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'How's the Stock Market Doing?' Using Absence of Arbitrage to Measure Stock Market Performance

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ABSTRACT

This paper provides a methodology for measuring stock market performance based on the restrictions provided by absence of arbitrage in security prices. Under the null hypothesis that the cumulative dividend-price process follows a geometric Brownian motion, a closed form related to the intertemporal marginal rate of substitution is derived and empirically evaluated for the S&P 500 stock index.. The stock market performance measure incorporates the level of risk adjustment required to compare the value of the stock index at the starting point with the interest rate deflated value at any given point in the time series. The paper provides empirical tests for the performance measure derived from the martingale property of arbitrage free price processes.

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‘How’s the Stock Market Doing?’
Using Absence of Arbitrage to
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The problem of measuring stock market performance has long perplexed Finance academics and practitioners. This problem is reflected in questions such as: ‘How’s the stock market doing?’ or ‘Is the current level of the S&P 500 stock index over or under valued relative to the long term trend?’ Because the process of valuation requires a notion of market equilibrium, a valid answer to such questions requires the risk aversion properties of stock investors to be specified. Given the level of interest rates and the mean and variance of the long term trend in stock prices, changes in market valuation reflect aggregate changes in the risk preferences of stock investors. Adapting theoretical results from Heaney and Poitras (1994), this paper exploits absence of arbitrage conditions to develop a measure of stock market performance based on these changing risk aversion preferences. The performance measure has the appealing property that it is specified as a closed form and can be implemented using the estimated drift and volatility of an appropriate specified state variable process. The properties of the performance measure are illustrated using time series plots. Results of empirical tests for the martingale property of the performance measure are provided. These tests indicate that the null hypothesis of martingale behavior is rejected for the full sample and relevant subsamples. The rejection of the null hypothesis is consistent with the conclusion that the stock market did not adhere to absence of arbitrage over the sample period.

The paper is composed of a theoretical and an empirical part. The first part of the paper provides a theoretical justification of the empirical procedures employed in the second part. Section I provides a brief review of the literature and an introduction to the general problem being examined. Section II states the relevant assumptions and provides two Propositions, one concerning arbitrage free pricing and another stating the differential restrictions that are exploited to derive the closed form for the performance measure. Section III develops a closed form for the performance measure associated with a geometric Brownian motion for the cumulative dividend-price process. Other examples for performance measures associated with alternative specifications of the wealth process are also provided. The second part of the paper, containing the empirical results, has two sections. Section IV provides parameter estimates required to implement the stock market performance measure and to construct time series plots. Section V provides empirical tests for the martingale hypothesis, estimated for the full 1960-2001 sample and some relevant subsamples. The paper concludes with Section VI which provides a summary of the results in the paper.

I. The Previous Literature

The performance measure developed in this paper has a number of roots. The theoretical results are based on the absence of arbitrage approach, initially suggested by Harrison and Kreps (1979) that introduced the concept of the equivalent martingale measure. As demonstrated by Heaney and Poitras (1994), it is possible to make a connection between the absence of arbitrage approach and notions introduced by Brennan (1979), Bick (1990) and He and Leland (1993) about the properties of stochastic stock price processes derived from representative investor models, e.g., Brennan (1979) and others have demonstrated that assuming the stock price process follows geometric Brownian motion requires a market equilibrium using a representative investor with a constant relative risk aversion utility function.[1] If an initial empirical assumption is made about the underlying wealth process, Heaney and Poitras (1994) demonstrated that it is possible to recover a closed form for the inter-temporal marginal rate of substitution. However, while the representative investor approach relies on a wealth process and a consumption process as arguments in the utility function, the absence of arbitrage approach permits a wider possible range of state variable processes. In particular, the approach in Section II uses equivalent martingale theory where the state variable is an interest-rate-deflated cumulative dividend-price process. In the absence of interest rate adjustment and using aggregate wealth as the state variable, this approach is related to the stochastic discount factor used in empirical tests of asset pricing models, e.g., Brav et al. (2002).

The theoretical approach taken in this paper can also be compared with the theory of rational stock pricing. This theory has a long history that includes Shiller (1981), Marsh and Merton (1986), Campbell and Shiller (1987), Evans (1991), Campbell and Kyle (1993), Sarno and Taylor (1999). These empirical studies all invoke some method to detrend price and dividend data, usually to achieve the statistical property of covariance stationarity that, in turn, is exploited to conduct hypothesis tests. It is well known that the method selected to detrend the data contains implicit equilibrium restrictions. Recognizing that absence of arbitrage is a fundamental requirement for security pricing models, it follows that it is desirable for detrending procedures to satisfy restrictions required for absence of arbitrage. As demonstrated in Section II, using absence of arbitrage restrictions for detrending is motivated by the theoretical result that, after deflating the stock price and dividend series by the cumulative interest rate process, the product of the 'intertemporal marginal rate of substitution' and the security price follows a martingale. For empirical testing purposes, exploiting the martingale property in this fashion involves a different econometric approach than that followed by conventional studies which exploit only the properties of a covariance stationary process obtained with an empirically based detrending process, e.g., West (1987), Evans (1991). More precisely, even though first differences of a martingale process do satisfy the essential requirements for a covariance stationary process, the arbitrage free detrending process imposes further equilibrium restrictions which involve specifying a fully parametric security pricing model. This produces a maximum likelihood estimation problem, where a specific hypothesis is tested parametrically. In contrast, conventional studies do not typically make such strong assumptions about the theoretical model generating asset prices.

There are many potential methods of detrending prices and dividends to achieve covariance stationarity. The presence of an equivalent martingale measure dictates that there are certain methods of detrending which both satisfy covariance stationarity and are consistent with absence of arbitrage opportunities in security prices. With this in mind, this paper investigates the following types of questions: What are the properties of arbitrage free detrenders? How restrictive are the equilibrium conditions imposed by absence of arbitrage? What are the limitations of using specific detrenders? The notion that the detrending procedure selected can impact the results of empirical tests is not new., e.g., Grossman and Shiller (1981), Evans (1991), Diba and Grossman (1988b). Despite this recognition of the potential importance of the detrending procedure selected, a number of somewhat different procedures have been employed in, say, studies of stock price rationality. A recent example is Campbell and Kyle (1993) which uses the Standard and Poor's Composite stock price index and the associated dividend series both detrended, initially, by the producer price index to get a "real stock price" and "real dividend". These series are then further detrended by "the mean dividend growth rate over the sample". The resulting series are, under the Campbell and Kyle method, required to be I(1) processes. With a number of minor improvements, this is the detrending procedure followed in Campbell and Shiller (1987). The failures of the less sophisticated detrending procedures used in early studies, e.g., Shiller (1981), contributed significantly to the "econometric difficulties" addressed in Campbell and Kyle (1993).

It is possible to develop a correspondence between the detrending process used in this paper and the specification of the indirect utility function for a representative investor. This connection has also been recognized in previous studies. Diba and Grossman (1988a) extend Lucas (1979) to demonstrate theoretically that the marginal utility of consumption will impact the determination of rational stock prices. Olivier (2000) is a recent theoretical example of this approach. At least since Scott (1989), the development of empirical procedures for estimating the marginal rate of substitution in capital asset pricing models have been of interest. Campbell et al. (1997, ch.8) discusses more recent studies along this line, with Otrok et al. (2002) being a recent empirical study. These studies almost always employ a utility function defined using the aggregate consumption or wealth process, leading to deflation of nominal prices and dividends by a consumption price index. Empirical problems associated with the use of consumption in asset return models has recently been recognized, e.g., Constantinides (2002). In contrast, the procedure employed in this paper deflates security prices and dividends by the cumulative interest rate process before detrending the cumulative price-dividend process by a function that accounts for the risk aversion properties associated with absence of arbitrage. This implies that specification of the detrending procedure requires some equilibrium assumption about risk aversion properties. As noted, following Heaney and Poitras (1994) risk aversion properties can be modelled by assuming a specific stochastic process for the state variables.

II. Absence of Arbitrage and Rational Security Valuation

The stock market performance measure examined in this paper has an intuitive foundation derivable from rational security valuation problems of the general form:

$$p(t) \;=\; V(t,T)\;p(T) \;+\; \int_t^T V(t,u)\;d(u)\;du \tag{1}$$

where $p(t)$ is the nominal security price observed at time t, $d(t)$ is the instantaneous nominal dividend or coupon paid at t, and T is the terminal or maturity date for the valuation problem ($T \geq t \geq 0$). Equation (1) is often referred to as a "no arbitrage" condition, e.g., Blanchard and Watson (1982) though the connection with the 'absence of arbitrage' associated with the equivalent martingale measure is more complicated.[2] The set of valuation operators $\{V(t,T)\}$ will, in general, involve both discounting and expected value operations. Two conditions almost always imposed on the $\{V(t,T)\}$ are:

$$\lim_{T \to t} V(t,T) \;=\; I \qquad\qquad \lim_{T \to \infty} V(t,T) \;=\; 0$$

The first condition ensures continuity and consistent pricing. The second condition ensures pricing convergence, e.g., Craine (1993).[3] It is well known that the convergence of the operator as $T \to \infty$ is required for the satisfaction of a transversality condition that, in turn, is needed to ensure that the difference equation identified by (1) will not have an infinite number of solutions. The basic notion of (1) is that, looking forward from an initial starting point at t to the end of the investment horizon at T, the return on a stock investment will be a combination of the cumulative dividends paid on the stock and the capital gain or loss due to price appreciation.

The importance of the transversality condition can be seen by interpreting (1) as a continuous time version of a price-dividend (stock) valuation model, where $p(T)$ is the anticipated nominal stock price at time T and $d(u)$ is the continuous dividend paid over $u \in (t,T]$. Progressive substitution for $p(T)$ produces the infinite horizon, discounted-dividend model. This model forms the basis of empirical tests for rational bubbles. Theoretically, a rational bubble is specified by observing that the security price $p(t)$ can be modelled as the sum of two components, a market fundamentals component $p_F(t)$ and a rational bubbles component $B(t)$, e.g., Evans (1991): $p(t) = p_F(t) + B(t)$, where $p_F(t)$ is associated with the infinite sum of the discounted value of expected future dividends and $B(t)$ can be any random variable that satisfies $B(t) = V(t,T)\ B(T)$. Because $p(t)$ can incorporate both fundamental and bubble information, progressive substitution for $p(T)$ produces restrictions on the bubble component. If the transversality condition:

$$\lim_{T \to \infty} V(t,T)\ p(T) = 0$$

is satisfied, then a rational bubble is ruled out, $B(t) = 0$. The current price will be determined solely by $p_F(t)$, the discounted value of expected future dividends.

To see this, consider the progressive substitution for $V(t)\ p(t)$ in (1) required for the discounted dividend model to hold. In the general case, this involves applying ***iterated expectations*** to expected future prices to derive a general solution of the form:

$$p(0) = \int_0^\infty V(0,u)\ d(u)\ du + B(0) = p_F(0) + B(0) \qquad (2)$$

where:

$$B(0) = \lim_{T \to \infty} V(0,T)\ B(T)$$

In other words, the market price of a security can deviate from its fundamental value, $p_F(t)$, without violating absence of arbitrage, providing the expected value of the rational bubble increases at a faster rate than that specified by the $\{V(t)\}$. For a rational bubble, $B(0) \neq 0$ and the transversality condition is not satisfied. The intuition behind the rational bubble is that there is a component of the price, the rational bubble, which is not related to the discounted dividends. In order to still be present in the limit, the bubble component is growing at a faster rate than the discounting process.[4]

The connection of the rational bubble model with the 'absence of arbitrage' model associated with the equivalent martingale measure can be made by observing that the valuation operators $\{V(t,T)\}$ can be decomposed into two parts, one associated with the time value of money and the other with the systematic risk of the security being priced. After removing the interest rate component from the security price and dividend processes, further adjustment for the risk associated with the state variable will produce a martingale process under the null hypothesis of absence of arbitrage. This result can be motivated theoretically by initially assuming that the cumulative dividend-price process is the only state variable. Given this, the first step in the process is to deflate the stock price and dividend processes by the cumulative (riskless) interest rate process. This deflation procedure is accomplished, for example, by choosing $t=0$ as the starting point for the series; price and dividend observations at $t=1$ are divided by $(1 + r(1))$; $t=2$ price and dividend observations are divided by $(1 + r(1))(1 + r(2))$ and so on; where $r(t)$ is the (riskless) interest rate for period t. A normalization condition that the interest rate process equal one at the pricing decision date, $t=0$, is imposed together with the assumption that the discounting process is strictly positive. The resulting stock price and dividend series are denoted as $\{S(t)\}$ and $\{D(t)\}$ to distinguish these series from the nominal un-discounted series $\{p(t)\}$ and $\{d(t)\}$ used in the rational bubble model.[5]

Considerable effort has been given to establishing that the presence of an equivalent martingale measure is sufficient to ensure absence of arbitrage in security prices. Arbitrage free pricing provides conditions such that, for $\{S(t)\}$ and $\{D(t)\}$, the interest-rate deflated price and dividend processes, the cumulative dividend-price process:

$$S(t) + \int_0^t D(u)\, du$$

is a martingale under the equivalent martingale measure, Q.[6] The precise role played by the risk detrending function, $Z(t)$, in achieving this result can be formalized in the following adaption of Girsanov's theorem (see also Ingersoll 1987, p.220-3):

Proposition 1: **Arbitrage Free Shadow Pricing**

Assuming an equivalent martingale measure Q exists, then there exists a positive martingale $\{Z(t)\}$ on the probability space $(\Omega,\mathbf{F},\{F(t)\},P)$, such that for the cumulative dividend-price process, the transformed process:

$$Z(t)\ S(t) + \int_0^t Z(u)\ D(u)\ du$$

is a martingale on $(\Omega,\mathbf{F},\{F(t)\},P)$ such that for $t,k \geq 0$:

$$Z(t)\ S(t) = E^{P}\left[\int_t^{t+k} Z(u)D(u)\ du + Z(t+k)\ S(t+k)\ \middle|\ F(t)\right] \tag{3}$$

To model current prices, it is appropriate to let $t = 0$ and observe that $Z(0) = 1$.

The significance of Proposition 1 can be clarified by comparing the $\{Z(t)\}$ with the valuation operators $\{V(t,T)\}$ arising from (1) and (2). Proposition 1 reveals that $\{Z(t)\}$ roughly corresponds to the systematic risk component of $\{V(t,T)\}$ that remains after adjusting for the interest rate deflating of S and D. There is a pedagogical connection of $\{Z(t)\}$ with the inter-temporal marginal rate of substitution arising in representative investor models. However, those models work with wealth or consumption as the state variable while Proposition 1 uses the cumulative dividend-price process as the state variable. The equivalent martingale measure provides the essential transformation to permit the expectation in (3) to be used for empirical testing purposes, i.e., the expectation is taken with respect to the empirical P measure. Following Heaney and Poitras (1994), this permits the theoretical assumptions made about the diffusions associated with the state variable processes to provide a nested null hypothesis under which a specific $\{Z(t)\}$ is the appropriate, arbitrage free detrender for prices and dividends. It is this detrender function $\{Z(t)\}$ that is used as the stock market performance measure. However, in order to be of practical value, a method is required to derive a specific closed form for $Z(t)$. Because $Z(t)$ is a function of the diffusion process assumed for the state variable, the precise method for doing this is not obvious. An intuitive solution procedure is to invoke Ito's lemma to specify partial differential equations for $Z(t)$ which can then be solved to determine a closed form for $\{Z(t)\}$. This is the approach used here.[7] To facilitate this process, the explicit connection of $Z(t)$ with the state variable $Y(t)$ will be recognized by introducing the notational convention $\{Z(t)\} \equiv \{Z(Y,t)\}$.

For purposes such as measuring stock market performance or empirically testing rational stock price models, Proposition 1 suggests a general outline for an arbitrage free procedure. Observed prices and dividends, p and d, are initially deflated by the cumulative interest rate process. The resulting S and D series are used to estimate the parameters of $Z(Y,t)$. The S and D series are then multiplied by the $Z(y,t)$ applicable to the valuation problem at hand where the function $Z(Y,t)$ is evaluated at $Y= y$. The specific functional form for the $Z(Y,t)$ used will depend on the diffusion process assumed for the underlying state variable processes $Y(t)$, i.e., it is possible for $Y(t)$ to be vector valued. The detrending procedure depends crucially on having a closed form for $\{Z(Y,t)\}$. Deriving this closed form involves starting from the $Y(t)$ diffusions and developing further restrictions needed to derive the closed form for $Z(t)$. Heaney and Poitras (1994) provides a theoretical result that can be used to derive the appropriate closed form. More precisely, recalling that $\{Z(t)\}$ is a martingale, if $Z(t)$ is assumed to obey the differentiability requirements needed to apply Ito's lemma, i.e., $Z(t)$ is twice differentiable in the state variables and once differentiable in time for each t in the interval $T \geq t \geq 0$, then it follows that $Z(Y,t)$ satisfies a set of partial differential equations that can be solved to get a specific functional form for $Z(y,t)$.

Given the functional restriction on $Z(Y,t)$ as well as the relatively weak diffusion assumptions on the $kx1$ vector dY required to derive Proposition 1, Heaney and Poitras (1994) provide the following restrictions on $Z(y,t)$ that can be used to restrict the coefficients in the diffusion processes for the state variables:

Proposition 2: **Differential Properties of Z**[8]

Given that $Z(y,t)$ obeys the K+1 first order partial differential equations:

$$\frac{\partial}{\partial y_i} Z(y,t) = h_i(t)\ Z(y,t) \qquad (4a)$$

$$\frac{\partial}{\partial t} Z(y,t) = f(t)\ Z(y,t) \qquad (4b)$$

Then the following integrability conditions:

$$\frac{\partial f}{\partial y_i} = \frac{\partial h_i}{\partial t} \qquad and \qquad \frac{\partial h_i}{\partial y_j} = \frac{\partial h_j}{\partial y_i} \qquad (5)$$

are necessary and sufficient for $Z(s,t)$ to satisfy (4), where:

$$h = -\Sigma^{-1}\lambda$$

$$\Sigma = \sigma\sigma'$$

$$f = -(\alpha' h + \frac{1}{2}h'\Sigma h + \frac{1}{2}Tr\ (\Sigma\frac{\partial h}{\partial y}))$$

and Σ is a full rank $K\times K$ variance-covariance matrix associated with the Brownian motions of the security price diffusions. *Tr* denotes the trace (the sum of the diagonal elements) of the matrix in brackets and $\partial h/\partial y$ is a KxK matrix with components $\partial h_i/\partial y_i$ where y_i is a defined point for the random variable $Y_i(t)$.

Given the implied restrictions on Σ and λ, conditions (4) can be integrated to obtain a solution for $Z(y,t)$. This resulting Z will depend on maintained null hypotheses about the parameters and functional form of the underlying price processes.

III. Specific Detrenders

This Section exploits (4) and (5) to derive closed form solutions for $Z(y,t)$. Following the discussion in Section II, it is appropriate to derive $Z(y,t)$ for a cumulative dividend-price process. As dividends are being cumulated into the state variable, this is modelled as a ***non-dividend paying*** state variable process. For pedagogical purposes, this cumulative price-dividend state variable process can be viewed as a proxy for the aggregate wealth process making a connection to the conventional interpretation of $Z(Y,t)$ as the intertemporal marginal rate of substitution, i.e., ratio of the marginal utilities of wealth at time t (> 0) and at $t=0$. However, this interpretation is only pedagogical as the cumulative dividend-price process combines both the conventional wealth and consumption processes used in representative investor models. The use of this state variable process is relevant because the absence of arbitrage relationship between prices and dividends in Proposition 1 applies to a cumulative dividend-price process.[9] Derivation of a specific closed form for Z requires precise specification of the diffusion process for the state variable, which then becomes a nested null hypothesis under which the *Z* is appropriate. To this end, let this 'aggregate wealth' process, *Y*, follow the lognormal (Black-Scholes) process:

$$dY = \mu Y\ dt + \sigma Y\ d\Theta \tag{6}$$

In (6), *Y* has been cumulative-interest-rate deflated. Recognizing that there is one state variable, the derivation of *Z* from the conditions associated with (4) require:

$$h = -\frac{\mu}{\sigma^2\ Y} \qquad and \qquad f = \frac{1}{2}[(\frac{\mu}{\sigma})^2 - \mu]$$

Verifying that (5) is satisfied, the *Z(Y,t)* can now be derived as:

$$Z(Y,t) = e^{\frac{1}{2}((\frac{\mu}{\sigma})^2 - \mu)(t)} \left(\frac{Y(t)}{Y(0)}\right)^{-\frac{\mu}{\sigma^2}} \tag{7}$$

where *t=0* is the initial starting time. Empirically, deflating the observed aggregate wealth process by the interest rate followed by detrending with *Z* will produce a martingale process, under the null hypothesis (6). As specified in Proposition I, under the null hypothesis of absence of arbitrage, *Z(t)* is also required to follow possess the martingale property. The *Z(t)* detrender requires two parameters to be estimated, μ and σ.

With this background, the rationale for interpreting *Z(t)* as a stock market performance can now be motivated using conventional notions from asset pricing theory. In particular, the representative investor framework identifies the state process *Y* as aggregate wealth. With some loss of content, the aggregate wealth process can be identified with the cumulative dividend-price process generated by the interest rate deflated 'S&P 500' price index and associated dividend payouts. Because *Z(y,t)* captures the adjustment for systematic risk at time *t* required to ensure absence of arbitrage, it follows that the time series $\{Z(y,t)\}$ can be characterized as a measure of the risk aversion propensity of the representative investor over time. Given that Proposition 1 requires dividends and prices deflated by the interest rate and then multiplied by the appropriate $Z(y,t)$ to be a martingale process, high values of Z can be characterized as reflecting a high degree of risk aversion for the representative investor at that point in time. A more precise statement that avoids the representative investor characterization would say that the value of the interest rate deflated cumulative dividend-price process is low relative to the long-term trend indicating that investors are pricing a high level of 'systematic risk' into stock prices. Conversely for low values of Z.

Interpreting whether specific values of Z are 'high' or 'low' follows from observing that $Z(t) = 1$ defines a situation where the return for the cumulative dividend-price process equals the return from the cumulative interest rate process. This can be seen from Proposition 1 where $Z(t) = 1$ indicates that deflating by the riskless interest rate is all that is required to obtain a martingale. Observing that the functional form of the arbitrage free detrender, $Z(Y,t)$, depends on a ***null hypothesis*** about the underlying stochastic 'wealth' process, it follows that performance measures based on Z (or empirical tests of models involving security prices) involve a joint hypothesis about the assumed stochastic structure of the underlying state variables and absence of arbitrage in security prices. Different stochastic assumptions will lead to different functional forms for Z. In other words, the method selected to detrend data implicitly imposes assumptions about the underlying security market equilibrium.

In order to further illustrate the connection to more conventional models, consider the case where the wealth process is identified with stock prices being the relevant state variable. In this case, it is possible to extend the analysis to allow dividends to be a constant fraction (δ) of the stock price: D

$= \delta S$. The asset price, S, is for simplicity assumed to follow a lognormal process: $dS = \theta S \ dt + \sigma S \ d\Theta$. Because there is still only one state variable, there is also only one risk premia, $\lambda = \theta S + \delta S$. While this approach to incorporating dividends is not fully consistent with observed dividend behaviour, it does provide the significant analytical simplification of retaining only stock prices as state variables. Based on Proposition 1, absence of arbitrage for a ***dividend paying asset*** requires the cumulative dividend-price process to be a martingale under Q. In this case:

$$h = -\frac{\theta + \delta}{\sigma^2 \ S} \equiv -\frac{\gamma}{S}$$

$$f = \gamma\theta - \frac{1}{2} \sigma^2 \gamma (1 + \gamma) = \frac{1}{2}(\frac{\theta^2 - \delta^2}{\sigma^2} - (\theta + \delta))$$

Verifying that (5) are satisfied, it is possible to derive Z as:

$$Z(S,t) = e^{f(t)} \left(\frac{S(t)}{S(0)} \right)^{-\gamma} = e^{\frac{1}{2}(\frac{\theta^2 - \delta^2}{\sigma^2} - (\theta + \delta))(t)} \left(\frac{S(t)}{S(0)} \right)^{-\frac{\theta + \delta}{\sigma^2}} \tag{8}$$

By construction, this Z is based on the null hypothesis of lognormal, interest rate deflated security prices and constant proportional payout dividends. Compared to (7), this detrender has an additional parameter to be estimated. Instead of the drift μ, there are now two values to be estimated, θ and δ. However, it can be verified by direct comparison of (8) with (7) that the solution is not substantively different.

The closed form for the detrender $Z(t)$ depends on the assumption model about the diffusion process assumed for the state variable. In order to illustrate the implications that assuming a different stochastic process for the state variable has on the functional form for Z, consider a more general form of (6), the constant elasticity of variance (CEV) process. Again taking Y to be the cumulative dividend-price process:

$$dY \;=\; \mu Y\; dt \;+\; \sigma Y^{\frac{\beta}{2}}\; d\Theta \qquad\qquad (9)$$

For this process:

$$h \;=\; -\frac{\mu}{\sigma^2}\; Y^{1-\beta} \qquad and \qquad f \;=\; \frac{1}{2}\Big(\frac{\mu}{\sigma}\Big)^2\; Y^{2-\beta} \;+\; \frac{\mu}{2}(1-\beta)$$

To satisfy the integrability conditions (5) now requires that either $\beta = 2$ or $\mu = 0$. Of the class of processes covered by the CEV, only the limiting lognormal, β=2 case is compatible with a non-zero drift, and $\lambda \neq 0$. For μ=0, because the asset involved does not pay dividends, this condition reduces to risk neutrality, $\lambda = 0$. The associated $Z(t) = 1$, a constant, indicating that deflating prices and dividends by the interest rate is all that is required for arbitrage free detrending. In the absence of a dividend process, the β ε [0,2) CEV processes does not appear compatible with absence of arbitrage.

IV. Empirical Results

IV.1 Description of the Data

The data file ie_data.xls was obtained from the Shiller website (www.aida.yale.edu/~shiller). This data file contains monthly data from January 1871 to September 2001 for four variables. These variables are: $\{p(t)\}$, nominal values of the Standard and Poor's (S&P) Composite stock price index; $\{d(t)\}$, the nominal dividends paid on the S&P Composite price index; $\{CPI(t)\}$, the consumer price index (1983=100); and, $\{e(t)\}$, earnings paid by companies in the S&P Composite price index. In addition to these series, monthly data series were obtained from the website for the Board of Governors of the Federal Reserve (www.federalreserve.gov) for the AAA bond yield from Jan. 1919 to Oct. 2002$\{r_t\}$.

Despite having such long data series available, for various reasons it is not feasible to use all of the observations available. One reason has to do with the substantive changes in the composition of the S&P Composite price index. For example, in the 19th and early 20th centuries, the index is heavily weighted by transportation stocks, especially railways, and it is not until the 1920's that industrial stocks are substantial component of the index. Even then, the number of stocks in the index was less than the number included in the post WWII era. In addition to the stocks included in the index, the legislative reform of securities markets in 1933 and 1934 also substantively changed the pricing environment. The market disruptions precipitated by WWI and WWII argue against the use of those observations. Finally, implementation of *Z(y,t)* requires estimates of the constant parameters μ and σ to be determined. In the absence of further theoretical structure specifying a method for allowing μ and σ to change over time, the full sample period selected for examination is the forty-plus year period from Jan. 1960 to Sept. 2001, amounting to 501 monthly observations. In addition to the full sample, this permits four ten year sub-samples to also be examined.

IV.2 Estimation of the Parameters for the GBM Detrender

The *Z(y,t)* selected for estimation is given in (7). Since the associated geometric Brownian motion process (6) is written on the interest rate deflated *Y*, the first step in the estimation process is to deflate the observed stock price and dividend series by the cumulative (riskless) interest rate process. This is done by multiplying each element of {*p(t)*} and {*d(t)*} by:

$$\prod_{i=0}^{t} \frac{1}{(1 + r(i))} \qquad \forall\ t = (0,1,2....(T-1))$$

As required, this interest-rate deflator equals unity at the initial date since *r(0)* = 0. The AAA bond yield is used because this rate captures both the level of the riskless interest rate and the term premium. The relationship between the risk premium and the term premium has been examined in Abel (1998). (Deflating by a short-term interest rate, such as the three or one month Treasury bill rate instead of $\{r(t)\}$ leave the term premium as a component of *Z(t)*.) The resulting series, *S(t)* and *D(t)* respectively, are given in Figures 1 and 2. A comparison can be made with the commonly used technique of deflating prices by the *CPI(t)*, given in Figure 3. S_t and D_t are used to estimate the parameters used in the detrender *Z(t)* and, in turn, to calculate *ZY(t)*, the *Z*-detrended value of *Y(t)*:

$$Y(t) = S(t) + \sum_{i=0}^{t-1} D(t-i) \qquad ZY(t) = Z(t)\ S(t) + \sum_{i=0}^{t-1} Z(t-i)\ D(t-i)$$

where *Y(t)* is the interest rate deflated, cumulative dividend-price process and *ZY(t)* is *Y(t)* detrended by *Z(t)* defined by (7).

An important step in estimating the detrender *Z(t)* is the estimation of the drift μ and the volatility σ of *dY/Y* from (7). Following Campbell et al. (1997), the maximum likelihood estimators of the drift and volatility are respectively, the (adjusted) mean and the standard deviation of the log-differences in the state variable, *Y*, in each pair of adjacent time periods. To formalize:

$$\hat{\alpha} = \frac{1}{T} \sum_{t=1}^{T} \ln(\frac{Y_t}{Y_{t-1}}) \qquad \hat{\sigma} = \sqrt{\frac{1}{T} \sum_{t=1}^{T} [\ln(\frac{Y_t}{Y_{t-1}}) - \hat{\alpha}]^2} \qquad \hat{\mu} = \hat{\alpha} + \frac{\hat{\sigma}^2}{2}$$

with *T* being the sample size. Table 1 reports the monthly drift and volatility estimates for the full sample and the four sub-samples:

Table 1

Estimated Monthly Drifts and Volatilities of Y_t

	α	μ	σ
1960.1-1970.12	.00218	.00253	.0266
1970.1-1980.12	-.00178	-.00124	.0325
1980.1-1990.12	.00285	.00335	.0318
1990.1-2001.9	.00318	.00361	.0293
Full: 1960.1-2001.9	.00093	.00113	.0201

The estimated monthly drifts are relatively 'small' because the prices and dividends have already been deflated by the cumulative interest rate. Given the values in Table 1, it is now possible to calculate the stock market performance measure, *Z(y,t)*.

Empirical estimates of *Z(y,t)* will vary with: the initial starting value selected, *Y(0)* ; the parameter estimates for μ and σ; and the length of the sample period. Because *Z(t)* measures the 'value' of *Y(t)* relative to *Y(0)* the starting value is important for interpreting the dimension of *Z(t)*. For example, if $Z(t = 20) = 1.23$ this means that the twentieth observation for *Y(t)* is has to be multiplied by 1.23 to be equal to the trend line value defined by μ and σ. If *Y(0)* is 'high' relative to the trend line, then values of *Y(t)* will be affected accordingly. Direct inspection of (7) reveals that if *Y(0)* is changed then the numerical value of *Z(t)* will also change. Given this, the *Z(t)* for the full 1960-2001 sample is provided in Figure 4. (The *Z(t)* for each of the four subsamples are provided in Figure 10.) Confronted with the empirical variation in μ and σ reflected in Table 1, it is sensible to proceed by estimating different values for the *Z(s,t)* detrender using the drift and volatility estimates for the full sample and various subsamples. This will produce different *ZY(t)* estimates, depending on the sample selected. In each case, the products of S_t and D_t with Z_t give the arbitrage-free detrended price (*ZS*) and dividend (*ZD*), respectively. In this context, *Z(t)* can be used to provide a relative comparison of *ZY* at different points in time. As expected from theory, higher values of *Z* are associated with bear markets and, conversely, the lowest values of Z arise in bull markets. For example, Figure 4 reveals that the market valuations circa 1997 were approximately at the levels attained by the bull market of the late 1960's.

Section V: Martingale Tests[10]

In Section II it was observed that, under the null hypothesis of absence of arbitrage in security prices, $\{Z(t)\}$ is a strictly positive martingale. This can be seen by applying Ito's lemma to (7) and using (6) to show the martingale requirement also applies to $Z(t)$. In general, the martingale property of $\{Z(t)\}$ is inherent in the theoretical framework captured in (4) and (5). Evidence that $\{Z(s,t)\}$ generated by (7) does not follow a martingale is a rejection of the joint null hypothesis (6) and (3). In particular, testing empirically whether $\{Z(t)\}$ follows a martingale is a specification test for the joint hypothesis of absence and arbitrage and the validity of (7). It is possible that, if the martingale hypothesis is not empirically validated, the null hypothesis of absence of arbitrage could still be true because (7) is not valid. Similarly, the joint null hypothesis of (6) and (3) requires $ZY(t)$ to also have the martingale property.

Various possible statistical methods are available for testing whether a time series is a martingale process, e.g., Campbell, et al. (1997, Sec. 2.1), Pantula et al. (1994). Tables 2-4 reports martingale tests derived from the result that first differences of a martingale process are orthogonal. Hence, under the null hypothesis that the $Z(t)$ is a martingale, the slope and intercept in an AR(1) regression for $\Delta Z(t)$ are expected to be insignificant. Because both the slope and intercept are required to be zero, the relevant statistic to examine is the F test. Because there is a lagged dependent variable in the regression, a test for autocorrelation is required. If autocorrelation is present then the statistical tests will be biased in favour of acceptance. Durbin’s h test is provided for the purpose of testing for autocorrelation in regressions with lagged dependent variables. In order to determine whether there is drift in the mean, regression results are presented both with and without a linear time trend. Similar tests are also done for *Y(t)* and *ZY(t)*. Under the joint null hypothesis of (6) with absence of arbitrage, *ZY(t)* is expected to be a martingale. If $Z(t) = 1$ for all t, then (3) implies that *Y(t)* is also expected to be a martingale. The properties of these two series reflect the difference between using the detrender given by (7), for *ZY(t)*, and assuming a detrender, $Z(t) = 1$. Results for both the full sample and the relevant subsamples are provided.

The results in Tables 2-3 reveal that *Z(t)* and *ZY(t)* generated by (7) do not meet the requirements of a martingale for the full 1960-2001 sample, i.e., $\Delta Z(t)$ exhibits statistically significant and positive autocorrelation. Results for the four 10 year subsamples are generally the opposite, with the F test for all coefficients jointly equal to zero being insignificant at the 1% level in all subsamples for *ZY(t)* and all but the 1980.1-1989.12 subsample for *Z(t),* i.e., it is not possible to reject the hypothesis of jointly zero coefficients at the 1% level with the F test. This rejection for the full sample, but not the subsamples, suggests that there are possible problems associated with achieving a martingale using the drift and volatility estimated over such a long time period. Because these parameters are essential components in the functional form for *Z(t)*, it may be necessary to incorporate parameter evolution in the specification of *Z(t)* in order to obtain a martingale for long sample periods. The possible impact of *Z(t)* on *ZY(t)*can be assessed by examining the case where $Z(t) = 1$applies, i.e., by examining the martingale properties of *Y(t)*. Comparing the results in Tables 3 and 4 reveals generally similar results. The *Y(t)*full sample results reject martingale behaviour, with the F test for three of the four subsamples being insignificantly different from zero. As with the *Z(t)* results, the F test for *Y(t)*is highly significant for the 1980.1-1989.12 subsample.

VI. Conclusions and Implications

Several issues are raised in this paper. Of general interest to empirical researchers, it is demonstrated that detrending procedures can impose significant equilibrium restrictions in empirical studies. In the specific case of rational stock pricing models, this is due to the implications that the null hypothesis of absence of arbitrage has for the empirical behaviour of security prices. Implementation of the detrending procedure proposed here involves making specific stochastic process assumptions about prices in order to generate a $Z(y,t)$. Because the selection of a closed form detrender embeds an assumption about how risk is reflected in security prices, the empirical tests involve a joint hypothesis concerning the detrending process and the security pricing model. Given this, the empirical results of this study provide mixed evidence about the null hypothesis of absence of arbitrage in aggregate stock prices. This conclusion was based on the tests for the martingale property, both in the detrender, $Z(y,t)$, and in the detrended state variable process, $ZY(t)$. The empirical results for the martingale hypotheses indicate a significant amount of positive autocorrelation in the first differences of both $Z(y,t)$ and $ZY(t)$ for the full 1960-2001sample. This rejection of absence of arbitrage is qualitatively different than finding evidence for a rational bubble where there is a component of the observed price that is not explained by the discounted value of expected future dividends. If there are rational bubbles, then discounted dividend stock pricing models, such as the Gordon model, are mis-specified. This conclusion is based on a rejection of the transversality condition. As the rejection of the martingale hypothesis relates to the properties of the path of security prices, the empirical results in Section V are not equivalent to finding evidence for a rational bubble.

The primary objective of this paper is to use the restrictions imposed by absence of arbitrage in security prices to develop a stock market performance measure, $Z(t)$. Rejection of the joint hypothesis involving absence of arbitrage begs an obvious question: does this rejection undermine the applicability of the performance measure? The answer to this question depends on how the performance measure is being used. Comparison of the subsample and full sample results indicates that fitting $Z(t)$ with constant parameters over a long time period generates positive autocorrelation in the measure. In practical terms, this says that the risk assessment reflected in aggregate stock market values is taken over a shorter time horizon than 40 years. The subsample results indicate that an assessment horizon of ten years is more appropriate. As such, measuring performance over long time periods requires some method for changing the parameter estimates. Yet, incorporating parameter evolution by introducing additional state variables substantively complicates the problem of deriving a closed form for the detrender. A practical, if somewhat ad hoc, solution would be to use a moving estimation window to determine the parameters required to calculate $Z(t)$. However, this raises the as yet unresolved problem of determining the optimal time period for estimating these parameters. This is a potentially fruitful avenue for future research on the properties of the stock market performance measure proposed in this paper.

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NOTES

1. As demonstrated by Decamps and Lazrak (2000), there is a semantic confusion in one of the proofs in Heaney and Poitras (1994) where it was not made sufficiently clear that separability of *Z(t)* in the forward and backward variables is used in the main Proposition 1.

2. This illustrates the potential confusions arising from the sometimes conflicting usage of "arbitrage". The convention in rational stock pricing models is that absence of arbitrage requires "assets are voluntarily held and that no agent can, given his private information and the information revealed by prices, increase his expected utility by reallocating his portfolio". This definition of absence of arbitrage is much weaker than that encountered in the pricing of derivative securities where an arbitrage is a riskless trading strategy which generates a positive profit with no net investment of funds.

3. One type of simple security valuation problem which has the form of (1) is the deterministic, continuous time bond pricing problem, where $p(t)$ is the current bond price, T is the maturity date, $p(T)$ is the principal, $c(t)$ is the continuous coupon paid at t and $\{V_b(t)\}$ is the associated discounting operator. Taking the valuation date to be t=0:

$$V_b(0,T) = \exp\{\int_0^T -r_u \, du\} = \exp\{\int_0^t -r_v \, dv\} \exp\{\int_t^{t+n} -r_w \, dw\} = V_b(0,t) \; V_b(t,t+n)$$

where r_i is the deterministic continuous interest rate for period i ($exp \equiv e$).

4. There are some models of rational bubbles which permit the bubble to collapse to zero prior to the limit being reached, e.g., Evans (1991). This subclass of bubbles is not being considered here. It is also possible that the *B(t)* will increase at such a rate that the stock price will be dominated by the bubble. However, this case is neither necessary or sufficient for there to be a rational bubble. Because stock prices cannot go negative, the possibility of the bubble dominating the price is sometimes used to rule out the possibility of a negative bubble.

5. Following the approach used in Harrison and Kreps (1979), the role of the discounting process in theoretical absence of arbitrage models is usually handled with the simplification of assuming that interest rates are zero, directly suppressing consideration of issues associated with the numeraire. This approach is impractical for empirical applications. Introducing deflating by the cumulative interest rate process into the definition of the state variables permits the arbitrage free shadow prices to be directly derived as the product of the marginal rate of substitution and the observed, interest rate detrended price process. To make the connection with empirical applications means that security prices and dividends deflated by the interest rate are the relevant state variables.

6. The cumulative dividend-price process could also be defined as the total return process. The change in notation from p and d to S and D is intended to recognize the difference between these variables: S and D are interest rate deflated while p and d are observed nominal prices and dividends.

7. The substitution of $Z(y,t)$ for $Z(Y,t)$, where y represents a specific point realization of Y, is deliberate. This follows because, much like a Taylor series, Ito's lemma involves expanding a function about a specific point.

8. The specification of f in Proposition 2 reveals that the assumption of differentiability on $Z(Y,t)$ also requires that the diffusion coefficients $\sigma(\cdot)$ and $\alpha(\cdot)$ be differentiable. This is substantively stronger condition than the Lipschitz and growth conditions required to get Proposition 1. From this it follows that Proposition 2 also requires, $\alpha(\cdot)$, $\sigma(\cdot)$ and $D(\cdot)$ to be second order differentiable in state and first order differentiable in time.

9. Models such as the consumption based CAPM would interpret the stock price process as the wealth process and the dividend process as the consumption process. As captured in Proposition 1, the absence of arbitrage approach permits a different choice of state variables.

10. Creating subsamples is not straight forward. Another practical problem which arises in constructing and interpreting the various Z-detrended series is associated with the presence of initial transients in the various data series: it takes a number of observations from the start of the detrended series for the process to damp down to an equilibrium state. This initial transient behaviour is expected, given that the $Z(t)$ are solved by integrating (5) which induces an initial starting value into the solution. The transient is generic to the pricing problem being examined and is independent of the starting value selected.