Midterm 18-3, Question II.1.a

Annualized duration for a Semi-annual Zero Coupon Bond

a) Simple solution (set Par value (M) = 1 and differentiate w.r.t the semi-annual rate)

$$P[y] = \frac{1}{(1 + \frac{y}{2})^{2T}} = (1 + \frac{y}{2})^{-2T}$$

$$\frac{dP}{d(1+\frac{y}{2})} = -\frac{2T}{(1 + \frac{y}{2})^{(2T + 1)}} = -2T\ (1 + \frac{y}{2})^{-(2T + 1)}$$

$$-(1 + \frac{y}{2})\frac{dP}{d(1 + \frac{y}{2})} = \frac{2T}{(1 + \frac{y}{2})^{(2T)}} \qquad -\frac{(1 + \frac{y}{2})}{P}\frac{dP}{d(1 + \frac{y}{2})} = 2T$$

The solution $2T$ is in ½ years. To get the annualized value divide by two, i.e., the annualized duration is T. For example, if the zero coupon bond has 10 years to maturity, then the simple solution duration will be 20 (in ½ years) with the annualized value being 20/2 = 10.

b) Differentiate with respect to the annual rate (again M = 1).

$$P[y] = \frac{1}{(1 + \frac{y}{2})^{2T}} = u^{-2T} \qquad where\ u = (1 + \frac{y}{2})$$

$$\frac{dP}{d(1 + y)} = \frac{dP}{dy} = \frac{dP}{du}\frac{du}{dy} \qquad \frac{dP}{du} = -(2T)\ u^{-(2T + 1)} \qquad \frac{du}{dP} = \frac{1}{2}$$

$$-(1 + y)\frac{dP}{d(1 + y)} = (1 + y)\frac{T}{(1 + \frac{y}{2})^{(2T + 1)}} \qquad -\frac{(1 + y)}{P}\frac{dP}{d(1 + y)} = \frac{1 + y}{1 + \frac{y}{2}}\ T$$

a) Annualized Duration for a semi-annual Term Annuity (set $C = 1$)

$$P[y] = \frac{1}{\frac{y}{2}} - \frac{1}{\frac{y}{2}(1 + \frac{y}{2})^{2T}} = \sum_{t=1}^{2T} \frac{1}{(1 + \frac{y}{2})^{t}}$$

Differentiating w.r.t. to (1 + y/2) gives (to see this set y/2 to x and diff wrt x):

$$-\frac{(1+\frac{y}{2})}{P} \frac{dP}{(1 + \frac{y}{2})} = \frac{1 + \frac{y}{2}}{\frac{y}{2}} - \frac{2T}{(1 + \frac{y}{2})^{2T} - 1}$$

Evaluating this duration for same values as for the annualized case given previously $T = 30$ and $y = .05$ and the formula value is 23.3518 dividing by 2 gives 11.6759 which can be compared with 11.9691 when the annually compounded interest rate is used.

b) Differentiate w.r.t. the annual interest rate:

$$\frac{dP}{dy} = -\left[\frac{2}{y^2} - \frac{2}{y^2 (1 + \frac{y}{2})^{2T}} - \frac{2T}{y(1 + \frac{y}{2})^{2T+1}}\right]$$

$$-\frac{1 + y}{P} \frac{dP}{d(1 + y)} = \frac{1 + y}{y} - \left[\frac{(1 + y)}{(1 + \frac{y}{2})} \frac{T}{\left(1 + \frac{y}{2}\right)^{2T} - 1}\right]$$

Evaluating this result for $T = 30$ and $y = .05$ gives 11.96 (see above).