that a company is worth more or less than its invested capital only to the extent that it earns more or less than its weighted average cost of capital. So the premium or discount relative to invested capital must equal the present value of the company's future economic profit. Because economic profit equals invested capital times the difference between the return on invested capital and the weighted average cost of capital

For purposes of equity valuation, the logic of the EVA-types models raises the question: are these approaches as superior to other *DCF* approach and other valuation techniques as the proponents claim? Unfortunately, the promise of superior performance for EVA as a security analysis tool compared to traditional measures such as net income does not have much empirical support. For example, Clinton and Chen (1998) found that other traditional accounting measures, such as *P/E*, *EPS* and *ROA*, tracked stock returns more reliably than *EVA*. More recently, Cordeiro and Kent (2001) considered whether analysts that adopted EVA outperformed other analysts in forecasting future EPS and found "***no significant relationship between EVA adoption and security analyst forecasts of future firm EPS performance***." Biddle et al. (1997, 1998) find similar results. For example, Biddle et al. (1997) conclude: "earnings [are] more highly associated with returns and firm values than EVA, residual income, or cash flow from operations. Incremental tests suggest that EVA components add only marginally to information content beyond earnings ... these results do not support claims that EVA dominates earnings in relative information content, and suggest rather that earnings generally outperform EVA." Similar results are reported by Kyriazis and Anastassis (2007).

C. Simplified Discounted Dividend Valuations

It is not too difficult to inspect the simplified *DCF* models of stock valuation, such as the Gordon growth model, and dismiss these models based on superficial analysis of the model structure. For example, without simplifying assumptions such as 'constant growth' the model is difficult to implement due to the larger number of terms that have to be estimated and calculated. Where the simplifying assumption of constant growth in dividends (earnings) is used, the empirical behaviour of dividends (earnings) does not support the assumption that dividends (earnings) grow at a constant rate over time. In addition, the market practice of low or no cash dividend payout would seem to argue against straight forward application of *DCF* models where the cash flows are defined as 'narrow' as opposed to 'broad' dividends. There are problems with obtaining estimates of k; there are problems with obtaining estimates of *D(1)*, and so it goes. However, adherents to Friedman's 'positivist' approach would argue it does not follow that just because the assumptions of a model seem impractical or unrealistic, that the model is necessarily invalid. Before dismissing the model out of hand, it would be appropriate to examine the performance of the model in practice.

Strong advocates of the simplified *DDM* model, such as Damodaran (1994) explicitly recognize ***essential characteristics of firms required for the Gordon growth model*** to be used for screening companies in order to produce adequate valuations. Damodaran identifies a number of the essential characteristics: a firm growth rate that is comparable to or lower than the nominal growth in the economy; the firm has a readily identifiable dividend-payout policy that is expected to continue into the future; and, the dividend payout of the firm has to be consistent with the assumption of stability, since stable firms generally pay substantial dividends. Damodaran (1994, p.103) optimistically estimates the average payout for large stable firms in the United States at about 60%. Large utilities, such as SW Bell, and high capitalization companies, such as Exxon, qualify as examples of a 'large stable firm'. Dresdner Bank, a German company, is used as an example of a large stable foreign company.

Calculating the Inputs

There are many subtle features that have to be resolved to do a simplified *DCF* valuation using the Gordon growth model. Consider the cost of equity k. It is conventional to estimate k using a form of the capital asset pricing model (CAPM): $k_i = E[R_i] = r + \beta_i \{E[R_M] - r\}$. To be operational this requires values for r, β_i and $E[R_M]$. In portfolio management applications of the CAPM, it is conventional to use a short-term interest rate, such as the 3 month Treasury bill rate, for ***the riskless interest rate*** (r). The maturity of the Treasury bill would be equal to the rebalancing frequency of the portfolio. However, application of the CAPM to determining the stock price using the Gordon growth model poses a different problem for defining the riskless interest rate. Following Damodaran (1994), the riskless security is identified "by matching the duration of the riskfree security with the duration of the asset being analysed". As a consequence, in the estimating the CAPM it is appropriate to use the longest maturity government bond available which, for the US, is the 30 year Treasury bond.

Another input required to use the CAPM is the risk premium on the market, $\{E[R_M] - r\}$, also known as ***the equity risk premium*** or equity premium. Some method is required to estimate this variable. Being an unobservable variable, it is difficult to determine the most appropriate procedure to estimate the expected return. However, the importance of the equity premium to modern Finance has generated a large number of studies attempting such an estimate (Oyefeso 2006). Using an 1889–1978 sample, Mehra and Prescott (1985) estimated an equity premium value of around 6% for

US data and argued that the size of the "equilibrium" equity risk premium was too large, marking the beginning of the 'equity premium puzzle' in modern Finance.[11] If correct, this puzzle has to sort out whether expected returns are too high or the risk-free rate is too low (Weil 1989) or both. Unfortunately, there is a considerable lack of both a theoretical and empirical consensus about the equity premium, e.g., Kocherlakota (1996). As Pastor and Stambaugh (2001, p.1207) observe: the equity premium remains "one of the most important but elusive quantities in finance".

What is the correct value to use for the equity premium in DDM valuations? Using Ibbotson Associates (1995) data, Brealey and Myers (1996), a widely used corporate finance textbook, reports an equity premium of between 8.2%–8.6% for the US market in 1995. This value is at the extreme end of the range of empirical estimates. The original estimate of Mehra and Prescott (1985) examined the average annual yield on the Standard and Poors 500 Index and found this was seven per cent over the period 1889–1978. Using an average yield for short-term debt of less than one per cent, the average equity premium of 6% is achieved. This estimate is also high. For a 1870-1989 sample, Shiller (1989) suggests an average equity premium of 4.3% for the US market. Expanding the search to international markets, using a 1985-1998 sample Claus and Thomas (2001) find equity premia as low as 3% for Canadian, French, German, Japanese, UK and US stock markets. Similarly, Dimson et al. (2002) find the following equity risk premia over the entire 20th century: France 7.7%; Switzerland 4.3%; Denmark 2.8%; Canada 4.6%; UK 4.9%; and USA 5.8%.

In Damodaran (1994), the equity premium was estimated by using the difference of the annualized geometric means for the S&P Composite stock market index (10.08%) and the Treasury bond rate (4.58%) -- values in brackets being the estimates over a 1926-1990 sample. ***The difference of 5.5% is used as the equity risk premium*** to estimate the cost of equity from the CAPM that, in turn, is used as the discount rate (k) in the Gordon model. As discussed in Sec. 1.1, the difference of the geometric mean is smaller than the difference of the arithmetic means (12.13% - 4.90% = 7.23%). The rationale for this choice is stated as: "where cash flows over a long time horizon are discounted back to the present, the geometric mean provides a better estimate of the risk premium" (p.22). Consistency would appear to require that geometric averages be used to estimate g but, despite recognition of this point, an arithmetic average is often used, e.g., Damodaran (1994, p.68).

A search for the 'best practices' approach to implementing the simplified *DCF* models leads to Damodaran (1994) where the implementation of the Gordon model and other types of *DCF* models reaches a relatively sophisticated stage of evolution.[12] ***Aswath Damodaran*** is a professor at the Stern School of Business at New York University (NYU) specializing in executive education and the author of a number of books along these lines. Given the just-around-the-corner proximity of the NYU business school campus to Wall Street, it is likely than many individuals in the New York financial community have been directly exposed to these ideas and have used, or attempted to use, the models in practical situations. Damodaran (1994) goes carefully over the appropriate procedures for estimating the discount rate, the cash flows and the growth rates. This background is then used to implement the dividend discount model and the free-cash-flow-to-equity discount model. Valuation results are provided for the common stocks of the firm types where the *DCF* model would be most likely to work.

Damodaran's Six DCF Valuation Myths
Damodaran (1994, p.2-4)

To be taken into account when assessing the validity of the *DCF* technique. These myths are:

"Since valuation models are quantitative, valuation is objective"

"A well-researched and well-done valuation is timeless"

"A good valuation provides a precise estimate of value"

"The more quantitative a model, the better the valuation"

"The market is generally wrong"

"The product of a valuation – the value – is what matters, the general process of valuation is not important".

Example 1: Exxon

Damodaran (1994) illustrates the use of ***the DDM for Exxon*** (now XOM), a large stable firm that is not a utility. The following information is provided in the valuation:

Exxon: Valuation date May 1993
Common Stock Price: $65.00
1992 earnings per share: $3.82 (% of earnings paid out as dividends 74%)
1992 Dividend: ($3.82)(.74) = $2.83
Earnings and dividend growth:
6% a year between 1988 and 1992 (expected to grow at same rate in long term)
Beta for Stock: 0.75 *30 year T-Bond rate*: 7%

Cost of equity = 7% + 0.75 x 5.5% = 11.13%
Value of equity per share = 2.83 x 1.06 / (0.1113 - 0.06) = $58.47.

The estimated value and the observed stock price are sufficiently close according to Damodaran, indicating that considerable pricing slippage is expected even when the *DDM* valuation is useful. Damodaran (1994) does not clarify whether Exxon selected more-or-less at random from the available group of 'large stable non-utility' firms, or whether Exxon was selected because the

Gordon growth model produced the most plausible price estimate from a group of such estimates? The choice of Exxon as an example of a large stable firm suitable for application of the Gordon growth model seems misplaced because of the sensitivity of Exxon's earnings to developments in the oil sector.

INSERT Exxon pages
Figure 4.2.b (XOM_10year chart)
Figure 4.2.c (XOM_Feb-08-08.pdf)

Poitras (2005, p.241) provides an updated *DDM* valuation of Exxon, now Exxon-Mobil (XOM), for March 2003. The beta was relatively unchanged at 0.91but the long Treasury bond rate had fallen to 5.375%. Solving for the cost of equity using the 5.5% long run risk premium on the market gives: 10.38% = 5.375% + 0.91(.055). Observing that for 2000-2003 dividends increased from .83 to .88 to .91 cents per share (2.68% dividend yield in 2003), gives a growth rate of g = 4.4%. Dividend growth is used in favour of the more variable earnings growth. Using earnings over the three years, the dividend payout is less than 50%. Evaluating the Gordon growth model estimate of the price of the stock gives: (.91)(1.044)/(.1038 - .044) = 15.89. This does not compare favorably to the observed stock price of $34.37. Raising the growth rate to 5.4% (or lowering the market risk premium by 1/.91%) only raises the price estimate to $19.08. Solving for the growth rate that is consistent with the observed price gives g = 7.7%.

INSERT S&P 500 (SPY) (SPY_1993-2007) Figure 4.2.d

As evidenced in Figures 4.2.b to 4.2.d, the *DDM* did not perform well in predicting the future price movement of XOM. The evidence of substantial overvaluation in March 2003 is brutally denied by the remarkable 237% increase the stock price has achieved by Feb. of 2008. Similarly, the 1994 assessment that the stock was fairly valued failed to miss a doubling of the stock price over a similar future interval. This is unfortunate because it would be difficult to find a more stylized example of a large, stable company. Formed in November 1999 by the merger of Exxon and Mobil, ExxonMobil (XOM) is the world's largest company by revenue (US$404.5 billion for 2007); by market capitalization (US$517.92 billion on July 20, 2007); and the most profitable, posting the largest annual profit by a U.S. company (US$40.6 billion) with a record quarterly net income for Q4 of 2007 ($11.7 billion). Due primarily to the dramatic increase in oil prices from 2003-08 $g > k$ for XOM, illustrating the difficulties of applying the *DDM* to companies where the future cash flows depend substantially on commodity prices – in the case of XOM, the price of crude oil.

Example 2: SBC/ATT

The original Gordon model (Gordon 1962) was developed for valuation of companies in regulated industries. At that time, these types of companies included telephones and public utilities. The regulation of rates provided these companies with stable and relatively predictable cash flows. With this historical application in mind, a more modern application of the Gordon growth model to Southwestern Bell is illustrated in the procedure followed by Damodaran (1994):

Southwestern Bell: Valuation Date: May 1993
Common Stock Price: $78.00
1992 Earnings per Share: $4.33 (% of earnings paid out as dividends: 63%)
1992 Dividend: ($4.33)(.63) = $2.73
Earnings and dividend growth:
6% a year between 1988 and 1992 (expected to grow at the same rate in the long term)
Beta for the stock: 0.95. *30 year T-bond rate*: 7%

Cost of equity = 7% + 0.95 x 5.5% = 12.23%
Value of equity = $2.73 x 1.06 / (0.1223 - 0.06) = $46.45

It is also possible to take the stock price as given, in this case $78.00, and to solve for *g* from the Gordon model as 8.43% (Damodaran 1994, p.103). This is interpreted as the expected growth rate embedded in the current price which is 2.43% higher than the estimated historical growth rate.

Updating the Damodaran (1994) valuation for Southwestern Bell is complicated. The world of regulated utilities that followed the breakup of AT&T in 1984, found one of the 'baby bells', Southwestern Bell Telephone Company, being managed by Southwestern Bell Corporation (SBC). In 1995, SBC decided to change its corporate name to SBC Communications, Inc. and aided by passage of the Telecommunications Act of 1996, proceed to acquire Pacific Telesis Group in 1997, Southern New England Telecommunications in 1998, and Ameritech in 1999. By 2002, SBC had integrated all the operating companies, and "SBC" emerged as a national telecommunications brand. On November 18, 2005, SBC Communications merged with AT&T Corp. and changed its name to AT&T Inc. The period from the breakup of AT&T to the reintegration of SBC and AT&T was marked by remarkable technological and regulatory changes in the telecommunications industry. The stable, regulated hard line business has been replaced by a sometimes fiercely competitive jungle of alternative technologies and sophisticated consumers.

Valuing Foreign Stocks

How does Damodaran use the Gordon growth model to value a foreign stock? As another case of a '***stable large firm***', Damodaran (1994, p.105) selects what was at the time ***the second largest German bank, Dresdner Bank***.

Dresdner Bank: Valuation Date: July 1993
Common Stock Price: 408 DM
1992 Earnings per Share: 34.05 DM (% of earnings paid out as dividends: 47.6%)
1992 Dividend: (34.05)(.476) = 16.21 DM
Earnings and dividend growth:
5% a year between 1983 and 1992 (expected to grow at the same rate in the long term)
Beta for the stock: 0.87. *10 year German bond rate*: 6.42%

Before the cost of equity can be calculated, a number of issues need to be addressed. In particular, the equity risk premium for a German stock cannot be measured relative to a US stock market index.

A German index is required. To this end, Damodaran uses 3.5% as the risk premium for the German DAX stock index over bonds. It follows

Cost of equity = 6.42% + (0.87 x 3.5%) = 9.45%
Value of equity per share = 16.21 DM x 1.05 / (0.0945 - 0.05) = 383.01 DM

Similar to the Exxon and SW Bell estimates, Damodaran obtains a relatively close estimate of the observed stock price for Dresdner using the Gordon growth model. No attention is given to substantive problems such as determining the US$ return or reconciling the need to use different equity premia in an internationally diversified portfolio.

Damodaran's favorable, if relatively limited, application of the discounted dividend model to specific stocks is supported by Sorensen and Williamson (1985) which provides an *ex ante* application of the dividend discount model to 150 stocks in the S&P Composite Index. Valuation using a form of the dividend discount model was done in Dec. 1980 and the stocks were held for two years with the result that stocks identified as 'undervalued' significantly outperformed 'overvalued' stocks. Further ***evidence in support of using the dividend discount model*** to identify undervalued stocks is provided by Haugen (1990) which examines the 1979-1991 performance of a fund that used the dividend discount model to select undervalued stocks. Over the 1979-1991 period, the quintile of stocks judged by the fund to be most undervalued using the dividend discount model outperformed the most overvalued by 1253% to 434%. Damodaran (1994, p.124) concludes: "The dividend discount model outperforms the market over five-year time periods, but there have been individual years when the model has significantly underperformed the market".

This empirical evidence in favour of the dividend discount model can be contrasted with the mountain of ***unfavourable evidence provided by the studies on rational and intrinsic bubbles***. A resolution of the seeming conflict between these disparate results can be found in Williams (1938, p.viii): "the present worth of future dividends, [is] of practical importance to every investor because it is the *critical* value above which he cannot go on buying or holding, without added risk". Studies of bubbles typically track the ability of aggregate dividends to explain aggregate stock prices and find that dividends do a poor job. Advocates of the dividend discount model depend on this result and use fluctuations in the pricing relationship to determine trading opportunities in individual equity securities. The properties of aggregate stock prices during periods of bubble behaviour, which are central to empirical analysis of rational and intrinsic bubbles, are of limited interest to discounted dividend investors who would see most equities as grossly 'over-valued' during these periods.

Ultimately, dividend discount models may perform well, not because the valuations are accurate, but because ***companies that pay stable and increasing dividends include many outstanding companies***. Recognizing that dividends reduce the funds available for investment, it is logical that high dividend payout will be accompanied by weaker future earnings growth than for comparable firms that have lower dividend payout. Yet, cross-sectional studies on the relationship between dividend payout and future earnings growth show that high-dividend-payout companies tend to experience stronger, not weaker, earnings growth in the future. Zhou and Ruland (2006) find the positive association between dividend payout and future earnings growth to be "robust to alternative measures of payout and earnings, sample composition, mean reversion in earnings, the effects of particular industries, time periods, and share repurchase".

4.3 Basic Theory of Interest

The discounted cash flow methods used in equity valuation are based on valuation methods initially developed for fixed income securities. In turn, with few exceptions preferred shares are appropriately valued as fixed income securities. Historically, the fixed income characteristics of common shares were important elements in equity valuation. As such, the basic theory of interest has relevance to the use of discounted cash flow to value equity securities. Recognizing that an adequate treatment of fixed income security valuation lies well outside the boundaries of equity security valuation, some topics of relevance have to be skipped in the tradeoff between brevity and completeness. In particular, no discussion is provided of the duration concept. Poitras (2005, p.213-4) demonstrates the basic result for the duration of a common stock by using the Gordon growth model to solve for the elasticity of stock price with respect to changes in the expected return on the stock. This produces a solution for the duration of common stock comparable to the duration of a perpetuity.

A. Different Possible Definitions

The number of possible definitions for 'the interest rate' is unexpectedly large. In practice, there is not even single interest rate for the same type of fixed income security, such as Treasury bonds, as there are different interest rates associated with the various maturities and coupons. Perhaps the important interest rate in fixed income analysis is the ***yield to maturity.*** This interest rate is a special case of the ***internal rate of return,*** the interest rate that equates the discounted value of the future net cash flows to the value of the initial investment. There are interest rates associated with the convention used to calculate the interest rate, such as the ***simple interest rate, annual percentage rate (APR), effective interest rate*** and ***continuously compounded interest rate.*** Other interest rate definitions include: ***coupon interest rate; zero coupon interest rate***; and, ***current yield***. This list is not exhaustive. In analyzing the term structure of interest rates, the ***spot interest rate (implied zero coupon interest rate)*** and ***implied forward (interest) rate*** are essential definitions. Each of these definitions has relevance to specific valuation problems.

Yield to Maturity

Perhaps the most widely used definition for 'the interest rate' is the yield to maturity. Although the ***yield to maturity*** can be used as the interest rate for a range of fixed income securities, the most important practical application is, arguably, to describe the interest rate for a bond. For a bond valuation problem where the bond pays annual coupons and is being valued on the issue date or a coupon payment date, the annualized yield to maturity (y) is obtained by solving:

$$P_B = \left\{ \sum_{t=1}^{T} \frac{C}{(1 + y)^t} \right\} + \frac{M}{(1 + y)^T}$$

In this formulation, T is the number of annual coupon payments remaining to be paid on the bond, (T = term to maturity in years), C is the annual coupon payment, M is the par value of the bond which is repaid at maturity and P_B is the price of the bond. Because the calculation assumes that future coupon cash flows can be reinvested at the stated y, it follows that y is only a ***promised*** yield to maturity. In other words, reinvestment of coupons at the promised yield is required in order for the bond to actually earn the stated yield if the bond is held to maturity. As such, the yield to maturity is an *ex ante* forecast of the *ex post* ***realized yield.*** When $C = 0$, the bond is referred to as a pure discount or zero coupon bond. If held to maturity, a default and option free zero coupon bond will have the promised yield to maturity equal to the realized yield.

In practice, the formula given for the price of an annual coupon bond is useful for pedagogical purposes. Only a relatively small number of bonds, e.g., Eurobonds, pay annual coupons. Most government issued bonds, such as those issued by the US Treasury, pay coupons semi-annually. As reflected in Tables 1-s and 4-a, the convention in the bond market is to express the coupon as the sum of the coupon payments made in one year. For example, the 11.75% coupon US Treasury bond maturing in November 2014 pays a \$5.875 coupon every six months for a bond with a par value of \$100. Valuing the semi-annual coupon bond on an issue date or coupon payment date, the valuation formula is:

$$P_B = \left\{ \sum_{t=1}^{2T} \frac{C/2}{\left(1 + \frac{y}{2}\right)^t} \right\} + \frac{M}{\left(1 + \frac{y}{2}\right)^{2T}}$$

The sum is now over $2T$ because there are now two payments per year (T is term to maturity measured in years). The convention of dividing the (annualized) yield to maturity creates a situation where the yield on annual coupon bonds is not directly comparable to the (annualized) yield to maturity calculated from this formula. The ***effective rate of interest*** is used to reconcile this difference.

The yield to maturity provides a measure that can be used to compare relative value across different types of bonds. For example, convention in the bond market is to execute trades using prices. Due to the variation in term to maturity and coupon, it is difficult to assess bond value by comparing prices. The yield to maturity provides a method to compare value across bonds. The simple rule of thumb for using yield to identify value is: "All other things equal, choose the bond with the highest yield to maturity". However, if all things are equal, then efficient pricing requires that the yields will be the same. Hence, the yield to maturity can be used as a measure to assess bonds that have one or more features that are different. This leads to the notion of what Fabozzi (1989) refers to as "***traditional yield spread analysis***" where the difference in yield for bonds with different features, e.g., different levels of default risk but the same term to maturity, is used as a measure of relative value.

Effective Interest Rate

In presenting the yield to maturity, it was observed that the convention used to calculate yields for semi-annual coupon bonds resulted in stated yields that were not directly comparable to yields calculated from annual coupon bonds. The ***effective interest rate*** or ***effective yield*** provides the annualized equivalent for a stated interest rate involving compounding at a greater than annual frequency. This problem of having the calculated yield changing with the compounding frequency is not specific to bonds but occurs with all fixed income transactions where compounding occurs. In particular, changing the compounding frequency is one of a number of possible methods of confusing consumers about the actual interest rate that is being charged on loans. In the US, truth in lending laws have codified the use of the effective yield where the concept is referred to as the ***annual percentage rate*** or **APR**. In others words, loan transactions are required to quote an effective yield when stating the lending rates for consumer loans. (In Canada, the APR has a different legal meaning.)

To illustrate the methodology for calculating the effective interest rate, consider the following problem: a 10% coupon government bond (paid semi-annually) currently sells at par. At current market prices, what coupon rate would be required if the coupons were paid annually? To solve this problem, step one requires equating future values for single payments: the future value of one payment received in T years, compounded m times per year at interest rate r produces:

$$FV^m = (1 + (r/m))^{Tm}$$

For annual compounding $m = 1$ at interest rate y this reduces to $FV^A = (1 + y)^T$. Equating the future values $FV^m = FV^A$ produces:

$$(1 + y)^T = (1 + (r/m))^{Tm} \quad \text{--->} \quad y = (1 + (r/m))^m - 1 \quad \text{---->} \quad r = [(1 + y)^{1/m} - 1]\, m$$

For a single cash flow occurring T years ahead, this result provides the method for transforming an interest rate for compounding m times a year to an interest rate compounded annually.

The next step involves converting the future value (FV) to a present value (PV). Present and future value represent the valuation of cash flows at different points in time. To translate a future value at time T to a present value involves discounting the value at the appropriate interest rate:

$$PV^A = FV/(1 + y)^T \qquad PV^{SA} = FV\,/\,(1 + (r/m))^{mT}$$

When the effective yield is used, then equating the future values is the same as equating the present values.

It is now possible to extend these results for single cash flows to coupon bonds. Working by example, consider a 1 year par bond with an 10% coupon. *In all cases, it is assumed that the cash flows from the bond are reinvested at the stated yield to maturity.* A semi-annual bond has cash flows of $C/2 = \$5 = (r/2)M$ which produces a time line of:

	\$5	\$105
t=0	t=6 months	t=1 year

The future value of this stream of cash flows is:

$$FV^S = \left(\frac{r}{2}\,M\right)\left(1 + \frac{r}{2}\right) + M\left(1 + \frac{r}{2}\right) = M\left(1 + 2\frac{r}{2} + \frac{r}{2}\right) = M\left(1 + \frac{r}{2}\right)^2$$

This can be compared with the future value for annual payments of $FV^A = M\,(1 + y)$. It follows that for one year bonds, the effective yield formula holds: $y = (1 + (r/2))^2 - 1$. For the annual coupon bond to be sold at par, the coupon on the bond would have to be 10.25%.

For a two year bond, the analysis is much the same. The time line for the semi-annual coupon bond is:

	\$5	\$5	\$5	\$105
t=0	t=6 months	t=1 year	t=1.5 years	t=2 years

To calculate the future value of the semi-annual cash flows:

$$FV^S = \frac{r}{2} M(1 + \frac{r}{2})^3 + \frac{r}{2} M(1 + \frac{r}{2})^2 + \frac{r}{2} M(1 + \frac{r}{2}) + M(1 + \frac{r}{2}) = M(1 + \frac{r}{2})^4$$

This can be compared with the future value of the annual coupons: $FV^A = M\,(1 + y)^2$. It follows that for two year bonds to have equal prices (from step two):

$$(1 + y)^2 = (1 + (r/2))^4 \ \text{--->}\ y = (1 + (r/2))^2 - 1$$

The effective yield result also holds in this case, and the annual coupon bond would have a coupon of 10.25% when the semi-annual coupon bond has a coupon of 10%, for both bonds to sell at par. By induction, this method for transforming the semi-annually compounded yield to an annually compounded yield extends to bonds with T years to maturity.

Current Yield and Dividend Yield

There are so many different approaches to calculating interest rates that it is not practical to give a detailed account of each method. Though not used much in recent years, prior to the advent of computerized calculations, the current yield was a commonly quoted approximation to the yield to maturity for bonds. In the form of the "dividend yield", the current yield is still the most common measure for valuing preferred stocks. The "current yield" is defined as: Current Yield = CY = (Annual Coupon or Dividend Paid)/(Bond or Stock Price) = C/P_B. The relationship between the current yield and the yield to maturity for a bond is useful to illustrate for pedagogical purposes. When $P_B = M$, the bond sells at par, then $CY = y$. When $P_B > M$, for premium bonds $CY > y$ with the difference increasing as the bond has a greater premium. When $P_B < M$, for discount bonds $CY < y$ with the difference increasing as the bond has a greater discount. For a zero coupon bond, $CY = 0$.

In general:

$$CY = \frac{C}{\sum_{t=1}^{T} \frac{C}{(1 + y)^t} + \frac{M}{(1 + y)^T}} \quad \rightarrow \quad CY\{\sum_{t=0}^{T-1} (1 + y)^t\} + \frac{M}{P_B} = (1 + y)^T$$

Some special cases follow, e.g., for $T = 1$:

$$CY + \frac{M}{P_B} = (1 + y) \quad \rightarrow \quad y = CY + \{\frac{M}{P_B} - 1\}$$

For $T = 2$:

$$CY\ (1 + y) + \frac{M}{P_B} = (1 + y)^2 \quad \rightarrow \quad y = CY + \{\frac{M}{P_B(1 + y)} - 1\}$$

As T increases, the size of the deviation shrinks to the point where, for a perpetuity:

$$CY = \frac{C}{\frac{C}{y}} = y$$

i.e., when T goes to infinity, the current yield equals the yield to maturity.

B. Examples of Fixed Income Valuation Problems

The history of interest rate calculations stretches back centuries, e.g., Poitras (2000, ch.4-5). In this history, the use of worked examples has played an important pedagogical role in illustrating concepts. An early example of the sophistication of such problems can be found in Witt (1613):

> A oweth to B £1200 to be paid in 6 yeares, in 12 equall payments, viz. at the end of each halfe yeare £100. They agree to cleare this debt in 3 yeares, in 6 equall payments, viz. at the end of each halfe yeare, one payment. The Question is, what each payment ought to be, reckoning interest after the rate of 10 per Cent per Ann. and int. upon int.

A conventional solution to this problem can be determined by equating the discounted value of the annuity stream of £100 for 12 half-year periods with the discounted value of £C for 6 half-year periods and solving for C. The exact solution requires recognizing Witt's practice of using $(1 + r)^{T/2}$ instead of the modern convention of $(1 + r/2)^T$ to discount the T period cash flow.

More precisely, the solution can be determined by solving:

$$\frac{100}{(1 + r)^{1/2}} + \frac{100}{(1 + r)} + ... + \frac{100}{(1 + r)^6}$$

$$= \frac{C}{(1 + r)^{1/2}} + \frac{C}{(1 + r)}... + \frac{C}{(1 + r)^3} = 100\ \{1 + (1 + r)^{1/2}\}\ \{\frac{1}{r} - \frac{1}{r\ (1 + r)^6}\}$$

$$= C\ \{1 + (1 + r)^{1/2}\}\ \{\frac{1}{r} - \frac{1}{r\ (1 + r)^3}\}$$

Solving this for $r = .10$ gives the solution stated by Witt of £175.13145 or £175. 2s. 7d. Yet, Witt is able to show that this solution can be obtained as:

$$100 + \frac{100}{(1 + r)^3} = £175.\ 2s.\ 7d.$$

Lewin (1970, p.126) describes the method Witt uses to arrive at this solution as "extremely elegant".

Though no longer used for government financing activities, during the early years of government finance perpetuity issues were commonly used. The perpetuity is still of theoretical interest today. What is a perpetuity? A perpetuity is a security that offers to pay a fixed or variable coupon, at regular intervals, forever (in perpetuity). If the coupon is variable, then it is referred to as a floating rate perpetuity. Almost all perpetuities issued in recent years have been floating rate perpetuities, e.g., the floating rate perpetuities issued by financial institutions in the Euromarkets during the 1980's. The most well known perpetuity is a ***consol***, originally issued by the British government in the 18th and early 19th century, which pays a fixed coupon. This perpetuity was so named because it originated from the consolidation of a number of different types of government debt issues, i.e., consol is a short form for consolidated debt issue. Consol issues traceable back to these early debt operations are still traded on the English exchanges.

The perpetuity is more than a historical curiousity. The pricing formula for this security is of theoretical value, if only to illustrate a geometric series. Consider deriving the pricing formula for such a security when coupons are fixed and paid annually:

$$P^{perp} = \sum_{t=1}^{\infty} \frac{C}{(1 + y)^t} = \frac{C}{(1 + y)} \{1 + \frac{1}{1+y} + \frac{1}{(1 + y)^2} + \frac{1}{(1 + y)^3} .... \}$$

Recall: $$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + ... \quad for\ |x| < 1.$$

$$\therefore \quad P^{perp} = \frac{C}{(1 + y)} \{ \frac{1}{1 - \frac{1}{(1+y)}} \} = \frac{C}{y}$$

It can be immediately verified that, when coupons are fixed and paid quarterly, the perpetuity has the same pricing formula. Because quarterly coupons are paid sooner than annual coupons, this result may seem odd. However, this result can be explained when it is observed that the annual and quarterly coupon perpetuities will sell for different prices. Hence, even though the pricing formula are the same, the yields will not be the same.

Another interesting result occurs for the default free floating rate perpetuity where the coupon is variable (floating) and equal to the current interest rate times the par value of the perpetuity, i.e., $C = y\, M$. It follows that this perpetuity will always sell at par because the coupon will adjust to keep the price of the perpetuity equal to par. This was the idea behind the floating rate perpetuities issued in the Euromarkets during the 1980's. Because financial institutions are regularly coming to the market to reissue short maturity debt, issuers could save on financing costs by offering a security that has a floating coupon. Similarly purchasers could save on commissions and other costs associated with rolling over short term debt issues. However, this analysis depends on the level of default risk staying relatively constant. In the face of actual default risk shocks, market makers were forced to absorb large amounts of these securities at falling prices. The loses incurred led to a collapse of the market for this type of security.

One interesting extension of the fixed coupon perpetuity pricing formula occurs with the pricing of fixed coupon, fixed term annuities. The basic formula for pricing such an annual pay annuity can be stated as:

$$PV_A = \sum_{t=1}^{T} \frac{\$A}{(1 + y)^t} = \$A \left\{ \frac{1}{y} - \frac{1}{y(1 + y)^T} \right\} = \frac{\$A}{y} \left\{ 1 - \frac{1}{(1 + y)^T)} \right\}$$

where *$A* is the annual coupon payment, *y* is the applicable interest rate and *T* is the term over which the annuity payment is received. This general pricing formula appears in numerous guises, such as in consumer loans for automobiles, house mortgages and the like. In these applications, the payment frequency is monthly. By observing that the cash flow stream from a fixed term annuity can be conceived as a perpetual annuity minus a perpetuity that starts at *T+1* (valued at *T*) the perpetuity formula can be used to reexpress this sum as a single closed form expression, i.e.:

$$PV_A = \frac{\$A}{y} - \left[\frac{\$A}{y} \frac{1}{(1 + y)^T} \right] = \frac{\$A}{y} \left[1 - \frac{1}{(1 + y)^T} \right] = \sum_{t=1}^{T} \frac{\$A}{(1 + y)^t}$$

In effect, the price of a fixed coupon annuity with maturity at T is equal to the price of a perpetuity minus the present value of a perpetuity that starts at *T+1* (with price taken at *T*).

In contrast to perpetuities that are something of an oddity in modern fixed income markets, much of the modern focus and discussion of fixed income securities is concerned with bond valuation. Various quirks can arise in bond pricing problems. One useful example arises with using the same yield to maturity to value bonds with different coupon payment frequencies. For example, consider the following problem: a bond offers eight annual coupon payments of $8 and will repay its face value of $100 at the end of eight years. You observe that other similar bonds have yields to maturity of 10%. How much is this bond worth? If the coupons are paid semi-annually, how much is the bond worth? To solve this basic problem, let C = \$8, M = \$100, and the term to maturity (T) be T = 8:

$$P_B^{\,A} = \$8 \sum_{t=1}^{8} \frac{1}{(1.1)^t} + \frac{\$100}{(1.1)^8} = \$8\ (5.335) + \$100\ (.467) = \$89.38$$

$$P_B^{\,SA} = \$4 \sum_{t=1}^{16} \frac{1}{(1.05)^t} + \frac{100}{(1.05)^{16}} = \$4\ (10.84) + \$45.80 = \$89.16$$

where $P_B^{\,A}$ is the price of the annual coupon bond and $P_B^{\,SA}$ is the price of the semi-annual bond.

The impact of increasing the coupon payments can be established with the following two problems:

if these bonds have coupon payments of \$12 annually, how much is the bond worth? If the coupons are paid semi-annually how much is the bond worth? These questions can be solved as: $P_B^{\ A} = \$12\ (5.335) + \$46.70 = \$110.72$ and $P_B^{\ SA} = \$6\ (10.84) + \$45.80 = \$110.84$. To assess the impact of increasing the term to maturity, compare these bond prices with the 8% coupon prices calculated with 10 years to maturity. Do the same for the bonds with the 12% coupon. What do you observe about the relationships between the prices? Solving for the bond prices gives for $C = \$8$, $T = 10$, $y = .10$, $P_B^{\ A} = \$8\ (6.145) + \$38.60 = \$87.76$ and $P_B^{\ SA} = \$4\ (12.46) + \$37.70 = \$87.54$. Similarly, for $C = \$12$, $T = 10$, $y = .10$, $P_B^{\ A} = \$12\ (6.145) + \$38.60 = \$112.34$ and $P_B^{\ SA} = \$6\ (12.46) + \$37.70 = \$112.46$.

This series of bond valuations illustrates the pricing differences that arise for discount and premium bonds. This illustration requires some definitions to be introduced. A '***straight bond***' requires a stream of fixed coupon payments paid at regular intervals plus a 'return of principal' at maturity that involves a payment on the maturity date equal to the stated ***par value*** (M) of the bond. If no coupons are offered the bond is said to be a ***zero coupon*** or ***pure discount*** bond. If the price of the bond $P_B > M$, then the bond is referred to as a ***premium*** bond. This occurs when the annual coupon payment C satisfies: $(C/M) > y$, i.e., the coupon rate exceeds the yield to maturity on the bond. If the price of the bond $P_B < M$, then the bond is referred to as a ***discount*** bond and $(C/M) < y$. If $P_B = M$, the bond sells at its par value, then $(C/M) = y$ and the bond is referred to as a ***par bond***.

In the bond valuation solutions, it can be observed that, for discount bonds with the same yield but different term to maturity, $P_B^{\ 10} < P_B^{\ 8}$. For premium bonds, with the same yield but different term to maturity, $P_B^{\ 10} > P_B^{\ 8}$. Shorter term bonds have higher prices when the bonds sell at a discount and longer term bonds sell at higher prices when the bonds sell at a premium. For discount bonds, with the same term to maturity and yield to maturity: $P_B^{\ \mathrm{SA}} < P_B^{\ \mathrm{A}}$. This result seems counter-intuitive because the semi-annual bond pays the coupon sooner than for the annual bond. For premium bonds, with the same term to maturity and yield to maturity: $P_B^{\ \mathrm{SA}} > P_B^{\ \mathrm{A}}$. All these results are somewhat misleading because with the same C and T a semi-annual coupon bond will always be preferred to an annual coupon bond, because a portion of the cash flows are received sooner. Hence, the semi-annual bond will sell for a higher price, and lower stated yield to maturity.

C. Term Structure of Interest Rates

In the situations involving coupon bonds, the relationship between term to maturity and the yield to maturity for the set of available bonds is referred to as the ***yield curve***. Different yield curves can be identified for different types of bonds, e.g., the corporate bond yield curve or Treasury bond yield curve. As will be discussed below, analysis of 'the yield curve' poses a range of problems. For example, bonds with the same term to maturity may have different coupons and, as a consequence, different yields. This leads to the introduction of the abstract relationship between term to maturity and the spot interest rate (implied zero coupon interest rate) referred to as the ***term structure of interest rates***. These definitions are not always adhered to in various texts and financial newspapers where the terminology 'term structure of interest rates' can be used synonymously with 'the yield curve'. However, the concepts are only equivalent if the yield curve is flat, i.e., the yield to maturity is equal across maturities.

Spot Interest Rate (Implied Zero Coupon Interest Rate)

As with any internal rate of return calculation, the yield to maturity has a number of limitations. One limitation involves applying the same interest rate to discount cash flows occurring at different points in time. This approach would only be valid if yield curves were flat, i.e., yields were the same for all terms to maturity. Casual inspection of real world bond markets reveals that near term cash flows are usually discounted at lower interest rates than longer term cash flows. This limitation leads directly to the concept of the ***spot interest rate*** or, more descriptively, the ***implied zero coupon interest rate*** (implied zero rate). Whereas there is a yield to maturity that can be calculated for every bond, the implied zero coupon interest rate applies to the term to maturity. There is an implied zero coupon interest rate for every fixed income payment date. For example, when the US Treasury issued 30 year bonds, there were at 60 implied zero coupon interest rates that could be calculated, one for each of the 60 coupon payment dates.

To see the connection between the implied zero rate and the yield to maturity, consider the following valuation formulas for bonds with annual coupons:

$$P_B = \left\{ \sum_{t=1}^{T} \frac{C}{(1 + y)^t} \right\} + \frac{M}{(1 + y)^T} = \left\{ \sum_{t=1}^{T} \frac{C}{(1 + z_t)^t} \right\} + \frac{M}{(1 + z_T)^T}$$

Recognizing that the implied zero rate or spot interest rate (z_t) is the interest rate applicable to cash flows occurring at time t, it follows that valuation with implied zero rates values the bond by treating each of the cash flows (coupon payments and return of principal) as zero coupon bonds. The price of the bond is then calculated as the sum of the prices of the zero coupon bonds, valued using the implied zero rate applicable to single cash flows for that term to maturity, i.e., the price of the bond is the sum of the appropriately discounted zero coupon bond prices. For default free (riskless) zero coupon bond prices, the spot interest rate (implied zero rate) is equal to the yield to maturity.

While each bond has an associated yield to maturity, each coupon payment date will have an implied zero rate. To make sense of such an interest rate, it is necessary to abstract from default risk and other features. Hence, the implied zero rates are extracted from the relevant default free bond prices, i.e., the US Treasury debt issues are used to extract the implied zero rates for the US debt market. Though it would be conceptually possible to use the observed interest rates for zero coupon bonds, such as the rates for US Treasury strip securities (STRIPS), for the implied zero rates, this raises a number of difficulties. Sundaresan (2002, p.231) specifically addresses this point: "It should be stressed that strips are not implied zeroes. Strips are traded securities directly subject to demand and supply. Implied zeroes are estimated pure discount functions derived from the prices of coupon-paying Treasury securities." As such, implied zeroes provide a benchmark for assessing the relative richness or cheapness of Treasury securities. Fabozzi (1989, p.192-3) identifies three reasons why stripped Treasury securities are not an adequate substitute for implied zeros: problems of liquidity in the Treasury strips market; maturity preferences in specific segments of the Treasury strip market may cause mispricing of certain maturities; and, differences in the tax treatment of stripped Treasuries and coupon Treasury bonds. Sundaresan (2002, p.237-241), Mason et al. (1995, p.48-55) and others demonstrate that, while not dramatically different, US Treasury strip rates do differ empirically from implied zero rates calculated from US Treasury coupon bonds.[13]

While each bond has an associated yield to maturity, each coupon payment date will have an associated implied zero coupon interest rate. As such, the method for calculating the implied zero rate differs from the method for calculating the yield to maturity. Mathematically, the yield to maturity calculation for a bond with T coupon payments involves one equation with one unknown. A bond with T coupon payments would have T unknown spot rates. This requires T equations to determine the T unknowns. Because the market used to calculate implied zero coupon rates is the US Treasury coupon bond market, a ***bootstrap*** technique is needed to extract the spot interest rates from the observed coupon yields. A bootstrap is the name given to a generic algorithm that uses an stepwise solution procedure to arrive at the solution for a number of unknowns. In particular, the first step solves for the first unknown using one equation with one unknown. This solution is then used to solve a second equation that is specified with the first unknown (now solved) and the second unknown. These two solutions are then used to solve the third equation that is specified with the first two unknowns (now solved) and the third unknown. This procedure continues until all the unknowns are solved.

Calculation of Implied Zero Coupon Interest Rates

The ***term structure of interest rates*** is concerned with the empirical relationship between term to maturity and the implied zero coupon rates calculated from the default free coupon bond market.[14] Because implied zero rates are not directly observed, it is necessary to estimate these variables from observed coupon bond prices. This involves bootstrapping spot interest rtes from semi-annual coupon bond prices. The bootstrapping technique, e.g., Fabozzi (2000); Sundaresan (2002), is an iterative process for calculating implied zero coupon interest rates (spot interest rates) from observed coupon bond rates. The process requires the observed yields for coupon bonds of each relevant term to maturity along the yield curve. In practice, spot rates would typically be extracted from the yield curve for federal government bonds, Treasury bonds in the US or Government of Canada bonds in Canada. Because these types of bonds pay semi-annual coupons, precision requires that the bootstrap be executed at semi-annual intervals.

For purposes of illustrating the bootstrap technique, assume that the relevant bonds are sold at par and pay coupons semi-annually. Further assume that the observed six month yield is 8.87%. For the US implied zero curve, this rate would be obtained by taking the observed six month Treasury bill discount rate and converting to a true yield basis. Because Tbills do not pay coupons this means that the quoted yield is for a six month zero coupon bond. If the observed yield on a one year semi-annual coupon bond is assumed to be 9.04, for a $100 par value bond this implies a semi-annual coupon payment of 4.52. Given this, the iteration for solving a sequence of implied zero coupon rates begins by discounting the first semi-annual coupon payment at the six month, zero coupon rate and solving for the implied one year zero coupon rate. For a bond sold at par this requires that:

$$100 = \frac{4.52}{1 + \frac{.0887}{2}} + \frac{104.52}{(1 + \frac{z_1}{2})^2}$$

where z_1 is the implied one year zero coupon rate, which can be calculated as 0.090438.

Having solved for z_1, the next step in the iteration involves using z_1 to solve the implied zero coupon rate, $z_{1.5}$, using a 1.5 year par coupon bond. If the observed rate on 1.5 year coupon bonds is 9.155, then this implies a semi-annual coupon payment on a \$100 par bond of 4.5775. This leads to:

$$100 = \frac{4.5775}{1 + \frac{0.0887}{2}} + \frac{4.5775}{(1 + \frac{z_1}{2})^2} + \frac{104.5775}{(1 + \frac{z_{1.5}}{2})^3}$$

Substituting the value for z_1 determined previously and solving gives $z_{1.5} = 0.091629$. The next step in the iteration involves solving for z_2. Taking the observed two year yield to be 9.2% produces:

$$100 = \frac{4.6}{1 + \frac{z_{.5}}{2}} + \frac{4.6}{(1 + \frac{z_1}{2})^2} + \frac{4.6}{(1 + \frac{z_{1.5}}{2})^3} + \frac{104.6}{(1 + \frac{z_2}{2})^4}$$

This formula can be used to solve for z_2, using the previously computed values for z_1 and $z_{1.5}$. This iterative process continues until the zero coupon rate for the desired term to maturity is calculated. The relevant zero coupon rate can be used to do calculations involving implied zero rates, e.g., solving for implied forward interest rates.

In certain cases, exactly precise implied zeros are not required. If this is the case, then it is possible to achieve computational simplifications by proceeding under a number of assumptions. In particular it is possible to reduce the number of computations by a factor of two by assuming that the observed government bond prices are for annual coupon bonds. Another simplification can be achieved by taking the nearest available bond instead of estimating a par bond yield curve. To see how this simplified calculation process works, consider the example in Poitras (2002, p.250) that involves solving for the Canadian spot interest rates from quotes obtained from the *Globe and Mail* for Aug. 28, 1994. This involves picking the following bills/bonds from the available maturities:

1 year tbill $z_1 = .0720$

6.5% 1 Aug 1996 $P_2 = 97.505$ ($y_2 = .07887$)
7.5% 1 Jul 1997 $P_3 = 98.225$ ($y_3 = .08197$)
6.5% 1 Sep 1998 $P_4 = 93.350$ ($y_4 = .08474$)
7.75 1 Sep 1999 $P_5 = 96.800$ ($y_5 = .08542$)

These bonds were selected because they were closest to the required maturity dates.

Two possible bootstrap solution techniques are available: the direct approach and the par bond approach. The direct approach involves using the observed price and coupon to solve for the spot interest rate. Solving for z_2:

$$97.505 = 6.5/(1 + z_1) + 106.5/(1 + z_2)^2$$

Using this method, $z_2 = .0792$. For the par bond approach, use the result that when the stated yield to maturity equals the *C/M* then the bond sells at par:

$$100 = 7.887/(1 + z_1) \; + \; 107.887/(1 + z_2)^2$$

Using this method $z_2 = .0791428$. The difference of .6 of a basis point is due to a combination of the assumption that the bond pays annual coupons, i.e., a semi-annual yield is used as an annual yield, and to the difference between the actual maturity date (Aug.1) and the required maturity date (Aug.28). Calculating the spot rates out to 5 years, it is evident that the differences involved are generally small:

Par bond: $z_3 = .082423 \quad z_4 = .0854697 \quad z_5 = .0861374$
Price/Coupon: $z_3 = .08232 \quad z_4 = .08595 \quad z_5 = .08630$

Even for the five year implied zero, the difference is only 1.7 basis points. Observe that when the yield curve slopes up, as in this case, the implied zero curve will be above the yield curve. Similarly, when the yield curve slopes down, the implied zero curve will be below the yield curve. When the yield curve is flat, then both curves will be equal. This relationship follows mathematically from observing that the yield to maturity acts as a form of geometric average of the spot rates.

Spot Rates and Implied Forward Interest Rates

The ***implied forward interest rate*** (implied forward rate) has a number of possible uses in theoretical modeling. This interest rate concept is an extension of the ***breakeven interest rate*** associated with comparing a rollover investment strategy with a buy and hold strategy. To see this, consider the following comparison involving a two period, two portfolio model where z_i is the zero coupon yield on a bond maturing *i* periods from now, $z_{i,j}$ is the (*i-j*) period interest rate starting at $t=i$ and maturing at $t=j$, e.g., $z_{1,2}$ is the one period interest rate that starts at t=1 and ends at t=2:

Portfolio A: Buy and hold a 2 year zero coupon bond. If the initial investment is \$1 the return at the end of 2 years is $(1 + z_2)^2$

Portfolio B: Buy and mature a 1 year zero coupon and use the proceeds to purchase another 1 year zero coupon bond, 1 year in the future. If the initial investment in this portfolio is \$1 then the expected return at end of year two is: $(1 + z_1)(1 + E[z_{1,2}])$

Portfolio A is referred to as the ***buy and hold*** portfolio and Portfolio B as the ***rollover*** portfolio.
To derive the ***breakeven interest rate***, assume that the expected returns on the two portfolios are equal, then:

$$(1 + z_2)^2 = (1 + z_1)(1 + E[z_{1,2}])$$

It follows that the breakeven expected interest rate can be calculated as:

$$(1 + E[z_{1,2}]) = \frac{(1 + z_2)^2}{(1 + z_1)}$$

Using this result, the correspondence between the breakeven interest rate and the definition of the implied forward rate $f_{1,2}$ follows appropriately: $f_{1,2} = E[z_{1,2}]$. (The notation for the implied forward rate differs across the various textbooks on the subject, i.e., there is no generally accepted notational convention.)

The use of $\{z_i\}$ to define $\{f_{i,j}\}$ is intentional. In the absence of representative zero coupon interest rates, implied forward rates are calculated using spot interest rates. The extension of the definition of an implied forward rate from the two period case to the n period case follows appropriately. For example, consider the one year implied forward interest rate that will apply from period t to $t+1$. This is given by:

$$(1 + f_{t,t+1}) = \frac{(1 + z_{t+1})^{t+1}}{(1 + z_t)^t}$$

by

It is also possible to define implied forward rates applying to the interest rates longer than one year. For example, the five year implied forward interest rate between t=5 and t=10 can be calculated using the 10 and five year spot rates:

$$(1 + f_{5,10}) = \sqrt[5]{\frac{(1 + z_{10})^{10}}{(1 + z_5)^5}}$$

Similarly the implied forward rate for a three year zero coupon bond that starts at t=2 and matures at t=5, using the two and five year spot rates, is specified:

$$(1 + f_{2,5}) = \sqrt[3]{\frac{(1 + z_5)^5}{(1 + z_2)^2}}$$

Given this, the general formula for the implied forward interest rate is specified:

$$(1 + f_{t,t+k}) = \sqrt[k]{\frac{(1 + z_{t+k})^{t+k}}{(1 + z_t)^t}}$$

It follows that an observed yield curve for, say, n maturities will produce $(n-1) + (n-2) + \ldots + 1$ implied forward rates.

NOTES

1. The stated solution is: for Piero, 138 ducats, 21 grossi, 11 pizoli and remainder; for Polo, 248 ducats, 0 grossi, 13 pizoli and remainder; and, for Zuanne, 176 ducats, 2 grossi, 7 pizoli and remainder. The *Treviso* proceeds to check the solution, so that 'no one has been cheated', by adding together the shares to verify that the total is 563 grossi.

2. The third problem is a more complicated variation of the second: 'Three men, Tomasso, Domenego, and Nicolo, entered into partnership. Tomasso put in 760 ducats on the first day of January, 1472, and on the first day of April took out 200 ducats. Domenego put in 616 ducats on the first day of February, 1472, and on the first day of June took out 96 ducats. Nicolo put in 892 ducats on the first day of February, 1472, and on the first day of March took out 252 ducats. And on the first day of January, 1475, they found that they had gained 3168 ducats, 13 grossi and 1/2. Required is the share of each, so that one shall be cheated.' The solution procedure is an extension of the rule-of-three procedure used to solve problem 2. However, due to crediting Nicolo with three months full investment instead of only one month, 'the solution stated does not satisfy the given conditions of the problem' (Swetz 1987, p.147). Ignoring the remainders, the solution is given for Tomasso as, 1052 ducats 11 grossi and 8 pizoli, for Domenego, 942 ducats 3 grossi and 21 pizoli, and for Nicolo, 1173 ducats 22 grossi and 17 pizoli.

3. A more precise specification of "such calculations" is needed to determine the precise meaning of the statement. Though primary sources for the 15th century are scarce, if profit was paid on profit then such arrangements would be conducted outside the glare of public scrutiny for fear of ecclesiastic sanction.

4. The requirement introduced by the NYSE in the1890's that traded firms provide financial statements was an early initial impetus to providing investors with accurate financial information, permitting the calculation of crude earnings numbers. While the securities laws of 1933 and 1934 were revolutionary, prior to this time US investors still received considerably better accounting information than investors in other jurisdictions. The situation in the UK was decidedly murkier. It was not until 1976 that turnover numbers had to be fully disclosed (Toms and Wilson 2003). Until the Companies Act was passed in 1948, balance sheet information was clouded by the absence of a requirement to provide consolidated accounts (Rutterford 2004).

5. Various motivations can be provided for bubble behavior. Following Flood and Garber (1980), an explosive bubble is generated by the need for prices to increase at an increasing rate in order to compensate new entrants to the market for the ever increasing risk of a price collapse that will eventually occur at some later date in the future.

6. Williams (1938, p.6) states: "investment value [is] the present worth of the future dividends in the case of a common stock, or the future coupon and principal in the case of a bond."

7. The convention in accounting is to make a distinction between valuation methods that involve 'cash flows' and 'accruals'. Using this distinction, discounted cash flow (*DCF*) techniques refer ***only*** to valuation methods that discount 'cash flows' such as free cash flow or dividends. Valuation methods such as the residual income method, which use accrual numbers, are not considered to be

DCF techniques, e.g., Penman and Sougiannis (1998). While this distinction is useful in accounting studies, it is terminological overkill when the primary objective is discussing the valuation of equity securities. The operative cash flow in the residual income model, i.e., accounting earnings/net income, can be viewed either as a proxy for a cash flow or taken to be an accounting cash flow. In both cases, a cash flow is being discounted and the residual income model can be viewed as a *DCF* technique. This interpretation is used in what follows.

8. Similar to the discounted dividend model, there is a transversality condition on the convergence of book value in the limit that needs to be satisfied for the progressive substitution to have a unique solution. Jiang and Lee (2005) provide more detail on the derivation.

9. Elements of the residual income model can be found in early work by Preinreich (1938) and Edwards and Bell (1961). More recently, Peasnell (1981, 1982) made useful contributions. The more recent popularity of the residual income model is primarily due to its formalization by Ohlson (1991, 1995) and Feltham and Ohlson (1995) (see also Lee 1999).

10. Common law countries include the important equity markets of: Australia, Canada, Hong Kong, Malaysia, Singapore, South Africa, UK and United States. The civil law countries include the important markets of: France, Germany, Italy, Japan, Spain, Sweden and Switzerland.

11. This is not to imply that the roots of the equity premium originate with Mehra and Prescott (1985). Such views can be found in Smith (1925) and, as demonstrated by Dimand (2007), were of interest of Irving Fisher.

12. Damodaran (1994) is of interest because of the intensive examination of the discounted dividend approach to *DCF* valuation. As such, Damodaran differs from other sources in the depth of coverage. Useful textbook level overviews of the different approaches to *DCF* analysis are available in a number of sources, e.g., Palepu et al. (2000) and Weston et al. (2001).

13. For the specific dates examined, both these sources report that implied zero and strip rates are approximately equal at short maturities with implied zeros being slightly below strip rates at intermediate maturities and strip rates being well below the implied zero rate at long maturities.

14. The ‘term structure of interest rates’ terminology can also be used to refer to implied zero coupon rates derived from bonds with default risk, e.g., Bierwag et al. (1992) refer to the duration of bonds from different term structures where the term structures differ due to default risk. However, where further qualification is not given, the term structure of interest rates refers to default free securities.