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Immunization Bounds, Time Value and Non-Parallel Yield Curve Shifts*

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ABSTRACT

Since Redington (1952) it has been recognized that classical immunization theory fails when shifts in the term structure are not parallel. Using partial durations and convexities to specify immunization bounds for non-parallel shifts in yield curves, Reitano (1991a,b) extended classical immunization theory to admit non-parallel yield curve shifts, demonstrating that these bounds can be effectively manipulated by adequate selection of the securities being used to immunize the portfolio. By exploiting properties of the multivariate Taylor series expansion of the spot rate pricing function, this paper extends this analysis to include time values permitting a connection to results obtained by Christiansen and Sorensen (1994), Chance and Jordan (1996), Barber and Copper (1997) and Poitras (2005) on the time value-convexity tradeoff.

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Immunization Bounds, Time Value and Non-Parallel Yield Curve Shifts

In the seminal work on fixed income immunization, Redington (1952) used a univariate Taylor series expansion to derive two rules for immunizing a life insurance company portfolio against a change in the level of interest rates: match the duration of cash inflows and outflows; and, set the asset cash flows to have more dispersion than the liability cash flows around that duration. From that beginning, a number of improvements to Redington's classical immunization rules have been proposed, aimed at correcting limitations in this classical formulation. Particular attention has been given to generalizing the classical model to allow for non-parallel shifts in the yield curve, e.g., Soto (2004, 2001), Nawalka et al. (2003), Navarro and Nave (2001), Crack and Nawalka (2000), Balbas and Ibanez (1998) and Bowden (1997). While most studies aim to identify rules for specifying portfolios that are immunized against instantaneous non-parallel shifts, Reitano (1992, 1996) explores the properties of the immunization bounds applicable to non-parallel shifts. In particular, partial durations and convexities are exploited to identify bounds on gains and losses for an instantaneous unit shift in the yield curve. The objective of this paper is to extend the partial duration framework by incorporating time value changes into the immunization bound approach. This extends the results of Christensen and Sorensen (1994), Chance and Jordan (1996), Barber and Copper (1997) and Poitras (2005) on the time value-convexity tradeoff.

I. Background Literature

Though Redington (1952) recognized that classical immunization theory fails when shifts in the term structure are not parallel, Fisher and Weil (1971) were seminal in situating the problem in a term structure framework. The development of techniques to address non-parallel yield curve shifts led to the recognition of a connection between immunization strategy specification and the type of assumed shocks, e.g., Boyle (1978), Fong and Vasicek (1984), Chambers et al. (1988). Sophisticated risk measures, such as M^2, were developed to select the best duration matching portfolio from the set of potential portfolios. Being derived using a specific assumption about the stochastic process generating the term structure, these theoretically attractive models encountered difficulties in practice. For example, "minimum M^2 portfolios *fail* to hedge as effectively as portfolios including a bond maturing on the horizon date" (Bierwag et al. 1993, p.1165). This line of empirical research led to the recognition of results such as the 'duration puzzle' (Ingersoll 1983; Bierwag et al. 1993; Soto 2001), where portfolios containing a maturity-matching bond have smaller deviations from the promised target return than duration matched portfolios not containing a maturity-matching bond. These results beg the question: are these empirical limitations due to failings of the stochastic process assumption underlying the theoretically derived immunization measures or is there some deeper property of the immunization process that is not being accurately modelled?

Instead of assuming a specific stochastic process and deriving the optimal immunization conditions, it is possible to leave the process unspecified and work directly with the properties of an expansion of the spot rate pricing function or some related transformation, e.g., Shiu (1987,1990). Immunization can then proceed by making assumptions based on the empirical behaviour of the yield curve. Soto (2001) divides these "empirical multiple factor duration models" into three categories. *Polynomial duration models* fit yield curve movements using a polynomial function of the terms to maturity, e.g., Crack and Nawalka (2000), Soto (2001), or the distance between the terms to maturity and the planning horizon, e.g., Nawalka et al. (2003). *Directional duration models* identify general risk factors using data reduction techniques such as principal components to capture the empirical yield curve behaviour, e.g., Elton et al. (2000), Barber and Copper (1996), Hill and Vaysman (1998), Navarro and Nave (2001). *Partial duration models,* including the *key rate duration models*, decompose the yield curve into a number of linear segments based on the selection of key rates, e.g., Ho (1992), Dattareya and Fabozzi (1995), Phoa and Shearer (1997). Whereas the dimension of polynomial duration models is restricted by the degree of the polynomial and the directional duration models are restricted by the number of empirical components or factors that are identified, e.g., Soto (2004), the number of key rates used in the partial duration models is exogenously determined by the desired fit of immunization procedure.

Reitano (1991a,b) provides a seminal, if not widely recognized, analysis of bond portfolio immunization using the partial duration approach.[1] Though Reitano evaluates a multivariate Taylor series for the asset and liability price functions specified using key rates, the approach is more general. Multiple factors derived from the spot rate curves, bond yield curves, cash flow maturities or key rates can be used. In particular, this paper uses a refined spot rate model where each cash flow is associated with a spot rate. Because this approach can be cumbersome as the number of future cash flows increases, practical applications involve factor models, key rates and interpolation schemes that exogenously determine the dimension of the spot rate space, as in Ho (1992) and Reitano (1992). The advantage of using a spot rate for each cash flow is precision in calculating the individual partial duration, convexity and time values for the elementary fixed income portfolios that are being examined. This is appropriate for the exploring the theoretical properties of the immunization problem where the spot rate curve can change shape, slope and location. While it is possible to reinterpret the refined spot rate curve change in terms of a smaller number of fixed functional factors, this requires some method of aggregating the individual cash flows. While such aggregation is essential where the number of the possibly random individual cash flows is large, as in practical applications, analytical precision is enhanced by having a one-to-one correspondence between spot rates and cash flows.

Defining a norm applicable to a unit parallel yield curve shift, Reitano exploits Cauchy-Schwarz and quadratic form inequality restrictions to identify bounds on the possible deviations from classical immunization conditions. In other words, even though classical immunization rules are violated for non-parallel yield curve shifts, it is still possible to put theoretical bounds on the deviations from the classical outcome and to identify the specific types of shifts that represent the greatest loss or gain. This general approach is not unique to Reitano. Developing the Gateaux differential approach introduced by Bowden (1997), Balbas and Ibanez (1998) rediscover the possibility of defining such bounds, albeit in an alternative mathematical framework. Balbas and Ibanez also introduce a novel innovation: a linear dispersion measure that, when minimized, permits identification of the 'best' portfolio within the class of immunizing portfolios. More precisely, a strategy of matching duration and minimizing the dispersion measure identifies the portfolio that will minimize immunization risk and, as a consequence, provides an optimal upper bound for possible loss on the portfolio.

Considering only the implications of instantaneous non-parallel yield curve shifts, it is difficult to argue that the partial duration approach to identifying immunization bounds is superior to the directional duration and polynomial duration models. However, when the analysis is extended to include the time-value convexity tradeoff identified by Christiansen and Sorensen (1994), Barber and Copper (1997) and Poitras (2005, ch.5), the partial duration approach has the desirable feature of providing a direct relationship between the convexity and time value elements of the immunization problem. This follows because convexity has a time value cost associated with the initial yield curve shape and the expected future path of spot rates for reinvestment of coupons and rollover of short-dated principal. Despite the essential character of the time value decision in overall fixed income portfolio management, available results on the time value-convexity tradeoff have been developed in the classical Fisher-Weil framework involving monotonic term structure shifts. By incorporating time value into the determination of immunization bounds, the partial duration approach has the desirable property of allowing attention to focus on the specifics of non-parallel yield curve shifts that are of practical interest.

II. The Reitano Partial Duration Model

Even though the ultimate objective will involve fund surplus immunization, the familiar bond price function specified with spot interest rates, *P(z)*, can be used as a basic starting point for presenting the model:

$$P(z) = \sum_{t=1}^{T} \frac{C}{(1 + z_t)^t} + \frac{M}{(1 + z_T)^T}$$

where: *P(z)* is the price of a bond paying annual coupon payments; *C* is the annual coupon; *M* is the bond par value; z_t is the spot interest rate (implied zero coupon interest rate) applicable to cash flows at time *t*; $t = (1,2,...T)$; $z = (z_1, z_2,...., z_T)'$ is the *Tx1* vector of spot interest rates; and *T* is the term to maturity in years. Recognizing that *P* is a function of the *T* spot interest rates contained in *z*, it is possible to apply a multivariate Taylor series expansion to this bond price formula, that leads immediately to the concepts of partial duration and partial convexity:

$$P(z) = P(z_0) + \sum_{t=1}^{T} \frac{\partial P(z_{t,0})}{\partial z_t}(z_t - z_{t,0}) + \frac{1}{2!} \sum_{i=1}^{T} \sum_{j=1}^{T} \frac{\partial^2 P(\cdot)}{\partial z_i \, \partial z_j}(z_i - z_{i,0})(z_j - z_{j,0}) + H.O.T.$$

$$\rightarrow \quad \frac{P(z) - P(z_0)}{P(z_0)} \cong -\sum_{t=1}^{T} D_t \, (z_t - z_{t,0}) + \frac{1}{2!} \sum_{i=1}^{T} \sum_{j=1}^{T} CON_{i,j} \, (z_i - z_{i,0})(z_j - z_{j,0}) \qquad (1)$$

where $z_0 = (z_{1,0}, z_{2,0},...., z_{T,0})'$ is the $Tx1$ vector of initial spot interest rates, D_t is the partial duration associated with z_t, the spot interest rate for time t, and $CON_{i,j}$ is the partial convexity associated with the spot interest rates z_i and z_j for i,j defined over $(1,2,....T)$.[2]

Observing that the partial durations at z_0 can be identified with a $Tx1$ vector $D_{\underline{T}} = (D_1, D_2, ... D_T)'$ and the partial convexities at z_0 with a TxT matrix C_T with elements $C_{i,j}$, the model proceeds by applying results from the theory of normed linear vector spaces to identify theoretical bounds on $D_{\underline{T}}$ and C_T. In the case of $D_{\underline{T}}$, the Cauchy-Schwarz inequality is used.[3] For C_T the bounds are based on restrictions on the eigenvalues of C_T derived from the theory of quadratic forms.[4] To access these results, the ***direction vector*** specified by Reitano is intuitively appealing. More precisely, taken as a group, the $(z_t - z_{0,t})$ changes in the individual spot interest rates represent shifts in yield curve shape. These individual changes can be reexpressed as the product of a direction shift vector N and a magnitude Δi:

$$(z_t - z_{t,0}) = n_t \, \Delta i \qquad where: \quad N = (n_1, n_2,...., n_T)'$$

It is now possible to express (1) in vector space form as

$$\frac{P(z) - P(z_0)}{P(z_0)} \cong -\Delta i \, [N' \, D_{\underline{T}}] + \Delta i^2 \, [N' \, C_T \, N]$$

From this, the spot rate curve can be shocked and the immunization bounds derived. The dimension of N provides a connection to alternative approaches to the immunization problem that reformulate the T dimensional refined spot rate curve in terms of a smaller ($< T$) number of fixed functional factors. The use of spot rates in the formulation does differ slightly from shocking the yield curve and then deriving the associated change in the spot rate curve. Using the spot rate approach, it follows that $N_0 = (1, 1,...., 1)'$ represents a parallel shift in the spot rate curve, with the size of the shift determined by Δi.

Having started with the bond price function, it is straight forward to have the analysis encompass the basic building blocks of the fixed income portfolio, the cash flows associated with individual assets and liabilities. This is accomplished using the objective of surplus immunization, e.g., Shiu (1987, 1990), Messmore (1990). This objective does produce a more general and complicated problem than calculating the partial durations and convexities for individual assets and liabilities in isolation. Surplus immunization involves explicit recognition of the balance sheet relationship: $A = L + S$, where A is the assets held by the fund, L is the fund liabilities and S is the accumulated surplus. The objective of immunizing the portfolio surplus, $S = A - L$, from changes in the term structure of interest rates requires:

$$S(z) = \sum_{t=1}^{T} \frac{CF_t}{(1 + z_t)^t} \quad \rightarrow \quad \frac{S(z) - S(z_0)}{S(z_0)} = 0 \cong -\Delta i \; [N' \, D_{\underline{T}}] + \Delta i^2 \; [N' \, C_T \, N] \qquad (2)$$

where $D_{\underline{T}}$ and C_T are now defined for the fund surplus, $S(z)$, and the cash flows at time t, CF_t, are the sum of the asset (+) and liability (-) cash flows that occur at time t. In this formulation, there is a singularity where $S(z_0) = 0$. In addition, because CF_t will be negative when the liability cash outflow exceeds the asset cash inflow at time t, it is possible for either the duration of surplus or convexity of surplus to take negative values, depending on the selected portfolio composition and yield curve shape.

To derive the classical immunization conditions, Redington (1952) uses a zero surplus fund – $S(z_0) = 0$ – where the present value of assets and liabilities are equal at $t=0$. In practice, this specification is consistent with a life insurance fund where the surplus is being considered separately. This classical immunization problem requires the maturity composition of an immunized portfolio to be determined by equating the duration of assets and liabilities. When the fund surplus function is generalized to allow non-zero values, the immunization conditions change to:

$$S = A - L \quad \rightarrow \quad \frac{1}{S}\frac{dS}{dt} = 0 = \frac{A}{S} D_A - \frac{L}{S} D_L \quad \rightarrow \quad D_A = \frac{L}{A} D_L$$

The classical zero surplus immunization result requires setting the duration of assets equal to the duration of liabilities. This only applies for a zero surplus portfolio. Immunization with a non-zero surplus requires the duration of assets to be equal to the duration of liabilities, multiplied by the ratio of the market value of assets to the market value of liabilities, $D_A = (L/A)\ D_L$, where L and A are the market values of assets and liabilities. As indicated, this more general condition is derived by differentiating both sides of $S = A - L$, dividing by S and manipulating. A similar comment applies to convexity, i.e., $CON_A > (L/A)\ CON_L$..

Allowing for a non-zero surplus changes the intuition of the classical duration matching and convexity conditions. Observing that the duration of a portfolio of assets is the value weighted sum of the individual asset durations, a positive fund surplus with a zero coupon liability allows surplus immunization using a combination of assets that have a shorter duration than that of the liability. As such, surplus immunization for a fund with a single liability having a duration that is longer than the duration of any traded asset can be achieved by appropriate adjustment of the size of the surplus. In general, a larger positive fund surplus permits a shorter duration of assets to immunize a given liability. Because yield curves typically slope upward, this result has implications for portfolio returns. In the classical immunization framework, such issues do not arise because the force of interest (Kellison 1991) is a constant and, in any event, the force of interest for a zero surplus fund is unimportant.[5] However, when the interest rate risk of the surplus has been immunized, the equity value associated with the surplus will earn a return that depends on the force of interest function. This return is measured by the "time value" function, e.g., Chance and Jordan (1996). Non-parallel shifts in the yield curve will alter the time value.

III. Convexity and Time Value

Following Redington (1952), classical immunization requires the satisfaction of both duration and convexity conditions: duration matching is required to be accompanied with higher portfolio convexity, e.g., Shiu (1990). The convexity requirement ensures that, for an instantaneous change in yields, the market price of assets will outperform the market price of liabilities. Yet, higher convexity does have a cost. In particular, when the yield curve is upward sloping, there is a tradeoff between higher convexity and lower time value (Christensen and Sorensen 1994, Poitras 2005, ch.5). This connection highlights a limitation of the Taylor series expansion in (1) and (2): the bond price and fund surplus functions depend on time as well as the vector of spot interest rates, i.e., $P = P[z,t]$ and $S = S[z,t]$. If yields do not change, higher convexity will likely result in a lower portfolio return due to the impact of time value. Though some progress has been made in exploring the relationship between convexity and time value (Chance and Jordan 1996, Barber and Copper 1997), the precise connection to the calculation of the extreme bounds on yield curve shifts is unclear. The extreme bounds associated with changes in convexity are distinct from those for duration. How shifts in extreme bounds for duration and convexity are associated with changes in the portfolio composition and, in turn, to the time value is, at this point, largely unknown.

Assuming for simplicity that cash flows are paid annually, evaluating the first order term for the time value in the surplus function, *S(z,t)*, produces:[6]

$$\frac{1}{S}\frac{\partial S}{\partial t} = -\{\sum_{t=1}^{T} \frac{CF_t \ \ln(1 + z_t)}{(1 + z_t)^t}\}\frac{1}{S} = N_0^{/} \ \Theta \qquad (3)$$

where $\Theta = (\theta_1, \theta_2, \ldots \theta_T)$ and $\theta_t = (CF_t / S)(\ln(1 + z_t) / (1 + z_t)^t)$. The sign on the time value can be ignored by adjusting time to count backwards, e.g, changing time from t=20 to t=19 produces Δt = -1. Taking the Δt to be positive permits the negative sign to be ignored. Using this convention and rearranging the expansion in (2) to incorporate time value gives the surplus change condition:

$$\frac{S(z) - S(z_0)}{S(z_0)} \cong -\Delta i \ [N^{/} \ D_{\underline{T}}] + \Delta i^2 \ [N^{/} \ C_T \ N] + \Delta t \ [N_0^{/} \ \Theta] \qquad (4)$$

In this formulation, the time value component is evaluated using (3) with the spot rates observed at the new location. With a non-zero surplus, satisfying the surplus immunization condition in (2) means that the value of the portfolio will increase by the time value.

One final point arising from the implementation of Reitano's partial duration model concerns the associated convexity calculation. Consider the direct calculation of the partial convexity of surplus, $CON_{i,j}^{S}$, where $i \neq j$:

$$S(z) = \sum_{t=1}^{T} \frac{CF_t}{(1 + z_t)^t} \rightarrow \frac{\partial S(z)}{\partial z_i} = \frac{i \ CF_i}{(1 + z_i)^{i+1}} \rightarrow \frac{\partial^2 S(z)}{\partial z_i \ \partial z_j} = 0 \quad for \quad i \neq j$$

where CF_t is the cash flow at time t for $t = (1,2,...T)$. From (1) and (2), it follows that the quadratic form $N' \ C_T^S \ N$ reduces to:

$$N^{/} \ C_T^{S} \ N = \sum_{t=1}^{T} n_t^2 \ CON_{t,t}^{S} = \sum_{t=1}^{T} n_t^2 \ \frac{1}{S(z_0)}\frac{\partial^2 S(z_0)}{\partial z_t^2}$$

In terms of the extreme bounds on convexity (see Appendix), this is a significant simplification. Because the *TxT* convexity matrix is diagonal, the extreme bounds are now given by the maximum and minimum diagonal ($CON_{i,i}$) elements. If the ith element is a maximal element, the associated optimal *N* vector for the convexity bounds is a *Tx1* with a one in the *i*th position and zeroes elsewhere. Similar to the duration adjustment, to compare $N_1' C_T N_1$ with either the classical convexity or $N_0' C_T N_0$ requires multiplication by *T*.

IV. Key Rates, Cash Flow Dates and the Norm

Reitano (1991a, 1992) motivates the analysis of surplus immunization with a useful example involving a portfolio containing a 5 year zero coupon liability (=\$63.97) together with a barbell combination of two assets, a 12% coupon, ten year bond (=\$43.02) and 6 month commercial paper (=\$25.65; surplus = \$9.28). The initial yield curve is upward sloping with the vector of yields being *y* = (.075, .09, .10) for the 0.5, 5 and 10 year maturities. Consistent with the key rate approach: "Yields at other maturities are assumed to be interpolated" (Reitano 1992, p.37). Reitano derives the vector of partial durations of surplus, $D_{\underline{T}}$, for the three relevant maturity ranges as (4.55, -35.43, 30.88).[7] It follows, for the parallel shift case, $N_0 = (1, 1, 1)'$, that $N_0' D_{\underline{T}} = 0$ corresponds to the classical surplus immunization condition: when the duration of assets equals the appropriately weighted duration of liabilities, the duration of surplus is zero. As a consequence, for a parallel yield curve shift, the change in the portfolio surplus equals zero. To derive the bounds for cases involving non-parallel shifts, Reitano selects the parallel yield curve shift case as the norming vector, that involves imposing a standard length of:

$$\| N_0 \| = \sqrt{N_0{}' N_0} = \sqrt{\sum_{t=1}^{3} n_{0,t}{}^2} = \sqrt{3}$$

From this Reitano is able to identify the extreme bounds on the change in the partial duration of surplus as ($N^{*\prime} \underline{D}_T =$) -81.78 $\leq N' \underline{D}_T \leq$ 81.78, that correspond to an estimate derived from the partial durations for the max $\%\Delta S$ from a shift of length $\sqrt{3}$. The extreme negative yield curve shift is identified as $N^* = (0.167, -1.3, 1.133)$ and the extreme positive shift as $-N^*$. Similar analysis for the convexity of surplus produces extreme positive and negative bounds of $-434.15 \leq N' C_T N \leq 424.04$ with associated shifts of (0.049, 0.376, 1.69)' and (-.306, -1.662, 0.379)'.

The specification of the three element norming vector for the key rate approach is motivated by making reference to market reality where it is not practical to match the dimension of the yield vector with the large number of cash flow dates, e.g., Ho (1992) and Phoa and Shearer (1997). Key rates are used to reduce the dimension of the optimization problem. Reitano selects an example with 3 relevant key rate maturities, one maturity applicable to a 5 year zero coupon liability and two maturities applicable to a 6 month zero coupon asset and a 10 year coupon bond. Even though there are partial durations and convexities associated with the regular coupon payment dates, only three maturity dates are incorporated into the analysis. In addition to empirical simplicity, another advantage of using key rates in Reitano’s model is that the elements of the extreme shift vector, as well as the individual partial durations and convexities, have a realistic appearance.[8] Because the partial durations, convexities and time values depend on the cash flow over a particular payment period, the aggregation of cash flows to key rate maturities permits comparable market value to be used across yield curve segments.

In contrast, the approach in this paper assigns a spot rate to each cash flow date, without regard to the size of the payment on that date. While this is cumbersome in practical applications where there are a large number of cash flow dates, there are distinct theoretical advantages. For example, consider Reitano's focus on a three element vector when there is actually *2T*=20 cash flow payment dates. While this has the apparent advantage that the calculated extreme bound vectors have the appearance of actual yield curve shifts, it gives no guidance as to how far these extreme shifts are from actual or 'most likely' shifts. To see this, consider the extreme shift vector calculated by Reitano, N^* = (0.167, -1.3, 1.133). The empirical likelihood of this particular shift, or a relatively similar shift, actually occurring is remote. Where the yield curve or spot rate vectors are specified with the actual number of cash flow payments, the optical appearance of the elements of the extreme shift vector often have a sawtooth pattern that seems unrealistic.[9] However, this is only a disadvantage if it is the (unlikely) extreme bounds that are of interest. If bounds arising from shift vectors that are empirically determined or 'most likely' are of interest, then assigning spot rates to cash flow dates along the yield curve is most natural. In this case, shift vectors are exogenously specified, either from empirical or *ex ante* estimates. This approach is developed in section VII below.

V. The Duration Bounds

Tables 1-3 provide results for the duration component of the Taylor series expansion: the individual partial durations; the elements, n_t^*, of the extreme shift vector N^*; and the calculated extreme duration bounds. Convexity and time value components are not considered in Tables 1-3. Solving for the partial durations, extreme shift vector and duration bounds requires the specification of *S*, *A* and *L*. Because there are a theoretically infinite number of potential combinations of *A* and *S* that can immunize a given *L*, some form of standardization is required. To this end, Tables 1-3 have been standardized to have the same market value for the liability. Tables 1 and 2 involve a 5 year zero coupon liability while Table 3 uses a 10 year annuity with the same market value as the

5 year zero. Table 1 immunizes the liability with only two assets, a six month zero and a 10 year, 12% semi-annual coupon bond. To illustrate the impact of surplus level on asset portfolio composition, results for a high surplus and a low surplus immunizing asset portfolio are provided. All Tables use the yield curve and spot rates from Fabozzi (1993) as the initial baseline.[10] This curve is upward sloping with a 558 basis point difference between the 6 month (.08) and ten year (.1358) spot interest rates.

Table 1 reports the partial durations, the n_t* and extreme duration bounds calculated from the Cauchy-Schwarz inequality (see Appendix). Comparison of the bounds between the low and high surplus cases depends crucially on the observation that the bounds relate to the percentage change in the surplus, e.g., an extreme bound of ±8.86 means the extreme change in surplus is 8.86%. Due to the smaller position in the 6 month asset, the larger bounds for the low surplus case also translate to a slightly larger extreme market value change when compared to the high surplus case. This result is calculated by multiplying the reported bound by the size of the surplus.[11] As expected, because all cash flow dates are used the extreme shift vector for duration, *N**, exhibits a sawtooth change, with about 80% of the worst shift concentrated on a fall in the 5 year yield and 17-20% on an increase in the 10 year rate.[12] This is an immediate implication of the limited exposure to cash flows in other time periods. However, even in this relatively simple portfolio management problem, the n_t* provide useful information about the worst case shift. With the proviso that the precise connection to unit shifts is obscured, there is not much loss of content to 'fill in' the sawtooth pattern as in the ‘key rate’ approach, due to the small partial durations in the intervening periods. Consistent with basic intuition, the worst type of shift has a sizeable fall in midterm rates combined with smaller, but still significant, rise in long term rates.

In using only the 6 month zero and 10 year bond as assets, Table 1 is constructed to be roughly comparable to the example in Reitano (1991a, 1992, 1996). A possible extension to the asset portfolio is to increase the number of assets to include a maturity matching bond. The impact of including such a bond directly addresses a variant of the 'duration puzzle', albeit in a situation where both portfolios being compared are surplus immunized. Table 2 provides results for two cases with similar surplus levels but with somewhat different asset compositions. One case increases the number of assets by including a par bond with a maturity that matches that of the zero coupon liability (T=5). This is referred to as the ***maturity bond*** portfolio. The other case does not include the maturity matching bond but, instead, increases the number of assets by including 3 and 7 year par bonds. This is referred to as the ***split maturity*** portfolio. For both portfolios the 1/2 year and 10 year bonds of Table 1 are included, with the position in the 10 year bond being the same in both of the Table 2 asset portfolios. In order to achieve surplus immunization, the 1/2 year bond position is permitted to vary, with the maturity matching portfolio holding a slightly higher market value of the 1/2 year asset. *A priori*, the split maturity portfolio would seem to have an advantage as four assets are being used to immunize instead of the three bonds in the maturity matching portfolio.

Given this, the results in Table 2 reveal that the portfolio with the maturity matching bond has a smaller surplus and much smaller extreme bounds even though more bonds are being selected in the split maturity portfolio. The partial durations reveal that, as expected, the presence of a maturity matching bond reduces the partial duration at T=5 compared to the split maturity case. The partial durations at T=3 and T=7 are proportionately higher in the split maturity case to account for the difference at T=5. The small difference in the partial duration at T=10 is due solely to the small difference in the size of the surplus. Examining the n_t* reveals that there is not a substantial difference in the sensitivity to changes in five year rates, as might be expected. Rather, the split maturity portfolio redistributes the interest rate sensitivity along the yield curve. In contrast, the maturity bond portfolio is relatively more exposed to changes in 10 year rates even though the market value of the 10 year bond is the same in both asset portfolios. This greater exposure along the yield curve by the split maturity portfolio results in wider extreme duration bounds because the norming restriction dampens the allowable movement in any individual interest rate. In other words, spreading interest rate exposure along the yield curve by picking assets across a greater number of maturities acts to increase the exposure to spot rate curve shifts of unit length.

Table 3 considers the implications of immunizing a liability with a decidedly different cash flow pattern. In particular, the liability being immunized is an annuity over T=10 with the same market value as the zero coupon liability in Tables 1-2. The immunizing asset portfolios are a ‘maturity matching' portfolio similar to that in Table 2 combining the 6 month zero coupon with 5 year and 10 year bonds. The market value of the 10 year bond is the same as in the Table 2 asset portfolios. The other case considered is a ‘low surplus’ portfolio, similar to that of Table 1, containing the 6 month zero and 10 year bond as assets. Table 3 reveals a significant relative difference between the extreme bounds for the two portfolios compared with the similar portfolios in Tables 1 and 2. The extreme bound for the low surplus portfolio has been reduced to about one third the value of the bound in Table 1 with the N^* vector being dominated by the n_t^* value for T=10. The extreme bound for the maturity bond portfolio has been reduced by just over one half compared to the optimal bound for the Table 2 portfolio with the N^* vector being dominated by the n_t^* values at T=5 and T=10. When the extreme bounds for the two portfolios in Table 2 are multiplied by the size of the surplus, there is not much difference in the potential extreme change in the value of the surpluses between the two portfolios in Table 3. This happens because, unlike the zero coupon 5 year liability of Table 2, the liability cash flow of the annuity is spread across the term structure and the addition of the five year asset provides greater coverage of the cash flow pattern. In the annuity liability case, the dramatic exposure to the T=10 rate indicated by the n_t* of the low surplus portfolio is a disadvantage compared to the maturity bond portfolio which distributes the rate exposure between the T=5 and T=10 year maturities.

VI. Convexity Bounds and Time Value

Table 4 provides incrementally more information on the portfolios examined in Tables 1-3. Certain pieces of relevant information are repeated from Tables 1-3: the surplus and the extreme bounds for duration. In addition, Table 4 provides the time value, the sum of the partial convexities ($N_0'C_T N_0$), the maximum and minimum partial convexities and the quadratic form defined by the duration-optimal-shift convexities, $N^*{}'C_T N^*$, where $N^* = (n_1^*,....,n_T^*)$ is the vector containing the optimal n^*'s from Tables 1-3 and C_T is a diagonal matrix with the $CON_{t,t}$ elements along the diagonal.[13] The quadratic form calculated using the N^* for the duration bound is of interest because it provides information about whether the convexity impact will be improving or deteriorating the change in surplus when the extreme duration shift occurs. Using these measures, Table 4 illustrates the importance of examining the convexity and time value information, in conjunction with the duration results. Of particular interest is the comparison between the maturity matching and the split maturity portfolios of Table 2.

The primary result in Table 2 was that the split maturity portfolio had greater potential exposure to spot rate (yield) curve shifts, as reflected in the wider extreme bounds associated with a spot rate curve shift of length one. Whether this was a positive or negative situation was unclear, as the extreme bounds permitted both larger potential gains, as well larger potential losses, for the split maturity portfolio. Yet, by identifying lower potential variability of the surplus of the maturity bond portfolio, this provides some insight into the duration puzzle. In this vein, Table 4 also reveals that, despite having a smaller surplus, the maturity bond portfolio has a marginally higher time value. This happens because, despite having a higher surplus and a smaller holding of the 1/2 year bond, the split maturity portfolio has to hold a disproportionately larger amount of the three year bond relative to the higher yielding seven year bond. With an upward sloping yield curve, this lower time value is combined with a ***higher*** convexity, as measured by $N_0'C_T N_0$. This is consistent with the results in Christensen and Sorensen (1994) where a tradeoff between convexity and time value is proposed, albeit for a classical stochastic model using a single interest rate process to capture the evolution of the yield curve.[14] As such, there is a connection in the maturity bond portfolio between higher time value, lower convexity and smaller extreme bounds that is directly relevant to resolving the ‘duration puzzle’.

Table 4 also provides a number of other useful results. For example, comparison of the high and low surplus portfolios from Table 1 adds to the conclusions derived from that Table. It is apparent that, all other things equal, the time value will depend on the size of the surplus. However, as illustrated in Table 4, the relationship is far from linear: the surpluses of the two portfolios from Table 1 differ by a factor of 10.9 and the time values differ by a factor of 2.5. High surplus portfolios permit a proportionately smaller amount of the longer term security to be held with corresponding impact on all the various measures for duration, convexity and time value. In addition, unlike the classical interpretation of convexity which is often associated with the single bond case where all cash flows are positive, convexity of the surplus can, in general, take negative values and, in the extreme cases, these negative values can be larger than the extreme positive values. However, this is not always the case, as evidenced in the Table 3 portfolios where the liability is an annuity. The absence of a future liability cash flow concentrated in a particular period produces a decided asymmetry in the Max *CON* and Min *CON* measures for individual $C_{t,t}$ convexities, with the Max values being much larger than the absolute value of the Min values. This is a consequence of the large market value of the 10 year bond relative to the individual annuity payments for the liability.

VII. Immunizing Against Specific Shifts

The appropriate procedure for immunizing a portfolio against arbitrary yield curve shifts is difficult to identify, e.g., Reitano (1996). Some previous efforts that have approached this problem, e.g,, Fong and Vasicek (1984), have developed duration measures with weights on future cash flows depending on a specific stochastic process assumed to drive term structure movements. This introduces 'stochastic process risk' into the immunization problem. If the assumed stochastic process is incorrect the immunization strategy will not perform as anticipated and may even underperform portfolios constructed using classical immunization conditions. In general, short of cash flow matching, it may not be possible to theoretically solve the problem of designing a ***practical*** immunization strategy that can provide "optimal" protection against arbitrary yield curve shifts. In the spirit of Hill and Vaysman (1998), it is possible to evaluate a specific portfolio's sensitivity to predetermined types of yield curve shifts. In practical applications, this will be sufficient for many purposes. For example, faced with a steep yield curve, a portfolio manager is likely to be more concerned about the impact of the yield curve flattening than with a further steepening. If there is some prior information about the expected change in location and shape of the yield curve, it is possible to explore the properties of portfolios that satisfy a surplus immunizing condition at the initial yield curve location.

The basic procedure for evaluating the impact of specific yield curve shifts requires a spot rate shift vector $(\Delta i)\ N_i = \{(z_{1,1} - z_{1,0}), (z_{2,1} - z_{2,0}),....(z_{T,1} - z_{T,0})\}$ to be specified that reflects the anticipated shift from the initial location at $z_0 = (z_{1,0}, z_{2,0}...z_{T,0})$ to the target location $z_1 = (z_{1,1}, z_{2,1}...z_{T,1})$. This step begs an obvious question: what is the correct method for adequately specifying N_i? It is well known that, in order to avoid arbitrage opportunities, shifts in the term structure cannot be set arbitrarily, e.g., Boyle (1978). If a stochastic model is used to generate shifts, it is required that N_i be consistent with absence of arbitrage restrictions on the assumed stochastic model. These restrictions, which apply to the set of all possible paths generated from the stochastic model, are not needed when the set of assumed future shifts is restricted to yield curve movements based on historical experience or *ex ante* expectations. Where such empirical shifts are notional, relevant restrictions relate to maintaining consistency between individual spot rates. In terms of implied forward rates, these restrictions take the form:

$$(1 + z_j)^j = (1 + z_i)^i (1 + f_{i,j})^{j-i} = (1 + z_1)(1 + f_{1,2})(1 + f_{2,3})....(1 + f_{j-1,j})$$

where the implied forward rates are defined as $(1 + f_{1,2}) = (1 + z_2)^2 / (1 + z_1)$ and $f_{j-1,j} = (1 + z_j)^j / (1 + z_{j-1})^{j-i}$ with other forward rates defined appropriately. This imposes a smoothness requirement on spot rates restricting the admissible deviation of adjacent spot rates.

In addition to smoothness restrictions on adjacent spot rates, the use of the partial duration approach requires that admissible N_i shifts satisfy the norming condition $\|N\| = 1$. In the associated set of unit length spot rate curve shifts, there are numerous shifts which do not satisfy the spot rate smoothness requirement. Because smoothing will allocate a substantial portion of the unit shift to spot rates that have small partial durations, restricting the possible shifts by using smoothness restrictions tightens the convexity and duration bounds compared to the extreme bounds reported in Tables 1-4. To illustrate this, three scenarios for shifting the initial yield curve are considered: flattening with an upward move in level, holding the T=10 spot rate constant; flattening with a downward move in level, holding the T=6 month rate constant; and, flattening with a pivoting around the T=5 rate, where the $T > 5$ year rates fall and the $T < 5$ year rates rise. In empirical terms, these three shift scenarios are plausible unit length shifts. Given these scenarios, what remains is to specify the elements of N_i for shifts of unit length. The (absence of arbitrage) smoothness restrictions require that changes in yield curve shape will distribute the shift proportionately along the yield curve. For example, when flattening with an upward move in level, the change in the T=6 month rate would be largest, with the size of the shift getting proportionately smaller as T increases, reaching zero at T=10.

Solving for a factor of proportionality in the geometric progression, subject to satisfaction of the norming condition, produces a number of possible solutions, depending on the size of the spot rate increase at the first step. The following three unit length N_i shift vectors were identified:

Time	**Flatten Up (YC1)**	**Flatten Down (YC2)**	**Pivot (YC3)**
1	0.400	0.000	0.371
2	0.368	-0.090	0.180
3	0.339	-0.097	0.164
4	0.312	-0.106	0.146
5	0.287	-0.115	0.126
6	0.264	-0.125	0.105
7	0.243	-0.136	0.082
8	0.224	-0.147	0.057
9	0.206	-0.160	0.030
10	0.189	-0.174	0.000
11	0.174	-0.189	-0.032
12	0.160	-0.206	-0.067

13	0.147	-0.224	-0.105
14	0.136	-0.243	-0.146
15	0.125	-0.264	-0.191
16	0.115	-0.287	-0.240
17	0.106	-0.312	-0.293
18	0.097	-0.339	-0.351
19	0.090	-0.368	-0.413
20	0.000	-0.400	-0.481

As in (1) and (2), the actual change in a specific spot rate requires the magnitude of the shift to be given. Observing that each of these three scenario N_i vectors is constructed to satisfy the norming condition $\| N \| = 1$, the empirical implications of this restriction are apparent. More precisely, unit length shifts do not make distinction between the considerably higher volatility for changes in short term rates compared to long term rates. Imposing both unit length shift and smoothness restrictions on spot rates is not enough to restrict the set of theoretically admissible shifts to capture all aspects of empirical consistency. While it is possible impose further empirically-based restrictions on the set of admissible shifts, for present purposes it is sufficient to work with these three empirically plausible spot rate curve shift scenarios.

Given the three unit length spot rate curve shifts, Table 5 provides the calculated values associated with (2) and (3) for the six portfolios of Tables 1-3. Because the initial duration of surplus is approximately zero and the portfolio convexity ($N_0' \, C_T \, N_0$) is positive in all cases, classical immunization theory predicts that the portfolio surplus will not be reduced by interest rate changes. The information in Table 5 illustrates how the partial duration approach generalizes this classical immunization result to assumed non-parallel spot rate curve shifts. Comparison of the size of $N_i'\underline{D_T}$ with the Cauchy bound reveals the dramatic reduction that smoothness imposes on the potential change in surplus value. In particular, from (2) it follows that a negative value for the partial duration measure $N_i'\underline{D_T}$ is associated with an ***increase*** in the value of the fund surplus projected by the duration component. All such values in Table 5 are negative, consistent with $N_i{'}D_{\underline{T}}$ indicating all three curve shifts produce an increase in the value of surplus. As in Tables 1-3, the change in the surplus from the duration component can be calculated by multiplying the surplus by the $N_i{'}D_{\underline{T}}$ value and the assumed shift magnitude Δi. For every portfolio, the YC3 shift produced a larger surplus increase from the duration component than YC1 and YC2. Given that the YC3 shift decreases the interest rate for the high duration 10 year asset and decreases the interest rate for the low duration 6 month asset, this result is not surprising. In contrast, while the YC2 shift increased surplus more than YC1 in most cases, the reverse result for the maturity bond portfolio with the annuity liability indicates that portfolio composition can matter when the spot rate curve shift is non-parallel.

Following Chance and Jordan (1996) and Poitras (2005, p.275), interpreting the contribution from convexity depends on the assumed shift magnitude Δi^2. A positive value for the convexity component ($N_i{'}\ C_T\ N_i$) indicates an improvement in the surplus change in addition to the increase from $N_i{'}D_{\underline{T}}$. For all portfolios, there was small negative contribution from convexity for the spot rate curve flattening up (YC1). When multiplied by empirically plausible values for Δi^2 the negative values are small relative to $N_i{'}D_{\underline{T}}$, indicating that the additional contribution from convexity does not have much additional impact. Results for the spot rate curve flattening down (YC2) and the pivot (YC3) produced positive convexity values for all portfolios. While for empirically plausible shift magnitudes the convexity values for the YC2 shift were also not large enough to have a substantial impact on the calculated change in surplus, the YC3 values could have a marginal impact if the shift magnitudes were large enough. For example, assuming Δi = .01, the 23.7% surplus increase predicted by $N_i{'}D_{\underline{T}}$ in the low surplus portfolio in Table 1 is increased by 1.12% from the convexity contribution. Also of interest is the magnitude of $N_i{'}\ C_T\ N_i$ relative to $N_0{'}\ C_T\ N_0$. While the calculated $N_i'D_T$ term is small in comparison to the Cauchy bound even for YC3, the calculated N_i' $C_T N_i$ is over half as large as $N_0' C_T N_0$ for YC3 and more than one third the value for YC2.

Results for partial duration and partial convexities are relevant to the determination of the change in surplus associated with various non-parallel yield curve shifts. Table 5 also reports results for the change in time value for the six portfolios and three spot rate curve shift scenarios. From (3) and (4), time value measures the rate of change in the surplus if the yield curve remains unchanged over a time interval. For portfolios with equal duration and different convexities, such as those in Table 2, differences in time value reflect the cost of convexity. For a steep yield curve, the cost of convexity is high and for a flat yield curve the cost of convexity is approximately zero. From (3), it is apparent that yield curve shifts will also impact the time value. While YC1-YC3 all reflect a flattening of the spot rate curve, the level of the curve after the shift is different. In addition, because the calculation of time value involves discounting of future cash flows, it is not certain that an upward flattening in the level of the spot rate curve (YC1) will necessarily produce a superior increase in time value compared to a flattening pivot of the curve (YC3).

Significantly, the YC3 shift produced the largest increases in surplus for all portfolios except the high surplus portfolio of Table 1 where the YC1 shift produced the largest increase in time value. In all cases, the YC2 shift produced the smallest increase in time value. These results are not apparent from a visual inspection of the different shifts, which appear to favour YC1 where spot rates increase the most at all maturity dates. To see how this occurs, consider the partial time values from the low surplus portfolio in Table 1. For the initial yield curve, θ_t associated with the largest cash flows are $\theta_1 = .18698$, $\theta_{10} = -.97942$ and $\theta_{20} = .30677$. For YC3, these values become $\theta_1 = .19513$, $\theta_{10} = -.97942$ and $\theta_{20} = .30990$ while for YC1 the values are $\theta_1 = .19576$, $\theta_{10} = -.98683$ and

θ_{20} = .30677. While the pivot leaves the time value of the liability unchanged and increases the time value of the principal associated with the10 year bond, the overall upward shift in rates associated with YC1 is insufficient to compensate for the negative impact of the rate increase for the liability. Comparing this to the high surplus case where YC1 had a superior increase in time value compared to YC3, the initial yield curve values for the high cash flow points are θ_1 = .056045, θ_{10} = -.090227 and θ_{20} = .026552 which change to θ_1 = .058678, θ_{10} = -.09091 and θ_{20} = .034144 for YC1 and θ_1 = .058489, θ_{10} = -.090227 and θ_{20} = .034493. Because the higher surplus portfolio has more relative asset value, the higher overall level of rates associated with YC1 can be reflected in the time value change

Finally, Table 5 provides further evidence on the duration puzzle. Based on the results in Table 4, comparison of the maturity matching portfolio with the split maturity portfolio reveals the time value-convexity tradeoff identified by Christensen and Sorensen (1994). The higher reported time value measure for the maturity bond portfolio was offset by the higher convexity values for the split maturity portfolio. Under classical conditions, this implies a superior value change for the split maturity portfolio if interest rates change sufficiently. Yet, in Table 5 the maturity matching portfolio has a larger change in surplus than the split maturity portfolio for the flattening up shift (YC1) while retaining the time value advantage across all three scenarios. However, for both the flattening down and the pivot shifts, the surplus increase for the split maturity portfolio does outperform the maturity matching portfolio as expected. This is another variant of the duration puzzle. Because the result does not apply to all three scenarios, this implies that the duration puzzle is not a general result but, rather, is associated with specific types of yield curve shifts. If this result extends to real time data, the presence of the duration puzzle can be attributed to the prevalence of certain types of yield curve shifts compared to other types. The presence of the duration puzzle in the selected scenarios is due to a complicated interaction between the partial durations, partial convexities and time values. As such, the partial duration approach is well suited to further investigation of this puzzle.

VIII. Conclusion

This paper extends the partial duration, surplus immunization model of Reitano (1992, 1996) to include a measure of the time value. This measure captures the impact of the force of interest function on the change in fund surplus. The extreme duration and convexity bounds for assessing the impact of non-parallel yield curve shifts on fund surplus provided in the Reitano model are examined and it is demonstrated that these bounds are considerably wider than actual surplus changes for shifts that are likely to occur. Recognizing that classical immunization is an idealized objective, the paper provides a range of measures – based on the partial durations, convexities and time values – that can be used to assess different aspects of portfolio immunization and return performance. These measures are used to examine a variant of the 'duration puzzle' where the performance of an immunizing portfolio with a maturity matching bond is compared with that of an immunizing portfolio without an asset that matches the maturity of the zero coupon liability. It is demonstrated that the superior performance of the maturity matching bond depends on the type of assumed shift and is not a general result. In addition, the partial duration, convexity and time value measures provide more detailed evidence on the tradeoff between convexity and time value identified by Chance and Jordan (1996), Barber and Copper (1997) and others.

In the spirit of Redington (1952), there is a considerable distance to travel from the simple portfolio illustrations of immunization theory presented in this paper to the complex risk management problems arising in financial institutions such as pension funds, life insurance companies, securities firms and depository institutions (e.g., Poitras 2006). In particular, cash flow patterns in financial institutions are decidedly more complicated. Not only are the cash flows more numerous, there is also an element of randomness that is not easy to model. This paper develops a theoretical method for capturing the impact that non-parallel yield curve shifts and time values have on fixed income portfolio returns. This is done by increasing the amount of data that is needed to implement the risk management strategy. Whereas classical duration can make use of the simplification that the portfolio duration is the value weighted sum of the durations of the individual assets and liabilities, the approach used here requires the net cash flows at each payment date to be determined. In simple portfolio illustrations this requirement is not too demanding. However, this requirement could present problems to a financial institution faced with large numbers of cash flows, a significant fraction of which may be relatively uncertain. Extending the partial duration approach to incorporate randomness in cash flows would provide further insight into practical immunization problems.

Appendix

Solution for the Optimal Duration Weights $n_1, n_2, ... , n_T$:

The optimization problem is to set the *N* weighted sum of the partial durations of surplus equal to zero, subject to the constraint that the *N'N* equals the norming value. Without loss of generality assume that the norming value is one, which leads to:

$$\underset{\{n_i\}}{opt}\ L = \sum_{i=1}^{k} n_i\ D_i - \lambda(\{\sum_{i=1}^{k} n_i^2\} - 1)$$

This optimization leads to $k + 1$ first order conditions in the $k\ n_i$ and λ. Observing that the first order condition for the n_i can be set equal to $\lambda/2$, appropriate substitutions can be made into the first order condition for λ that provides the solution for the individual weights.

Solution for the Optimal Convexity Weights $n_1, n_2, ... , n_T$:

The optimization problem for convexity involves the objective function:

$$\underset{\{n_i\}}{opt}\quad L_c = N'\ C_T\ N - \lambda\ (N'N - 1)$$

Recognizing that C_T is a real symmetric matrix permits a number of results from Bellman (1960, Sec.4.4, Sec. 7.2) to be accessed. In particular, if *A* is real symmetric then the characteristic roots will be real and have characteristic vectors (for distinct roots) that are orthogonal. Ordering the characteristic roots from smallest to largest, the following bounds apply to the quadratic form $N'C_T\ N$:

$$\lambda_1 \geq \lambda_2 \geq ..... \geq \lambda_t \qquad \rightarrow \qquad \lambda_1 \geq N'\ C_T\ N \geq \lambda_T$$

Morrison (1976, p.73) develops these results further by recognizing that the solution to the optimization problem corresponds to the defining equation for characteristic vectors: $[C_T - \lambda I]N = 0$, where premultiplication by *N'* and use of the constraint gives: $\lambda = N'\ C_T\ N$. Hence, for the maximum (and minimum) characteristic roots of C_T, the optimum shift vector is the characteristic vector associated with that characteristic root.

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Table 1

Partial durations, $\{n_t\}$ and extreme bounds for the 5 Year Zero Coupon Liability Immunized with High and Low Surplus Examples*

	High Surplus		Low Surplus	
Date	D_t	n_t*	D_t	n_t*
0.5	0.687	0.0775	2.292	0.0236
1.0	0.075	0.0085	0.870	0.0089
1.5	0.107	0.0121	1.238	0.0128
2.0	0.136	0.0153	1.569	0.0162
2.5	0.161	0.0182	1.863	0.0192
3.0	0.183	0.0206	2.111	0.0217
3.5	0.201	0.0226	2.318	0.0239
4.0	0.214	0.0241	2.472	0.0254
4.5	0.226	0.0255	2.611	0.0269
5.0	-7.933	-0.89545	-86.115	-0.88658
5.5	0.244	0.0275	2.815	0.0290
6.0	0.246	0.0277	2.837	0.0292
6.5	0.247	0.0279	2.854	0.0294
7.0	0.246	0.0278	2.845	0.0293
7.5	0.243	0.0275	2.810	0.0289
8.0	0.242	0.0273	2.792	0.0287
8.5	0.239	0.0269	2.758	0.0284
9.0	0.231	0.0260	2.664	0.0274
9.5	0.221	0.0249	2.551	0.0263
10.0	3.786	0.42733	43.742	0.45033
Extreme Duration Bounds:	Cauchy = $\|D\|$ ± 8.860%		Cauchy = $\|D\|$ ± 97.131%	
Surplus:	50.75		4.66735	

* The market value of the High Surplus Portfolio is composed of ($68.3715) 1/2 year zero coupon and ($69.89445) 10 year semi-annual coupon bonds. The market value of the Low Surplus Portfolio is composed of ($17.8382) 1/2 year and ($74.343) 10 year bonds. The liability for both the High Surplus and Low Surplus Portfolios is a 5 year zero coupon bond with $150 par value and market value of $87.51. The extreme Cauchy bounds are derived using $\| N \| = 1$.

Table 2

Partial durations, $\{n_t\}$ and extreme bounds for the Five Year Zero Coupon Liability Immunized with the Maturity Bond and Split Maturity Examples*

	Maturity Bond		Split Maturity	
Date	D_t	n_t*	D_t	n_t*
0.5	0.476	0.0183	0.251	0.0063
1.0	0.454	0.0174	0.446	0.0112
1.5	0.646	0.0248	0.635	0.0159
2.0	0.818	0.0314	0.804	0.0201
2.5	0.971	0.0373	0.955	0.0239
3.0	1.101	0.0422	7.737	0.1939
3.5	1.208	0.0464	0.834	0.0209
4.0	1.288	0.0494	0.889	0.0223
4.5	1.361	0.0522	0.939	0.0235
5.0	-23.866	-0.91558	-36.998	-0.92716
5.5	0.636	0.0244	1.012	0.0254
6.0	0.641	0.0246	1.020	0.0256
6.5	0.645	0.0247	1.026	0.0257
7.0	0.643	0.0247	8.177	0.2049
7.5	0.635	0.0244	0.601	0.0151
8.0	0.631	0.0242	0.597	0.0150
8.5	0.623	0.0239	0.590	0.0148
9.0	0.602	0.0231	0.569	0.0143
9.5	0.577	0.0221	0.545	0.0137
10.0	9.884	0.37921	9.350	0.23430
Extreme Duration Bounds:	Cauchy = ‖D‖ ± 26.07%		Cauchy = ‖D‖ ± 39.905%	
Surplus:	10.32685		10.91804	

* The market value of the Maturity Bond Portfolio is composed of ($5.13) 1/2 year, ($55.54) 5 year and ($37.172) 10 year bonds. The market value of the Split Maturity Portfolio is composed of ($0.4105) 1/2 year, ($33.8435) 3 year, (27.0) 7 year and ($37.172) 10 year bonds. The liability is a 5 year zero coupon bond with $150 par value and market value of $87.51. The extreme Cauchy bounds are derived using ‖ N ‖ = 1.

Table 3

Partial durations, $\{n_t\}$ and extreme bounds for the 10 Year Annuity Liability Immunized with the Maturity Bond and Low Surplus Examples*

	Maturity Bond		Low Surplus	
Date	D_t	n_t*	D_t	n_t*
0.5	1.034	0.0841	2.856	0.0869
1.0	-0.275	-0.0224	-0.705	-0.0214
1.5	-0.392	-0.0319	-1.003	-0.0305
2.0	-0.497	-0.0404	-1.271	-0.0387
2.5	-0.590	-0.0480	-1.509	-0.0459
3.0	-0.668	-0.0543	-1.710	-0.0520
3.5	-0.734	-0.0597	-1.877	-0.0571
4.0	-0.782	-0.0636	-2.001	-0.0609
4.5	-0.826	-0.0672	-2.115	-0.0643
5.0	8.257	0.67159	-2.201	-0.0670
5.5	-1.401	-0.1139	-2.280	-0.0694
6.0	-1.412	-0.1148	-2.298	-0.0699
6.5	-1.420	-0.1155	-2.311	-0.0703
7.0	-1.415	-0.1151	-2.304	-0.0701
7.5	-1.398	-0.1137	-2.276	-0.0693
8.0	-1.389	-0.1130	-2.262	-0.0688
8.5	-1.372	-0.1116	-2.234	-0.0680
9.0	-1.325	-0.1078	-2.157	-0.0656
9.5	-1.269	-0.1032	-2.066	-0.0629
10.0	7.855	0.63888	31.647	0.9629
Extreme Duration Bounds:	Cauchy = ‖D‖ ± 12.29%		Cauchy = ‖D‖ ± 32.87%	
Surplus:	10.5979		4.68	

* The market value of the Maturity Bond Portfolio is composed of ($25.97) 1/2 year, ($34.956) 5 year and ($37.172) 10 year bonds. The market value of the Low Surplus Portfolio is composed of ($31.388) 1/2 year and ($60.791) 10 year bonds. The liability has market value of $87.51 with annual coupon, paid semi-annually, of $14.96. The extreme bounds are derived using ‖ N ‖ = 1.

Table 4

Time Values, Convexity and Other Measures for the Immunizing Portfolios*

TABLE 1 (5 Year Zero Liability)	High Surplus	Low Surplus
Surplus	50.75	4.667
Time Value = 2 $N_0'\Theta$	0.07118	0.0278
$N_0'\ C_T\ N_0$	17.18	221.94
$N^{*\prime}\ C_T\ N^*$	-25.34	-259.59
Max CON_t	37.23	430.09
Min CON_t	-41.34	-448.79
Cauchy Duration Bound	± 8.86%	± 97.13%
$N_0'D_{\underline{T}}$	-0.000	-0.102

TABLE 2 (5 Year Zero Liability)	Maturity Bond	Split Maturity
Surplus	10.327	10.918
Time Value = 2 $N_0'\Theta$	0.0664	0.0648
$N_0'\ C_T\ N_0$	42.37	44.21
$N^{*\prime}\ C_T\ N^*$	-85.39	-142.06
Max CON_t	97.19	91.93
Min CON_t	-124.38	-192.81
Cauchy Duration Bound	± 26.07%	± 39.90%
$N_0'D_{\underline{T}}$	-0.025	-0.022

TABLE 3 (10 Year Annuity Liability)	Maturity Bond	Low Surplus
Surplus	10.598	4.68
Time Value = 2 $N_0'\Theta$	0.0724	0.0510
$N_0'\ C_T\ N_0$	11.96	109.34
$N^{*\prime}\ C_T\ N^*$	21.61	287.65
Max CON_t	77.23	311.17
Min CON_t	-11.89	-19.36
Cauchy Duration Bound	±12.29%	±32.86%
$N_0'D_{\underline{T}}$	-0.019	-0.077

* See Notes to Tables 1-3. Multiplying by 2 to make appropriate adjustment to convert semiannual to annual rates, the time value $N_0'\Theta$ is defined in (3). The sum of the partial convexities is $N_0'C_T\ N_0$ The quadratic form, $N^{*\prime}\ C_T\ N^*$, is the sum of squares for the relevant N* from Tables 1-3 multiplied term-by-term with the appropriate partial convexities. Max CON and Min CON are the maximum and minimum individual partial convexities.

Table 5

Partial Durations and Convexities
for the Immunizing Portfolios under
Different Yield Curve Shift Assumptions*

TABLE 1 (5 Year Zero Liability)

	High Surplus			Low Surplus		
Surplus		50.75			4.667	
	YC1	YC2	YC3	YC1	YC2	YC3
$N_i'D_T$	-0.606	-.879	-1.875	-8.205	-11.127	-23.75
$N_i'C_T N_i$	-0.828	6.082	9.778	-9.398	71.151	112.23
$2 N_0'\Theta_i$	.09144	.0878	.08922	.0436	.0408	.0554
Cauchy Bound ($\|D_T\|$)		±8.86%			±97.13%	
$N^{*\prime} C_T N^*$		-25.34			-259.59	
$N_0' C_T N_0$		17.18			221.94	
Time Value = $N_0'\Theta$		.07118			.0274	

TABLE 2 (5 Year Zero Liability)

	Maturity Bond			Split Maturity		
Surplus		10.327			10.918	
	YC1	YC2	YC3	YC1	YC2	YC3
$N_i'D_T$	-1.532	-2.328	-4.955	-1.451	-2.478	-5.289
$N_i'C_T N_i$	-1.976	15.667	25.489	-2.039	16.281	25.51
$2 N_0'\Theta_i$	.07274	.06864	.07398	.07108	.06674	.07182
Cauchy Bound ($\|D_T\|$)		±26.06%			±39.90%	
$N^{*\prime} C_T N^*$		-85.39			-142.06	
$N_0' C_T N_0$		42.37			44.21	
Time Value = $N_0'\Theta$		.0664			.0648	

TABLE 3 (10 Year Annuity Liability)

	Maturity Bond			Low Surplus		
Surplus		10.598			4.68	
	YC1	YC2	YC3	YC1	YC2	YC3
$N_i'D_T$	-0.874	-0.638	-1.360	-5.086	-5.287	-11.298
$N_i'C_T N_i$	-0.847	5.902	12.01	-5.039	36.520	62.55
$2 N_0'\Theta_i$	.07736	.07408	.07592	.0616	.0600	.07156
Cauchy Bound ($\|D_T\|$)		±12.29%			±32.86%	
$N^{*\prime} C_T N^*$		21.61			267.49	
$N_0' C_T N_0$		11.96			109.34	
Time Value = $N_0'\Theta$		.0724			.0510	

* See Notes to Tables 1-3. YC1 has the spot rate curve flattening up, with the T=10 rate constant; YC2 has the spot rate curve flattening down with the T=.5 (6 month) rate constant; and, YC3 has a flattening pivot with the T=5 year rate constant. $2N_0'\Theta_i$ annualizes (3) evaluated using the N_i shifted spot rates.

NOTES

1. Reitano can be credited with introducing the terms "partial duration", "partial convexity" and the "direction vector".

2. *H.O.T.* refers to higher order terms which will be ignored. This assumption, which could be problematic for bonds with special features, such as call provisions, implies that the discussion centres on default free, straight bonds. At this point, it is possible to reduce the dimension of the vector of spot rates, and the associated partial derivatives, using key rates or other factors in place of spot rates.

3. In applying the Cauchy-Schwarz inequality, Reitano recognizes that the extreme bounds are achieved when the vectors are collinear. In effect, the extreme yield curve shift is calculated and these values are used to calculate the bounds using $N^{*\prime}\underline{D}_T$. The result is tighter bounds than those provided by direct application of the Cauchy-Schwarz values. In Reitano's case, the upper and lower Cauchy-Schwarz bounds, calculated from the product of the inner products of *N* and *D*, are 303.6 and -303.6, respectively.

4. D'Antonio and Cook (2004) provide a useful overview of various approaches to the classical convexity formulas.

5. Shiu (1990, p.171-2) demonstrates that, in the classical immunization framework, because the force of interest is constant, the time value will equal a constant.

6. In actuarial science, the terminology 'force of interest' (function) is used in place of 'time value' (function), e.g., Kellison (1991). Observing the (4) has the appearance of a Taylor series expansion of *S[z,t]*, exact determination of the second order Taylor series expansion would involve the inclusion of the second derivative information for the time values, i.e., the second cross derivative terms and the second derivative with respect to time, e.g., Poitras (2005, p.279-83). Consistent with *a priori* reasoning, Chance and Jordan (1996) and Poitras (2005) indicate that these terms are likely to be relatively small. Hence, only information from the first derivative with respect to time is used.

7. These values can be determined directly from the formula for the partial duration, taking account of the relevant par values and, where applicable, the coupon cash flows of the various assets and liabilities as given in Reitano.

8. Identifying key rates as "factors", it is possible to extend the discussion to include the directional duration models where, instead of key rates, empirically determined estimates of yield curve slope, shape and location are used.

9. This is because the size of individual partial durations and convexities depend on the size of the cash flow on the payment date. By using key rate durations and interpolating yields off key rates, the sawtooth pattern associated with the extreme upper and lower bounds will be smoothed out. If key rates are not used, then the n_t* will reflect the size of the cash flow in period *t*, small cash flow periods will have n_t* close to zero and large cash flow periods will have relatively large n_t*.

Consistent with the approach, in Tables 1-3 it would be possible to aggregate the partial duration and *N* elements across cash flow dates and report a smaller number of these values.

10. This ***par bond*** curve has semi-annual yields from 6 months to 10 years, i.e., *y* = (.08, .083, .089, .092, .094, .097, .10, .104, .106, .108, .109, .112, .114, .116, .118, .119, .12, .122, 124, .125)'. The 6 month and 1 year yields are for zero coupon securities, with the remaining yields applying to par coupon bonds. This yield curve produces the associated spot rate curve, *z* = (.08, .083, .0893, .0925, .0946, .0979, .1013, .106, .1083, .1107, .1118, .1159, ,1186, .1214, .1243, .1256, .1271, .1305, .1341, .1358)'.

11. The relevant calculation for the change in surplus value with a shift magnitude of $\Delta i = .01$ using the extreme Cauchy bound for the low surplus portfolio would be (.01)(97.131)(4.66735) = 4.5334. For the high surplus portfolio the solution is (.01)(8.86)(50.75) = 4.49645.

12. To see this, recall that the length of the shift vector is one. As a consequence, the sum of the squares will equal one and the square of each n^* is the percentage contribution of that particular rate to the extreme directional shift vector.

13. This follows from the previous discussion in section III which identified the off-diagonal elements as being equal to zero.

14. In these one factor models, the spot interest rates for various maturities are constructed by taking the product of the generated one period interest rates along each specific paths, treating each future short term interest rate as a forward rate. This will produce the predicted spot rate structure for each path, e.g., Fabozzi (1993, Chp.13). While useful for generating analytical results, it is well known that such processes cannot capture the full range of potential term structure behaviour. Without restrictions on the paths, some paths may wander to zero and the average over all the paths may not equal the actual observed spot interest rate curve.