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# ‘How’s the Stock Market Doing?’ Using Absence of Arbitrage to Measure Stock Market Performance

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## ABSTRACT

This paper provides a methodology for ***measuring stock market performance*** based on the restrictions provided by ***absence of arbitrage in security prices.***

Under the null hypothesis that the ***aggregate cumulative dividend-price process*** follows a ***geometric Brownian motion***, a closed form related to the intertemporal marginal rate of substitution is derived and empirically evaluated.

The stock market performance measure is based on the level of ***risk adjustment*** required to compare the value of the stock index at the starting point with the interest rate deflated value at any given point in the time series.

The paper concludes with ***empirical tests*** for the martingale property of the performance measure.

## II. Absence of Arbitrage and Rational Security Valuation

The stock market performance measure has an intuitive foundation derivable from rational security valuation problems of the general form:

$$p(t) = V(t,T)\ p(T) + \int_t^T V(t,u)\ d(u)\ du \tag{1}$$

where *p(t)* is the nominal security price observed at time *t*, *d(t)* is the instantaneous nominal dividend or coupon paid at *t*, and *T* is the terminal or maturity date for the valuation problem ($T \geq t \geq 0$).

(1) is often referred to as a "no arbitrage" condition, e.g., Blanchard and Watson (1982) though the connection with the 'absence of arbitrage' associated with the equivalent martingale measure is more complicated.

**RATIONAL BUBBLES**

--Two conditions almost always imposed on the $\{V(t,T)\}$ are:

$$\lim_{T \to t} V(t,T) = I \qquad \lim_{T \to \infty} V(t,T) = 0$$

The first condition ensures continuity and consistent pricing. The second condition ensures pricing convergence, e.g., Craine (1993).[1] It is well known that the convergence of the operator as $T \to \infty$ is required for the satisfaction of a transversality condition that, in turn, is needed to ensure that the difference equation identified by (1) will not have an infinite number of solutions.

--A rational bubble is specified by observing that the security price $p(t)$ can be modelled as the sum of two components, a market fundamentals component $p_F(t)$ and a rational bubbles component $B(t)$, e.g., Evans (1991):

$$p(t) = p_F(t) + B(t)$$

where $p_F(t)$ is associated with the infinite sum of the discounted value of expected future dividends and $B(t)$ can be any random variable that satisfies $B(t) = V(t,T)\, B(T)$.

RATIONAL BUBBLES (cont'd)

--Because $p(t)$ can incorporate both fundamental and bubble information, progressive substitution for $p(T)$ produces restrictions on the bubble component. If the transversality condition:

$$\lim_{T \to \infty} V(t,T)\ p(T) = 0$$

is satisfied, then a rational bubble is ruled out, $B(t) = 0$.

--In the general case, this involves applying ***iterated expectations*** to expected future prices to derive a general solution of the form:

$$p(0) = \int_0^\infty V(0,u)\ d(u)\ du + B(0) = p_F(0) + B(0) \qquad (2)$$

where:

$$B(0) = \lim_{T \to \infty} V(0,T)\ B(T)$$

The market price of a security can deviate from its fundamental value, $p_F(t)$, without violating absence of arbitrage, providing the expected value of the rational bubble increases at a faster rate than that specified by the $\{V(t)\}$. For a rational bubble, $B(0) \neq 0$ and the transversality condition is not satisfied.

**THE ABSENCE OF ARBITRAGE CONNECTION**

--**The first step in the process is to deflate the stock price and dividend processes by the cumulative (riskless) interest rate process.**

--This deflation procedure is accomplished, for example, by choosing $t=0$ as the starting point for the series; price and dividend observations at $t=1$ are divided by $(1 + r(1))$; $t=2$ price and dividend observations are divided by $(1 + r(1))(1 + r(2))$ and so on; where $r(t)$ is the (riskless) interest rate for period $t$. A normalization condition that the interest rate process equal one at the pricing decision date, $t=0$, is imposed together with the assumption that the discounting process is strictly positive.

--**The resulting stock price and dividend series are denoted as $\{S(t)\}$ and $\{D(t)\}$ to distinguish these series from the nominal un-discounted series $\{p(t)\}$ and $\{d(t)\}$ used in the rational bubble model.**

ABSENCE OF ARBITRAGE (cont's)

The cumulative dividend-price process:

$$S(t) + \int_0^t D(u)\ du$$

is a martingale under the equivalent martingale measure, $Q$

*Proposition 1*: **Arbitrage Free Shadow Pricing**

Assuming an equivalent martingale measure $Q$ exists, then there exists a positive martingale $\{Z(t)\}$ on the probability space $(\Omega, \boldsymbol{F}, \{F(t)\}, P)$, such that for the cumulative dividend-price process, the transformed process:

$$Z(t)\ S(t) + \int_0^t Z(u)\ D(u)\ du$$

is a martingale on $(\Omega, \boldsymbol{F}, \{F(t)\}, P)$ such that for $t,k \geq 0$:

$$S(t) = E^P \left[ \int_t^{t+k} Z(u) D(u)\ du + Z(t+k)\ S(t+k) \mid F(t) \right]$$

To model current prices, it is appropriate to let $t = 0$ and observe that $Z(0) = 1$.

HEANEY AND POITRAS (1994)

*Proposition 2*: **Differential Properties of Z**

Given that $Z(y,t)$ obeys the $K$+1 first order partial differential equations:

$$\frac{\partial}{\partial y_i} Z(y,t) = h_i(t)\ Z(y,t) \qquad (4a)$$

$$\frac{\partial}{\partial t} Z(y,t) = f(t)\ Z(y,t) \qquad (4b)$$

Then the following integrability conditions:

$$\frac{\partial f}{\partial y_i} = \frac{\partial h_i}{\partial t} \qquad and \qquad \frac{\partial h_i}{\partial y_j} = \frac{\partial h_j}{\partial y_i} \qquad (5)$$

are necessary and sufficient for $Z(s,t)$ to satisfy (4), where:

$$h = -\Sigma^{-1}\lambda$$

$$\Sigma = \sigma\sigma'$$

$$f = -(\alpha' h + \frac{1}{2}h'\Sigma h + \frac{1}{2}Tr\ (\Sigma\frac{\partial h}{\partial y}))$$

$\Sigma$ is a full rank $K\times K$ variance-covariance matrix associated with the Brownian motions of the security price diffusions. $Tr$ denotes the trace (the sum of the diagonal elements) of the matrix in brackets and $\partial h/\partial y$ is a $KxK$ matrix with components $\partial h_j/\partial y_i$ where $y_i$ is a defined point for the random variable $Y_i(t)$.

## III. Specific Detrenders

-- Let the cumulative dividend-price process, $Y$, follow the lognormal (Black-Scholes) process:

$$dY = \mu Y\ dt + \sigma Y\ d\Theta \tag{6}$$

In (6), $Y$ has been cumulative-interest-rate deflated.

--The conditions associated with (4) require:

$$h = -\frac{\mu}{\sigma^2\ Y} \qquad and \qquad f = \frac{1}{2}[(\frac{\mu}{\sigma})^2 - \mu]$$

Verifying that (5) is satisfied, the $Z(Y,t)$ can now be derived as:

$$Z(Y,t) = e^{\frac{1}{2}((\frac{\mu}{\sigma})^2 - \mu)(t)} \left(\frac{Y(t)}{Y(0)}\right)^{-\frac{\mu}{\sigma^2}} \tag{7}$$

where $t=0$ is the initial starting time.

AN ALTERNATIVE DETRENDER WITH THE STATE VARIABLE AS THE PRICE

Allow dividends to be a constant fraction ($\delta$) of the stock price:

$$D = \delta S$$

Absence of arbitrage for a ***dividend paying asset*** requires the cumulative dividend-price process to be a martingale under $Q$. In this case:

$$h = -\frac{\theta + \delta}{\sigma^2 S} \equiv -\frac{\gamma}{S}$$

$$f = \gamma\theta - \frac{1}{2}\sigma^2 \gamma (1 + \gamma) = \frac{1}{2}(\frac{\theta^2 - \delta^2}{\sigma^2} - (\theta + \delta))$$

Verifying that (5) are satisfied, it is possible to derive $Z$ as:

$$Z(S,t) = e^{f(t)} \left(\frac{S(t)}{S(0)}\right)^{-\gamma}$$

$$= e^{\frac{1}{2}(\frac{\theta^2 - \delta^2}{\sigma^2} - (\theta + \delta))(t)} \left(\frac{S(t)}{S(0)}\right)^{-\frac{\theta + \delta}{\sigma^2}} \tag{8}$$