

part 2

Fixed Income Valuation

Since the chief emphasis must be placed on the avoidance of loss, bond selection is primarily a negative art. It is a process of exclusion and rejection, rather than of search and acceptance. In this respect the contrast with common-stock selection is fundamental in character. The prospective buyer of a given common stock is influenced more or less equally by the desire to avoid loss and the desire to make profit. The penalty for mistakenly rejecting the issue may conceivably be as great as that for mistakenly accepting it. But an investor may reject any number of good bonds with virtually no penalty at all, provided he does not eventually accept an unsound issue.

Graham and Dodd (1934) on the selection of corporate bonds

The difficulties connected with the problem of arriving at a concept of duration are, indeed, extremely great.

Macauley (1938, p. 52) on the concept of duration

It should always be kept in mind that corporate managements generally do not have bondholders' interests at heart; they are elected by, and beholden to, the owners of the business, namely the common shareholders. Their duty is to increase shareholder wealth, not that of bondholders. They do not owe any fiduciary duty to bondholders, their only responsibility is contractual.

Wilson and Fabozzi (1996, p. 114)



Chapter Summary

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chapter 4

Basics of Fixed Income Valuation

4.1 BASICS OF FIXED INCOME SECURITIES¹

The philosophy of fixed income analysis

Section 1.3 dedicated considerable attention to expanding the notion that Finance is a human science. For this purpose, the uncertainty associated with common stock valuation provided a useful basis for illustrating why the approach and techniques of the natural sciences are not fully adequate to develop understanding of problems being examined in Finance. Other than the historical discussion of life annuity valuation, only passing attention was given to fixed income securities. This omission was intentional, as fixed income security valuation poses a different shade of the epistemological problems than were raised for common stock valuation. Fixed income security analysis is based on valuation methods, derived from discounted cash flow models, that are "accurate" when applied appropriately. These valuation methods also are the basic tools for, say, actuarial science, which is a branch of applied mathematics. Is it possible that the epistemology of the natural sciences is appropriate for the fixed income element of Finance?

Sorting out the philosophy of fixed income analysis poses something of a quandary. The methods and procedures used to value these securities differ in form, if not in substance, from those used to value common stocks. Though still present, uncertainty plays a different and seemingly less pervasive role in fixed income analysis. To value the return to common stocks in excess of the riskless rate of return, modern Finance uses the concept of systematic risk that is measured relative to an "asset market" proxied by a broadly based domestic stock market index such as the S&P 500. In contrast, the return on fixed income securities is decomposed into a combination of the riskless return and components associated with term to maturity, coupon, default risk, option features, tax status and the like. The notion of an "asset market return" similar to that for common stocks is difficult to identify in the case of fixed income securities. Instead, the comparable concept for fixed income securities focuses on the level and term structure of a set of default free securities, usually proxied by available maturities for US Treasury debt.

Examined in isolation, with no reference to other securities, the valuation of default free fixed income securities could be approached as a purely mathematical problem. However, this would require unrealistic assumptions to be made. For example, if the default free security were a fixed coupon bond, then some assumption would have to be made about the interest rate at which the future coupon payments were reinvested. Alternatively, it could be assumed that the default free securities were zero coupon bonds, but this still raises the problem of comparing the certain return on a k period bond with the uncertain return from investing in k successive one period zero coupon bonds. It does not take much consideration to see that the semantics of "fixed income" securities is misleading. All that is fixed is the size of the coupon payment, and for some "fixed income" securities even this may not be fixed, e.g., tontines, mortgage pass throughs, floating rate notes. In other cases, the term over which the fixed payments are made may be unknown, e.g., life annuities, callable bonds.

Though the cash flow patterns of fixed income securities differ from common stocks in being more well defined, the problem of dealing with uncertainty is still present. In addition, situations where a decision has to be made about, say, whether to purchase a common stock or a bond or how to allocate capital invested in a portfolio between bonds and stocks, there is still the problem of assessing the uncertainty associated with common stock valuation. In other words, investment strategy requires assessments of relative valuation. Even in the extreme case where one type of security has a purely deterministic return, the decision about whether to purchase such a security and, if so, how much, still has to deal with the other securities that are available for purchase. As such, it is not possible to develop a philosophy of fixed income analysis that would conform to the epistemological approach of the natural sciences.

The contrast between actuarial science and fixed income analysis is revealing. There are some close correspondences. However, actuarial science, the mathematical science of insurance, is not concerned with the range of risks that are conventionally encountered in securities markets. Actuaries are concerned with the probability of loss versus no loss. Many of the risks encountered in securities markets are *speculative risks*, where there is a possibility of loss, as well as a possibility of gain. Such risks can be distinguished from *pure risks* that involve situations with only the chance of loss or no loss (Vaughan 1982, p. 8): "The distinction between pure and speculative risks is an important one, because normally only pure risks are insurable. Insurance is not concerned with the protection of individuals against those losses arising out of speculative risks. Speculative risk is voluntarily accepted because of its two-dimensional nature, which includes the possibility of gain." It is apparent that the valuation problems being modeled differ between actuarial science and fixed income analysis. The problem of determining the actuarially sound premium for a life insurance policy is substantively different than determining whether, say, a corporate bond is a desirable investment at a given price.

Given this, actuarial science and fixed income analysis still have some close correspondences. As Graham and Dodd (1934) observe: "Since the chief emphasis must be placed on the avoidance of loss, bond selection is primarily a negative art." The methods of valuation for fixed income securities are more mathematical and empirically precise than for common stocks. If an investor has a well defined investment horizon, it is even possible to buy a fixed income security that provides a return that is known

with relative certainty. For fixed income securities issued by corporations, the uncertainty associated with both the cash flow and the asset base sustaining the security is less than for common stocks because of the higher priority of claim. As such, the inductive method of the natural sciences explains substantially more of security analysis for fixed income securities than for common stocks. Yet, there is still the human element in fixed income security valuation and the linear progression of knowledge is still unavailable. In the end, fixed income valuation still lies within the realm of the human sciences.

Given that analysis of fixed income securities has to account for the human aspect, this analysis does proceed much farther towards an understanding of the valuation problem than is the case with common stocks. The essential characteristics of fixed income securities are more readily adaptable to the techniques of discounted cash flow analysis. While there are problems with determining the cash flows and discount rates, there is relatively little debate about the applicability of the discounted cash flow model. The same cannot be said about common stock valuation where it is not clear whether discounted cash flow analysis can be effectively applied. This leads to a fundamental question confronting security analysis: to what extent can the discounted cash methods used to value fixed income securities be applied in the valuation of common stocks? Is security analysis a subject with a homogeneity of technique or is there two or more distinct subjects involved, each requiring a distinct approach to the valuation problem?

Types of securities and markets

Section 1.1 gave a cursory overview of the types of fixed income securities that are available. Various classification schemes are available: by type of issuer, e.g., corporate vs. government; by country of issue, e.g., national vs. international; by credit rating, e.g., investment grade vs. speculative; classification by features, e.g., callable, puttable, convertible, and tax status; and, by term to maturity, e.g., bond market vs. money market. In the US, a breakdown by type of issuer reveals the relative importance of mortgage related issues, effectively collateralized mortgage obligations and mortgage pass throughs. This category and the corporate debt category are the two largest and are the two categories most characterized by issues with embedded options. The municipal government debt category, which includes both state and local issuers, is characterized by issues with favorable tax status. Including other issues with special features, e.g., callable US Treasury bonds, it is apparent that a large part of the US publicly traded debt market is characterized by special features.

Table 4.1 is only given as a crude reference point. There are substantial difficulties in assembling numbers on the relative size of the outstanding issues. For example, as illustrated in table 4.2, the outstanding amount reported for the US Treasury ignores non-marketable debt, such as US Savings bonds, and other types of US federal government debt obligations such as those for intra-government holdings of Government Account Series issues that are held by entities such as the Social Security trust fund, the Civil Service Retirement and Disability Fund, the Federal Hospital Insurance Trust Fund, and the Federal Retirement Thrift Savings Fund (G Fund).² There are also implied and

Table 4.1 Outstanding level of public and private debt 1985–2002:Q3^a (\$ billions)

Year	Municipal	US Treasury ¹	Mortgage-related ²	Corporate ^a	Federal agencies	Money market ³	Asset-backed ^{a4}	Total
1985	859.5	1,437.7	372.1	776.5	293.9	847.0	0.9	4,587.6
1986	920.4	1,619.0	534.4	959.6	307.4	877.0	7.2	5,225.0
1987	1,010.4	1,724.7	672.1	1,074.9	341.4	979.8	12.9	5,816.2
1988	1,082.3	1,821.3	772.4	1,195.7	381.5	1,108.5	29.3	6,391.0
1989	1,135.2	1,945.4	971.5	1,292.5	411.8	1,192.3	51.3	7,000.0
1990	1,184.4	2,195.8	1,333.4 ^b	1,350.4	434.7	1,156.8	89.9	7,745.4
1991	1,272.2	2,471.6	1,636.9	1,454.7	442.8	1,054.3	129.9	8,462.4
1992	1,302.8	2,754.1	1,937.0	1,557.0	484.0	994.2	163.7	9,192.8
1993	1,377.5	2,989.5	2,144.7	1,674.7	570.7	971.8	199.9	9,928.8
1994	1,341.7	3,126.0	2,251.6	1,755.6	738.9	1,034.7	257.3	10,505.8
1995	1,293.5	3,307.2	2,352.1	1,937.5	844.6	1,177.3	316.3	11,228.5
1996	1,296.0	3,459.7	2,486.1	2,122.2	925.8	1,393.9	404.4	12,088.1
1997	1,367.5	3,456.8	2,680.2	2,346.3	1,022.6	1,692.8	535.8	13,102.0
1998	1,464.3	3,355.5	2,955.2	2,666.2	1,296.5	1,978.0	731.5	14,447.2
1999	1,532.5	3,281.0	3,334.2	3,022.9	1,616.5	2,338.2	900.8	16,026.4
2000	1,567.8	2,966.9	3,564.7	3,372.0	1,851.9	2,661.0	1,071.8	17,056.1
2001	1,688.4	2,967.5	4,125.5	3,818.2	2,143.0	2,542.4	1,281.1	18,566.1
2002:Q3 ^a	1,782.4	3,121.4	4,589.8	3,987.6	2,282.8	2,516.2	1,489.9	19,770.1

^a The Bond Market Association estimates

^b Denotes break in series due to the inclusion of additional source data on private-label MBS/CMOs;

(1) interest bearing marketable public debt; (2) includes GNMA, FNMA, and FHLMC mortgage-backed securities and CMOs and private-label MBS/CMOs; (3) includes commercial paper, bankers' acceptances, and large time deposits; (4) includes public and private placements.

Sources: US Department of Treasury; Federal Reserve System; Federal National Mortgage Association; Government National Mortgage Association; Federal Home Loan Mortgage Corporation; Bond Market Association

explicit federal obligations in the form of guarantees on other debt issues. Another example of the vagaries of the aggregate debt numbers in Table 4.1 are the reported outstanding amounts for mortgage-backed securities that are only for the agencies solely issuing that type of security, i.e., the Government National Mortgage Association (GNMA), the Federal National Mortgage Association (FNMA) and the Federal Home Loan Mortgage Corporation (FHLMC). Such issues by other agencies are included under Federal Agencies.

The composition of marketable US federal government debt by maturity and type of security is given in tables 4.2 and 4.3. Information on pricing of specific issues was given in chapter 1, table 1.9. In addition to offering debt such as the Tbills, Tnotes and Tbonds discussed in chapter 1, the US Treasury also offers debt with a number of different features. Savings bonds, inflation-indexed notes and bonds, STRIPS (Separate Trading of Registered Interest and Principal Securities), in addition to the outstanding

Table 4.2 Summary of Treasury securities outstanding, November 30, 2002 (US \$ millions)

Type of debt	Amount Outstanding		Total
	Debt held by public	Intragovernmental holdings	
<i>Marketable</i>			
Bills	901,417	26	901,444
Notes	1,568,838	26	1,568,864
Bonds	588,465	310	588,775
Inflation-indexed notes	101,250	0	101,250
Inflation-indexed bonds	45,357	0	45,357
Total marketable	3,205,328	362	3,205,690
<i>Non-marketable</i>			
Domestic series	29,995	0	29,995
Foreign series	12,519	0	12,519
State and local govt series	154,090	0	154,090
US savings securities	194,443	0	194,443
Government account series	48,883	2,693,746	2,742,628
Other	4,095	0	4,095
Total nonmarketable	444,025	2,693,746	3,137,771
Total public debt outstanding	3,649,353	2,694,108	6,343,460

Source: US Bureau of Public Debt

Table 4.3 Maturity distribution and average length of marketable US Treasury debt held by private investors (US \$ millions)

End of fiscal year	Amount outstanding privately held	Maturity classes				Average length	
		Within 1 year	1-5 years	5-10 years	10-20 years	20 years or more	
1998	2,856,637	940,572	1,105,175	319,331	157,347	334,212	5 yrs 10 months
1999	2,728,011	915,145	962,644	378,163	149,703	322,356	6 yrs 0 months
2000	2,469,152	858,903	791,540	355,382	167,082	296,246	6 yrs 2 months
2001	2,328,302	900,178	650,522	329,247	174,653	273,702	6 yrs 1 month
2002	2,492,821	939,986	802,032	311,176	203,816	235,811	5 yrs 6 months

Source: US Treasury Bulletin obtained from Office of Market Finance, Office of the Under Secretary for Domestic Finance

long term bonds with call features, all qualify as being debt with special features.³ STRIPS are not actually a different security but, rather, STRIPS is a US Treasury program that lets investors hold and trade the individual interest and principal components of eligible Treasury notes and bonds as separate securities. When a Treasury

fixed-principal or inflation-indexed note or bond is stripped, each coupon payment and the principal payment becomes a separate zero-coupon security. For example, consider a Treasury note with 10 years remaining to maturity that consists of a single principal payment at maturity and the 20 semi-annual coupon payments that are made every 6 months for 10 years. When this note is converted to STRIPS form, each of the 20 coupon payments and the principal payment becomes a separate security.

All marketable Treasury notes and bonds are eligible for STRIPS though, as yet, there is no trading in STRIPS for inflation indexed bonds and notes. Only the conventional fixed coupon notes and bonds are currently traded in STRIPS format. It is important to recognize that the US Treasury does not issue or sell STRIPS directly to investors. STRIPS are traded only through eligible financial institutions and government securities brokers and dealers. As indicated in table 1.9, the traded zero coupon securities that originate from STRIPS are identified by the payment date and whether the zero coupon security originated from a coupon or a principal payment. All STRIPS from coupon payments that occur on the same day are considered the same security but principal payments occurring on the same day are distinguished as different securities. This distinction is important because STRIPS components can be reassembled or "reconstituted" into a fully constituted security through the commercial book-entry system.

Reconstituting a security requires the financial institution or government securities broker or dealer to obtain the principal component and all unmatured interest components for the security being reconstituted where the principal and interest components satisfy the appropriate minimum or multiple amounts for a security to be reconstituted. It is usually the case that market prices of STRIPS fluctuate more than the prices of fully constituted securities of the same maturity. This is a consequence both of the longer duration of the STRIPS for the same term to maturity and the liquidity of the STRIPS market relative to the fully constituted market. As the STRIPS market has grown in size and volume, the flexibility to strip and reconstitute securities can allow investors to take advantage of trading strategies that exploit discrepancies between the pricing of STRIPS and fully constituted Treasury securities.

The STRIPS program is only one of a number of interesting initiatives that the US Treasury has pursued in an effort to minimize US federal government interest costs and to increase the accessibility of Treasury securities to individual investors. One such initiative has been the introduction and expansion of the Treasury-online program that will allow individual investors to purchase Treasury securities on-line. Other initiatives include the introduction of inflation-indexed Treasury securities (I-bonds, I-notes), a type of security that has been used in a number of other countries, e.g., England, Canada, Israel. An I-security has a floating rate coupon that is equal to the fixed "real" rate of interest that is set at the time the security is issued plus the value of a reference inflation index, such as the CPI. The US Treasury has also introduced 4 week Tbills and has made a number of adjustments to the US Savings bond.

Determining the value of outstanding debt for government entities is relatively easy compared to the difficulties of determining a number for the outstanding amount of corporate debt issues. Unlike government entities that report outstanding debt issues in a systematic fashion through well defined public channels, there is no formal public entity responsible for collecting data on corporate debt. In addition, there are also

classification problems such as whether to include issues by foreign corporations in the US market or US dollar issues by US corporations in offshore markets, such as the Eurobond market. There is also the problem of whether to include foreign government owned corporations, e.g., Ontario Hydro, or the corporate divisions of the supranational government entities, e.g., the International Financing Corporation of the World Bank. As such, any value for outstanding corporate debt is a best-guess estimate. The number reported was obtained from the Bond Market Association (www.bondmarkets.com) and is on the high side of the range of reported estimates.

Unlike government and agency debt that is relatively homogeneous in terms of the types of features, corporate debt offers a wide array of special features. In general, the use of specific features will be associated with the economic category of the issuer. These general categories are: utilities; industrial companies; transportation companies; and finance and banking companies. Using "bond" to refer to all debt issues including notes and debentures as well as bonds, a taxonomy of some important features together with brief definitions includes:

- **Call feature:** a callable bond has a provision that permits the issuer to buy back the issue at a predetermined price (the call price) at a prespecified future date (the call date). The predetermined price is usually either par value, par value plus coupon or primary issue offering price. If there is more than one call date, then it is typical for the call price to be higher for closer call dates, say, starting at par plus coupon for the first call date and scaling down to par on the last call date. If the issue has no protection against an immediate call, the issue is referred to as *currently callable*. In some cases, the call provision may be combined with *refunding protection* that restricts the firm from paying for the redemption of the issue by issuing a new series of bonds with a lower coupon, i.e., by doing a refunding.
- **Sinking fund provision:** in order to reduce the credit risk of a bond issue, the bond indenture may require the issuer to retire a prespecified portion of the bond issue at regular intervals up to the term to maturity. If the sinking fund provision does not require the issue to be fully retired by the maturity date, the unpaid balance is referred to as the *balloon maturity*. Depending on the form of the sinking fund provision, the bonds to be retired may be selected by lottery or by purchases in the open market. If selected by lottery, the repurchase price is usually the primary issue price. This feature is commonly used for bond issues by industrial companies. It is seldom used for issues by banking and finance companies. A sinking fund provision done by lottery has an expected term to maturity that is less than the stated term to maturity.
- **Put feature:** a puttable (retractable) bond has a provision that permits the bondholder to sell back the issue at a predetermined price (the put price) at a prespecified future date (the put date). The *encashability* feature of some government savings bonds is a form of put option. Between 1986 and 1996, 203 puttable bond issues were made in the investment grade corporate bond market raising \$39 billion (Crabbe and Nikoulis 1997). Most issues set the put price at par and have one put date. Two reasons for exercising the put are higher market rates and a deterioration in credit quality.

- **Extendible feature:** this feature is not found on many issues. Extendible bonds permit the bondholder to redeem the bonds at a later date than the stated maturity date. One example of this provision was the extendible bond issues made by the government of Canada during the high interest rate period of the late 1970s. These issues permitted bondholders to extend bonds that were, say, 15 years to maturity at original issue for another 5 years at the same coupon. Extendible bonds with above market coupons would be extended.
- **Conversion feature:** a convertible bond permits the bondholder to convert the bond issue to another security, usually the common stock for corporate bonds, at a predetermined price (the conversion price) at a prespecified future date (the conversion date). A typical convertible bond issue would have 10 to 15 years to maturity with the conversion price set 15–25% above the common stock price on the primary issue date. It is possible for a convertible bond to have a number of conversion dates and conversion prices. As with common stock warrants, the indenture for a convertible bond has to contain information about the treatment of the conversion feature in the event of recapitalization, a stock split, a stock dividend, an issuance of warrants or additional amounts of common stock.
- **Warrant bond feature:** unlike a convertible bond where the conversion feature is embedded in the bond, a warrant bond feature permits the convertibility feature to be detached and sold as a warrant, e.g., Vaidya et al. (1995). Unlike convertible bonds where the conversion price at primary issue is set so that the “call option” to convert is deep out of the money, a warrant bond sets the exercise price to be either at the money or in the money. A typical warrant bond would have approximately 30–40% of the value of the purchase price associated with the value of the warrant. Such issues are popular in East Asian markets such as Singapore, Hong Kong and Japan.
- **Floating rate feature:** a bond with a floating rate has a provision to reset the coupon at regular intervals to track a well-defined reference rate, such as LIBOR. As most debt issues with the floating rate feature are 2 to 7 years to maturity, the reference to FRN (floating rate notes) is commonly used to refer to issues with this feature. At primary issue, coupon rates are set at a spread to the reference rate that reflects the credit status of the borrower and liquidity of the issue. Typical reset rates are equal to the maturity of the reference rate, e.g., reset every 3 months when 3 month LIBOR is the reference rate. Due to changes in credit status and liquidity, the value of a floating rate note for corporate borrowers can deviate substantially from par. Default free issues will have prices close to par (equal to par on reset/coupon payment dates).
- **Cap, floor and collar features:** a bond with a floating rate may have an additional feature that sets either a ceiling (cap) on the upward adjustment of the coupon, or a lower limit (floor) on the downward adjustment of the coupon, or both a cap and a floor (collar).

This is only a partial listing of possible features, albeit those that are most commonly observed. It is possible for more than one feature to appear, e.g., a floating rate note with a collar that is both puttable and callable.

In practice, retail investors may find it more attractive to purchase bond funds rather than purchase the underlying bonds directly. For example, this would be the case where the objective is to purchase high yield, below investment grade debt issues. Due to limitations on capital, a retail investor will often be unable to purchase enough issues to achieve sufficient diversification to manage firm specific default risk, an important element in high yield debt securities. In addition, liquidity and availability concerns may make it difficult for the retail investor to purchase such securities. In addition to liquidity and capital considerations, the element of professional money management is also an important incentive for purchasing bond funds. This rationale has been recognized going back at least to Irving Fisher. While the benefits of professional management are generally less for bond funds than equity funds, this is usually reflected in the lower management expense ratios for bond funds. In addition, there are still the vagaries of specialized types of bonds, e.g., international bond funds, where specialized knowledge may be valuable.

Investors purchasing bond funds can choose between a range of *open-end* and *closed-end funds*. An open-end bond fund is much the same as an open-end equity fund. Transactions in open-end funds are done directly with the fund company, e.g., Vanguard or Fidelity, or through an intermediary that deals with the fund company. Various types of fees are associated with the purchase, sale and management of these funds, e.g., front-load fees that are paid at the time of purchase and back-load fees that are paid at the time of sale. The fee structure is an important consideration in the purchase of an open-end fund. The Investment Company Institute reports that the number of open-end funds at year end 2001 was 2,091 bond funds, 4,717 equity funds, 484 hybrid (bond/equity combination) funds and 1,015 taxable/non-taxable money market funds. Total net assets in these funds were (in US\$ billions): \$925.1 bond, \$3,418 equity, \$346.3 hybrid, and \$2,285.3 in money markets. The growth in open-ended funds has been staggering. For comparison, in 1991 the number of funds and net assets (in \$US billions) were: 1,180 funds with \$393.8 for bonds; 1,191 funds with \$404.7 for equity; 212 funds with \$52.2 for hybrids; and, 820 funds with \$543.5 money market funds.

Within the general class of bond funds, there is considerable variation in the fund objectives. Though the Investment Company Institute does not provide a breakdown by fund objectives for open-end bond funds, a general picture can be taken from the closed-end bond funds for which data is provided. Unlike open-end funds that increase (decrease) security holdings as investors invest (withdraw) money from the fund, closed-end funds involve an initial public offering (IPO) of securities that represent claims against the fund. The capital raised by the IPO is then invested and managed according to the stated fund objectives. As such, closed-end fund shares are similar to common stock. Closed-end funds are also traded in much the same fashion as common stocks, with most issues being traded on the NYSE or AMEX. Closed-end bond funds can be classified into three groups, according to the tax treatment of the income distribution.⁴ Some of these funds, which hold bonds from issuers such as corporations or the US Treasury, have income distributions that are fully taxable (taxable bond funds). Other funds, which hold issues of municipal bonds from a number of states, are exempt from

federal tax, but subject to state taxes (national municipal funds). Finally, there are funds that hold municipal bonds from a single state (state municipal funds). For qualifying investors, state municipal funds are exempt from both federal and state taxes.

As indicated in Table 4.4, the financial press provides a further decomposition of closed-end bond fund types by identifying the specific types of securities held, e.g., fully taxable bond funds can be classified as: US Treasury; mortgage-backed securities; investment grade (mainly corporate); high yield; and hybrid. Within the taxable bond fund group, the Investment Company Institute also identifies convertible bond funds.

Table 4.4 Closed-end bond funds

STOCK (SYM)	EXCH	NAV	CLOSE	NET CHG	VOL 100s	PREM /DISC	DIV	12 MO YIELD
U.S. Gov't. Bond Funds								
ACM OppFd AOF	N	8.78	9.10	0.06	58	3.4	0.72	8.0
ACM IncFd ACG	N	8.40	8.51	0.05	2745	1.3	0.81	10.5
MFS GvMkTr MGF	N	7.50	6.75	-0.02	1403	-10.1	0.36	5.4
MS GvtIn GVT	N	9.87	9	0.00	178	-8.7	0.41	4.7
♣ScudderGvt KGT	N	7.43	6.72	0.00	245	-9.6	0.30m	5.3
U.S. Mortgage Bond Funds								
AmIncmFd MRF c	N			-0.03	174	NA	0.66	7.7
AmSelPort SLA c	N			-0.00	25	NA	1.05	7.9
AmStrat ASP c	N			-0.02	9	NA	0.87	6.8
AmStratII BSP c	N			-0.02	31	NA	1.14	8.3
AmStratIII CSP c	N			0.03	272	NA	1.05	8.0
BlkRkAdvTr BAT a	N	11.41	11.22	-0.06	33	-1.6	0.70	6.2
BlkRkBIG Tr BCT a	A	16.01	15.91	-0.01	4	-0.7	0.90	7.6
BlkRkInco Tr BKT a	N	7.10	7.57	0.02	444	6.3	0.61	13.5
BlkRkInv Tr BQT a	N	9.64	9.60	0.01	160	-0.4	0.03	1.3
Hyperion05 HTO c	N	9.75	9.66	0.06	8	-0.8	0.30	4.2
Hyperion HSM	N	14.47	14.60	0.09	203	0.9	1.30	8.8
HyperionFd HTR c	N	9.34	10.38	0.09	269	10.9	0.90	8.9
NationsGov04 NGF	N	10.04	10	0.00	4	-0.4	0.33	3.4
♣PIMCO Comrd PCM c	N	12.63	14.41	-0.14	90	14.1	1.13	9.8
TCW/DW 03 TMT	N				NA	NA		4.0
Investment Grade Bond Funds								
BlkRkCB Tr BHK a	N	14.89	13.79	0.08	592	-7.5	1.05	8.6
BlkRkIncoOpp Tr BNA ac	N	12.03	11.07	0.01	73	-8.2	0.84	7.4
CignaInv IIS a	N	18.96	17.04	-0.12	29	-10.2	0.92	5.5
EtnVncLtdFd EVV a	A	19.15	18.44	0.03	2342	-3.8	1.61	NS
♣1838BondFd BDF	N			0.23	308	NA	1.25m	7.1
♣FtDeaborn FTD	N	16.47	14.85		NA	-9.9	0.80	5.6
♣HnckJ IncSec JHS	N	16.56	15.41	0.13	66	-6.9	0.92	4.5
HrtfrdIncoFd HSF a	N	8.25	7.63	-0.07	27	-7.6	0.55	7.6
HatterasSec HAT	N	14.84	13.22	-0.02	3	-10.9	0.78	5.4

Table 4.4 (Cont'd)

STOCK (SYM)	EXCH	NAV	CLOSE	NET CHG	VOL 100s	PREM /DISC	DIV	12 MO YIELD
♣MotgmrySt MTS	N	19.67	18.41	-0.01	47	-6.4	1.63	6.9
MS InSec ICB	N	17.62	16.16	0.00	52	-8.2	0.96	5.9
PacAmShis PAI	N	16.60	15.16	0.03	3	-8.7	0.96m	6.5
♣PIMCO OppFd PTY a	N	17.19	17.22	0.02	1452	0.2	1.65	NS
PionrIntrst MUO a	N	12.61	11.47	0.01	20	-9.0	0.68	6.2
TransamInco TAI	N	24.21	23.69	0.04	53	-2.1	1.80	7.2
♣VnKmBond VBF	N	19.77	17.97	0.10	51	-9.1	1.04	6.5
VestaurSec VES c	A	14.26	13.22	0.00	11	-7.3	1.14	7.0
WesternAssetPremBd WEA a	N	15.02	15.66	NA	NA	4.2	NA	8.4
High Yield Bond Funds								
40/86StrInco CFD	N	11.09	11.13	0.10	91	0.4	1.02	8.7
BlkRkHiYld Tr BHY a	N			-0.01	98	NA	1.01	10.7
BlkRkDurlnco Tr BLW a	N	19.93	18.82	0.12	464	-5.6	1.50	NS
CignaHiInco HIS a	N	2.85	2.91	0.02	385	2.1	0.28	9.7
CIM HiYld CIM a	A	4.56	4.85	0.06	41	6.1	0.45	9.2
♣ColonialIntr CIF	N	3.69	3.69	-0.02	336	-0.3	0.32	9.3
CpHiYld COY	N	8.82	9.76	0.05	462	10.5	1.07	10.5
CpHiYld III CYE	N	8.69	8.79	-0.01	424	1.2	1.00	11.1
CpHiYld V HYV	N	16.25	16	-0.05	789	-1.5	1.82	11.0
CpHiYld VI HYT	N	15.72	14.85	0.06	1018	-5.5	0.68	NS
CrSuisHighYld DHY	N	4.44	5.29	0.05	495	18.9	0.69	12.3
DDJ Cndn Hi Yld HYB.U y	T			NA	NA	NA	NA	12.7
DebtStratFd DSU	N	6.65	7.04	0.03	1036	5.7	0.88	11.7
DryfsHiYldFd DHF a	N	4.77		-0.03	1097	NA	0.59	12.2
EvrgrnIncoFd EAD	A	15.73	15.66	-0.02	889	-0.5	1.65	NS
FrnklnUnvlTr FT	N	6.35	5.75	-0.02	196	-9.3	0.36	7.5
HiIncoFd HIO	N	7.31	7.13	0.00	1490	-2.3	0.60	9.4
HiYldFd HYI	N	5.60	6	0.04	139	7.0	0.51	9.4
HiYldPlsFd HYP	N	4.04	4.72	0.09	255	16.8	0.42	9.0
LBFIRSTTrIncOpp LBC a	N	15.46	15.51	NA	NA	0.2	NA	NS
MgdHiInc MHY	N	7.12	6.97	0.03	1003	-2.3	0.60	9.4
MgdHiYldPI HYF a	N	5.17	5.75	0.01	758	11.0	0.63	11.1
MS HiYld MSY a	N	6.94	6.54	-0.01	185	-5.8	1.56	7.5
NewAmFd HYB	N	2.20	2.16	0.01	2689	-1.4	0.21	10.6
♣PachldrHi PHF c	A	8.52	9.22	0.07	162	8.3	0.90	10.2
♣PIMCO Hilnco PHK	N	15.65	14.94	0.03	1684	-4.5	1.46	NS
PionrHI Tr PHH	N	15.97	16.35	-0.04	471	2.3	1.65	10.2
♣ProspectSt PHY	N	2.68	3.22	0.02	283	20.2	0.33	9.9
PutnmMgdYld PTM a	N	9.04	8.66	-0.05	121	-4.3	0.65	8.6
RMK HilncoFd RMH a	N	14.83	16.13	-0.11	808	8.8	1.56	NS
SIBrHIF FIF	N	10.43	11.47	0.00	10	10.0	0.96	8.4
SIBrHIF II HIX	N	11.36	12.97	-0.02	1019	14.3	1.38	10.9

Table 4.4 (Cont'd)

STOCK (SYM)	EXCH	NAV	CLOSE	NET CHG	VOL 100s	PREM /DISC	DIV	12 MO YIELD
♣ScudderHigh KHI	N	5.96	7.47	0.03	193	25.3	0.64	8.9
SrHighInc ARK	N	6.02	6.35	-0.03	396	5.2	0.71	10.4
♣VnKmHiInc VIT	N	3.64	4.29	0.00	183	17.6	0.36	9.4
♣VnKmTrII VLT	N	4.56	5.19	0.12	135	13.8	0.44	8.7
ZenixFd ZIF	N	3.13	3.94	-0.05	242	25.6	0.39	11.3

Source: Wall Street Journal, January 2004



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There are also funds that hold international bonds. These funds can be further classified as single country or multi-country. The Investment Company Institute reports that at year-end 2001 there was in the US (millions US\$) \$1,510.8 in outstanding closed-end multi-country bonds funds and \$1,296.6 in single country funds. Finally, it is also possible to classify bond funds according to the frequency of the distributions with most funds making monthly payouts, though some funds (< 20%) make quarterly payouts. Much as with open-end funds, the aggregate US market for closed-end funds experienced considerable growth during the 1990s, with much of this growth concentrated in domestic bond funds.⁵ The Investment Company Institute reports that, at year-end 1990, the four general classes of closed-end funds had market values (in billions) of \$9.6 for domestic equity funds, \$28 for domestic bond funds, \$5.5 for international and global equity funds and \$9.3 for international and global bond funds. By year-end 2000, these values were domestic equity \$23.6, domestic bond \$89.3, global equity \$12.9 and global bond \$8.7, respectively.

Types of interest rates

Due to the large number of possible definitions, reference to "the interest rate" can be confusing. The number of possible definitions that appear in fixed income analysis include: *yield to maturity*; *spot interest rate (implied zero coupon interest rate)*; *implied forward (interest) rate*; *simple interest rate*; *continuously compounded interest rate*; *coupon interest rate*; *discount rate*; *annual percentage rate (APR)*; *effective interest rate*; *zero coupon interest rate*; and, *current yield*. This list is not exhaustive. Of these variations, each has particular relevance, depending on the valuation problem at hand. Perhaps the most widely used definition for "the interest rate" is the yield to maturity. This concept of the interest rate is a direct application of the internal rate of return to the valuation for the cash flow from a fixed income security. The internal rate of return (*IRR*) is an important interest rate concept that appears, for example, in capital budgeting theory, e.g., Brealey and Myers (1991). The *IRR* is the rate of interest that equates the discounted future net cash flows produced by a capital asset with the value of the initial investment in the capital asset.

YIELD TO MATURITY

Although the *yield to maturity* can be used as the interest rate for a range of fixed income securities, the most important practical application is, arguably, to describe the interest rate for a bond. For a simple bond valuation problem, where the bond pays annual coupons and is being valued on the issue date or a coupon payment date, the annualized yield to maturity (y) is obtained by solving:

$$P_B = \left\{ \sum_{t=1}^T \frac{C}{(1+y)^t} \right\} + \frac{M}{(1+y)^T}$$

In this formulation, T is the number of annual coupon payments remaining to be paid on the bond, (T = term to maturity in years), C is the annual coupon payment, M is the par value of the bond which is repaid at maturity and P_B is the price of the bond. Because the calculation assumes that future coupon cash flows can be reinvested at the stated y , it follows that y is only a *promised* yield to maturity. In other words, reinvestment of coupons at the promised yield is required in order for the bond to actually earn the stated yield if the bond is held to maturity. As such, the yield to maturity is an *ex ante* forecast of the *ex post realized* yield. When $C = 0$, the bond is referred to as a pure discount or zero coupon bond. If held to maturity, a default and option free zero coupon bond will have the promised yield to maturity equal to the realized yield.

In practice, the formula given for the price of an annual coupon bond is useful for pedagogical purposes. Only a relatively small number of bonds, e.g., Eurobonds, pay annual coupons. Most government issued bonds, such as those issued by the US Treasury, pay coupons semi-annually. As reflected in tables such as 1.10 and 4.5, the convention in the bond market is to express the coupon as the sum of the coupon payments made in one year. For example, the 11.75% coupon US Treasury bond maturing in November 2014 pays a \$5.875 coupon every six months for a bond with a par value of \$100. Valuing the semi-annual coupon bond on an issue date or coupon payment date, the valuation formula is:

$$P_B = \left\{ \sum_{t=1}^{2T} \frac{C/2}{\left(1 + \frac{y}{2}\right)^t} \right\} + \frac{M}{\left(1 + \frac{y}{2}\right)^{2T}}$$

The sum is now over $2T$ because there are now two payments per year (T is term to maturity measured in years). The convention of dividing the (annualized) yield to maturity creates a situation where the yield on annual coupon bonds is not directly comparable to the (annualized) yield to maturity calculated from this formula. The *effective rate of interest* is used to reconcile this difference.

The importance of the yield to maturity in practical applications is illustrated in table 4.5. The yield to maturity provides a measure that can be used to compare relative value across different types of bonds. For example, convention in the bond market is to execute trades using prices. Due to the variation in term to maturity and coupon, it is difficult to assess bond value by comparing prices. The yield to maturity provides a method to compare value across bonds. The simple rule of thumb for using yield to identify value is: "All other things equal, choose the bond with the highest yield to maturity." However, if all things are equal, then efficient pricing requires that the yields will be the same. Hence, the yield to maturity can be used as a measure to assess bonds that have one or more features that are different. This leads to the notion of what Fabozzi (1989) refers to as "*traditional yield spread analysis*" where the difference in yield for bonds with different features, e.g., different levels of default risk but the same term to maturity, is used as a measure of relative value.

In the situations involving coupon bonds, the relationship between term to maturity and the yield to maturity for the set of available bonds is referred to as the *yield curve*.

Table 4.5 Canadian bonds

Federal	Coupon	Mat. date	Bid \$	Yld%
Canada	12.000	Mar. 01/05	110.38	2.44
Canada	3.500	Jun. 01/05	101.39	2.45
Canada	6.000	Sep. 01/05	105.33	2.59
Canada	12.250	Sep. 01/05	115.11	2.59
Canada	8.750	Dec. 01/05	110.89	2.70
Canada	12.500	Mar. 01/06	119.28	2.99
Canada	5.750	Sep. 01/06	106.81	3.01
Canada	14.000	Oct. 01/06	128.06	3.05
Canada	7.000	Dec. 01/06	110.56	3.11
Canada	7.250	Jun. 01/07	112.43	3.30
Canada	4.500	Sep. 01/07	103.71	3.40
Canada	13.000	Oct. 01/07	133.03	3.39
Canada	10.000	Jun. 01/08	125.53	3.61
Canada	6.000	Jun. 01/08	109.42	3.64
Canada	4.250	Sep. 01/08	102.30	3.70
Canada	11.000	Jun. 01/09	134.23	3.86
Canada	5.500	Jun. 01/09	107.64	3.90
Canada	4.250	Sep. 01/09	101.41	3.97
Canada	10.750	Oct. 01/09	134.43	3.93
Canada	9.750	Mar. 01/10	130.40	4.07
Canada	9.500	Jun. 01/10	130.04	4.08
Canada	5.500	Jun. 01/10	107.62	4.12
Canada	9.000	Mar. 01/11	128.91	4.24
Canada	6.000	Jun. 01/11	110.50	4.32
Canada	8.500	Jun. 01/11	126.30	4.29
Canada	5.250	Jun. 01/12	105.55	4.45
Canada	5.250	Jun. 01/13	105.38	4.54
Canada	10.250	Mar. 15/14	145.82	4.56
Canada	5.000	Jun. 01/14	103.15	4.61
Canada	11.250	Jun. 01/15	158.81	4.56
Canada	9.750	Jun. 01/21	155.36	4.96
Canada	9.250	Jun. 01/22	150.97	4.98
Canada	8.000	Jun. 01/23	135.78	5.07
Canada	9.000	Jun. 01/25	149.65	5.14
Canada	8.000	Jun. 01/27	138.32	5.16
Canada	5.750	Jun. 01/29	108.30	5.16
Canada	5.750	Jun. 01/33	109.35	5.13

Source: *Globe and Mail*, Jan. 2004; supplied by RBC Dominion Securities Inc.

Different yield curves can be identified for different types of bonds, e.g., the corporate bond yield curve or Treasury bond yield curve. As will be discussed below, analysis of "the yield curve" poses a range of problems. For example, bonds with the same term to maturity may have different coupons and, as a consequence, different yields. This leads to the introduction of the abstract relationship between term to maturity and the spot interest rate (implied zero coupon interest rate) referred to as the *term structure of interest rates*. These definitions are not always adhered to in various texts and financial newspapers where the terminology "term structure of interest rates" can be used synonymously with "the yield curve." (The end of chapter questions deal with the plotting of the yield curve from actual data, as well as an analysis of the relationship between the yield curve and the spot rate curve.)

EFFECTIVE INTEREST RATE

In presenting the yield to maturity, it was observed that the convention used to calculate yields for semi-annual coupon bonds resulted in stated yields that were not directly comparable to yields calculated from annual coupon bonds. The *effective interest rate* or *effective yield* provides the annualized equivalent for a stated interest rate involving compounding at a greater than annual frequency. This problem of having the calculated yield changing with the compounding frequency is not specific to bonds but occurs with all fixed income transactions where compounding occurs. In particular, changing the compounding frequency is one of a number of possible methods of confusing consumers about the actual interest rate that is being charged on loans. In the US, truth in lending laws have codified the use of the effective yield where the concept is referred to as the *annual percentage rate* or *APR*. In others words, loan transactions are required to quote an effective yield when stating the lending rates for consumer loans. (In Canada, the APR has a different legal meaning.)

To illustrate the methodology for calculating the effective interest rate, consider the following problem: a 10% coupon government bond (paid semi-annually) currently sells at par. At current market prices, what coupon rate would be required if the coupons were paid annually? To solve this problem, step one requires equating future values for single payments: the future value of one payment received in T years, compounded m times per year at interest rate r produces:

$$FV^m = (1 + (r/m))^m$$

For annual compounding $m = 1$ at interest rate y this reduces to $FV^A = (1 + y)^T$. Equating the future values $FV^m = FV^A$ produces:

$$(1 + y)^T = (1 + (r/m))^m \rightarrow y = (1 + (r/m))^m - 1 \rightarrow r = [(1 + y)^{1/m} - 1]m$$

For a single cash flow occurring T years ahead, this result provides the method for transforming an interest rate for compounding m times a year to an interest rate compounded annually.

The next step involves converting the future value (FV) to a present value (PV). Present and future value represent the valuation of cash flows at different points in time. To translate a future value at time T to a present value involves discounting the value at the appropriate interest rate:

$$PV^A = FV/(1+y)^T \quad PV^{SA} = FV/(1+(r/m))^{mT}$$

When the effective yield is used, equating the future values is the same as equating the present values.

It is now possible to extend these results for single cash flows to coupon bonds. Working by example, consider a 1 year par bond with an 10% coupon. *In all cases, it is assumed that the cash flows from the bond are reinvested at the stated yield to maturity.* A semi-annual bond has cash flows of $C/2 = \$5 = (r/2)M$ which produces a time line of:

$$\begin{array}{cccc} \$5 & & \$105 & \\ t = 0 & t = 6 \text{ months} & t = 1 \text{ year} & \end{array}$$

The future value of this stream of cash flows is:

$$FV^S = \left(\frac{r}{2}M\right)\left(1 + \frac{r}{2}\right) + M\left(1 + \frac{r}{2}\right) = M\left(1 + 2\frac{r}{2} + \frac{r}{2}\right) = M\left(1 + \frac{r}{2}\right)^2$$

This can be compared with the future value for annual payments of $FV^A = M(1+y)$. It follows that for 1 year bonds, the effective yield formula holds: $y = (1 + (r/2))^2 - 1$. For the annual coupon bond to be sold at par, the coupon on the bond would have to be 10.25%.

For a 2 year bond, the analysis is much the same. The time line for the semi-annual coupon bond is:

$$\begin{array}{cccccc} \$5 & & \$5 & & \$5 & & \$105 \\ t = 0 & t = 6 \text{ months} & t = 1 \text{ year} & t = 1.5 \text{ years} & t = 2 \text{ years} & \end{array}$$

To calculate the future value of the semi-annual cash flows:

$$FV^S = \frac{r}{2}M\left(1 + \frac{r}{2}\right)^3 + \frac{r}{2}M\left(1 + \frac{r}{2}\right)^2 + \frac{r}{2}M\left(1 + \frac{r}{2}\right) + M\left(1 + \frac{r}{2}\right) = M\left(1 + \frac{r}{2}\right)^4$$

This can be compared with the future value of the annual coupons: $FV^A = M(1+y)^2$. It follows that for 2 year bonds to have equal prices (from step two):

$$(1+y)^2 = (1 + (r/2))^4 \rightarrow y = (1 + (r/2))^2 - 1$$

The effective yield result also holds in this case, and the annual coupon bond would have a coupon of 10.25% when the semi-annual coupon bond has a coupon of 10%, for

both bonds to sell at par. By induction, this method for transforming the semi-annually compounded yield to an annually compounded yield extends to bonds with T years to maturity.

SPOT INTEREST RATE (IMPLIED ZERO COUPON INTEREST RATE)

As with any internal rate of return calculation, the yield to maturity has a number of limitations. One limitation involves applying the same interest rate to discount cash flows occurring at different points in time. This approach would only be valid if yield curves were flat, i.e., yields were the same for all terms to maturity. Casual inspection of real world bond markets reveals that near term cash flows are usually discounted at lower interest rates than longer term cash flows. This limitation leads directly to the concept of the *spot interest rate* or, more descriptively, the *implied zero coupon interest rate* (implied zero rate). Whereas there is a yield to maturity that can be calculated for every bond, the implied zero coupon interest rate applies to the term to maturity. There is an implied zero coupon interest rate for every fixed income payment date. For example, when the US Treasury issued 30 year bonds, there were 60 implied zero coupon interest rates that could be calculated, one for each of the 60 coupon payment dates.

To see the connection between the implied zero rate and the yield to maturity, consider the following valuation formulas for bonds with annual coupons:

$$P_B = \left\{ \sum_{t=1}^T \frac{C}{(1+y)^t} \right\} + \frac{M}{(1+y)^T} = \left\{ \sum_{t=1}^T \frac{C}{(1+z_t)^t} \right\} + \frac{M}{(1+z_T)^T}$$

Recognizing that the implied zero rate or spot interest rate (z_t) is the interest rate applicable to cash flows occurring at time t , it follows that valuation with implied zero rates values the bond by treating each of the cash flows (coupon payments and return of principal) as zero coupon bonds. The price of the bond is then calculated as the sum of the prices of the zero coupon bonds, valued using the implied zero rate applicable to single cash flows for that term to maturity, i.e., the price of the bond is the sum of the appropriately discounted zero coupon bond prices. For default free (riskless) zero coupon bond prices, the spot interest rate (implied zero rate) is equal to the yield to maturity.

While each bond has an associated yield to maturity, each coupon payment date will have an implied zero rate. To make sense of such an interest rate, it is necessary to abstract from default risk and other features. Hence, the implied zero rates are extracted from the relevant default free bond prices, i.e., the US Treasury debt issues are used to extract the implied zero rates for the US debt market. Though it would be conceptually possible to use the observed interest rates for zero coupon bonds, such as the rates for US Treasury strip securities (STRIPS), for the implied zero rates, this raises a number of difficulties. Sundaresan (2002, p. 231) specifically addresses this point: "It should be stressed that strips are not implied zeroes. Strips are traded securities directly subject to demand and supply. Implied zeroes are estimated pure discount functions derived from

the prices of coupon-paying Treasury securities. Yet, as expected, implied zeroes provide a benchmark for assessing the relative richness or cheapness of Treasury securities."

Fabozzi (1989, pp. 192–3) identifies three reasons why stripped Treasury securities are not an adequate substitute for implied zeros: problems of liquidity in the Treasury strips market; maturity preferences in specific segments of the Treasury strip market may cause mispricing of certain maturities; and, differences in the tax treatment of stripped Treasuries and coupon Treasury bonds. Sundaresan (2002, pp. 237–41), Mason et al. (1995, pp. 48–55) and others demonstrate that, while not dramatically different, US Treasury strip rates do differ empirically from implied zero rates calculated from US Treasury coupon bonds.⁶

While each bond has an associated yield to maturity, each coupon payment date will have an associated implied zero coupon interest rate. As such, the method for calculating the implied zero rate differs from the method for calculating the yield to maturity. Mathematically, the yield to maturity calculation for a bond with T coupon payments involves one equation with one unknown. A bond with T coupon payments would have T unknown spot rates. This requires T equations to determine the T unknowns. Because the market used to calculate implied zero coupon rates is the US Treasury coupon bond market, a *bootstrap* technique is needed to extract the spot interest rates from the observed coupon yields (see section 4.3). A bootstrap is the name given to a generic algorithm that uses an stepwise solution procedure to arrive at the solution for a number of unknowns. In particular, the first step solves for the first unknown using one equation with one unknown. This solution is then used to solve a second equation that is specified with the first unknown (now solved) and the second unknown. These two solutions are then used to solve the third equation that is specified with the first two unknowns (now solved) and the third unknown. This procedure continues until all the unknowns are solved.

IMPLIED FORWARD INTEREST RATES

The *implied forward interest rate* (implied forward rate) has a number of possible uses in theoretical modeling. This interest rate concept is an extension of the *breakeven interest rate* associated with comparing a rollover investment strategy with a buy and hold strategy. To see this, consider the following comparison involving a two period, two portfolio model where z_i is the zero coupon yield on a bond maturing i periods from now, $z_{i,j}$ is the $(i-j)$ period interest rate starting at $t = i$ and maturing at $t = j$, e.g., $z_{1,2}$ is the one period interest rate that starts at $t = 1$ and ends at $t = 2$:

- **Portfolio A:** buy and hold a 2 year zero coupon bond. If the initial investment is \$1 the return at the end of 2 years is $(1 + z_2)^2$.
- **Portfolio B:** buy and mature a 1 year zero coupon and use the proceeds to purchase another 1 year zero coupon bond, 1 year in the future. If the initial investment in this portfolio is \$1 then the expected return at end of year two is: $(1 + z_1)(1 + E[z_{1,2}])$.

Portfolio A is referred to as the *buy and hold* portfolio and Portfolio B as the *rollover* portfolio.

To derive the *breakeven interest rate*, assume that the expected returns on the two portfolios are equal, then:

$$(1 + z_2)^2 = (1 + z_1)(1 + E[z_{1,2}])$$

It follows that the breakeven expected interest rate can be calculated as:

$$(1 + E[z_{1,2}]) = \frac{(1 + z_2)^2}{(1 + z_1)}$$

Using this result, the correspondence between the breakeven interest rate and the definition of the implied forward rate $f_{1,2}$ follows appropriately: $f_{1,2} = E[z_{1,2}]$. (The notation for the implied forward rate differs across the various textbooks on the subject, i.e., there is no generally accepted notational convention.)

The use of $\{z_i\}$ to define $\{f_{i,j}\}$ is intentional. In the absence of representative zero coupon interest rates, implied forward rates are calculated using spot interest rates. The extension of the definition of an implied forward rate from the two period case to the n period case follows appropriately. For example, consider the one year implied forward interest rate that will apply from period t to $t + 1$. This is given by:

$$(1 + f_{t,t+1}) = \frac{(1 + z_{t+1})^{t+1}}{(1 + z_t)^t}$$

It is also possible to define implied forward rates applying to the interest rates longer than one year. For example, the five year implied forward interest rate between $t = 5$ and $t = 10$ can be calculated using the 10 and 5 year spot rates:

$$(1 + f_{5,10}) = \sqrt[5]{\frac{(1 + z_{10})^{10}}{(1 + z_5)^5}}$$

Similarly the implied forward rate for a three year zero coupon bond that starts at $t = 2$ and matures at $t = 5$, using the two and five year spot rates, is specified:

$$(1 + f_{2,5}) = \sqrt[3]{\frac{(1 + z_5)^5}{(1 + z_2)^2}}$$

Given this, the general formula for the implied forward interest rate is specified:

$$(1 + f_{t,t+k}) = \sqrt[k]{\frac{(1 + z_{t+k})^{t+k}}{(1 + z_t)^t}}$$

It follows that an observed yield curve for, say, n maturities will produce $(n - 1) + (n - 2) + \dots + 1$ implied forward rates (see end of chapter questions).

CURRENT YIELD

There are so many different approaches to calculating interest rates that it is not practical to give a detailed account of each method. Though not used much in recent years, prior to the advent of computerized calculations, the current yield was a commonly quoted approximation to the yield to maturity. The "current yield" is defined as: Current Yield = $CY = (\text{Annual Coupon Paid})/(\text{Bond Price}) = C/P_B$. The relationship between the current yield and the yield to maturity is useful for pedagogical purposes. When $P_B = M$, the bond sells at par, then $CY = y$. When $P_B > M$, for premium bonds $CY > y$ with the difference increasing as the bond has a greater premium. When $P_B < M$, for discount bonds $CY < y$ with the difference increasing as the bond has a greater discount. For a zero coupon bond, $CY = 0$.

In general:

$$CY = \frac{C}{\sum_{t=1}^T \frac{C}{(1+y)^t} + \frac{M}{(1+y)^T}} \rightarrow CY \left\{ \sum_{t=0}^{T-1} (1+y)^t \right\} + \frac{M}{P_B} = (1+y)^T$$

Some special cases follow, e.g., for $T = 1$:

$$CY + \frac{M}{P_B} = (1+y) \rightarrow y = CY + \left\{ \frac{M}{P_B} - 1 \right\}$$

For $T = 2$:

$$CY(1+y) + \frac{M}{P_B} = (1+y)^2 \rightarrow y = CY + \left\{ \frac{M}{P_B(1+y)} - 1 \right\}$$

As T increases, the size of the deviation shrinks to the point where, for a perpetuity:

$$CY = \frac{C}{\frac{C}{y}} = y$$

i.e., when T goes to infinity, the current yield equals the yield to maturity.

MONEY MARKET CALCULATIONS: THE DISCOUNT RATE

Some basic background is required to interpret the interest rate quotes for money market securities provided in the financial press. Unlike bonds, money market securities are quoted and traded using interest rates. When comparing interest rates derived from US money market securities with other fixed income securities, it has to be recognized

that most US money market securities, Tbills, BA's, commercial paper and term repos, are quoted on a *discount rate* and *not* a true yield basis, e.g., Stigum (1990). In addition to using a different pricing formula, the discount rate calculation also involves calculating the year as though it has 360 days. To see this, consider the US method for determining the purchase price of a 1 year, \$1 million par value Tbill sold at a discount rate of 8%:

$$\text{Discount} = (\$1,000,000)(.08)(364/360) = \$80,888.89$$

As a result, the price paid for this Tbill maturing in one year is the maturity (par) value minus the discount or \$919,111.11. The general formula for arriving at the discount is:

$$D = dF(tsm/360)$$

where D is the discount, F is the maturity value, d is the discount rate and tsm is the time from settlement to maturity.⁷ Similarly, for a Tbill with 3 days to maturity, par value of \$1 million and discount rate of 9%:

$$D = (.09)(\$1,000,000)(3/360) = \$750$$

The bill is purchased for \$999,250 and matures in 3 days for \$1 million, returning \$750 in interest.

The method for valuing an 01 or one basis point (.0001 expressed as a decimal) for the 3 month \$1 million par value Tbill position is:

$$D = (.0001)(\$1,000,000)(90/360) = \$25$$

The value of an 01 (one basis point) for a \$1 million, 6 month Tbill:

$$D = (.0001)(\$1,000,000)(180/360) = \$50$$

Observing that the Eurodollar (and Tbill) futures contracts are written on 1 million dollar par value 3 month deliverable securities, it is necessary to use two Eurodollar futures contracts to provide a dollar value equivalent movement for a 6 month, \$1 million dollar par value Eurodollar cash position.

It should be recognized that the discount rate understates the simple interest, true yield or bond equivalent yield formula which is used in Canada and other countries for money market securities. The formula for calculating the annualized true yield (y) is the familiar:

$$r = \frac{F - P}{P} \frac{365}{tsm} \quad \text{or} \quad P = \frac{F}{1 + \left(r \frac{tsm}{365} \right)}$$

where P is the price paid or $D = F - P$. Substituting this result gives:

$$y = (D/(F - D)) (365/tsm)$$

This can be solved in terms of the discount rate by substituting the method for calculating D for discount securities to give:

$$y = \frac{(365)d}{360 - d(tsm)}$$

This is the formula for converting a discount rate into a true or simple interest yield.

To see how this works, consider the discount rate for the US BA quoted at .0477 or 4.77%. Using the conversion formula:

$$y = [365(.0477)]/[360 - \{(.0477)90\}] = .0489462$$

Similarly for the CP rate of .0480:

$$y = [365(.0480)]/[360 - \{(.0480)90\}] = .0492577$$

The general formula can also be used to show that the quoted discount rate will always be below the true yield, $y > d$, as well as showing that the discrepancy between y and d will be greater the higher is the rate of discount and the longer is the term to maturity. Some manipulation gives the discount rate associated with a given true yield:

$$d = \frac{360y}{365 + y(tsm)}$$

In recent years, the relationship between d and y is reported in the financial press, as reflected in the US Treasury bill rates reported in table 1.9.

To verify the formula for deriving the discount rate associated with an observed yield, consider the November 3, 1994 bill observed on August 4, 1994. This three month Tbill would typically be indicated in bold in the quotes in the financial press because on August 4 it is the most recently issued 3 month bill and, due to the focus of market trading on the most recently issued maturities, this bill will tend to be the most liquid. The tsm for this bill is 12 weeks plus 3 days or 87 days with a reported yield of 4.56. Evaluating for d gives:

$$d = [360(.0456)]/[365 + \{(.0456)(87)\}] = .0445$$

As an exercise, calculate and verify the values for the current three month Tbill reported in table 1.9.

Some examples of valuation problems

The history of interest rate calculations stretches back centuries, e.g., Poitras (2000, chs 4-5). In this history, the use of worked examples has played an important role in

illustrating concepts. An early example of the sophistication of such problems can be found in Witt (1613): "A oweth to B £1,200 to be paid in 6 yeares, in 12 equall payments, viz. at the end of each halfe yeare £100. They agree to cleare this debt in 3 yeares, in 6 equall payments, viz. at the end of each halfe yeare, one payment. The Question is, what each payment ought to be, reckoning interest after the rate of 10 per cent per Ann. and int. upon int." A conventional solution to this problem can be determined by equating the discounted value of the annuity stream of £100 for 12 half-year periods with the discounted value of £C for 6 half-year periods and solving for C. The exact solution requires recognizing Witt's practice of using $(1+r)^{t/2}$ instead of the modern convention of $(1+r/2)^T$ to discount the T period cash flow.

More precisely, the solution can be determined by solving:

$$\begin{aligned} & \frac{100}{(1+r)^{1/2}} + \frac{100}{(1+r)} + \dots + \frac{100}{(1+r)^6} \\ &= \frac{C}{(1+r)^{1/2}} + \frac{C}{(1+r)} \dots + \frac{C}{(1+r)^3} = 100\{1 + (1+r)^{1/2}\} \left\{ \frac{1}{r} - \frac{1}{r(1+r)^6} \right\} \\ &= C\{1 + (1+r)^{1/2}\} \left\{ \frac{1}{r} - \frac{1}{r(1+r)^3} \right\} \end{aligned}$$

Solving this for $r = .10$ gives the solution stated by Witt of £175.13145 or £175. 2s. 7d. Yet, Witt is able to show that this solution can be obtained as:

$$100 + \frac{100}{(1+r)^3} = £175. 2s. 7d.$$

Lewin (1970, p. 126) describes the method Witt uses to arrive at this solution as "extremely elegant."

Though no longer used for government financing activities, during the early years of government finance perpetuity issues were commonly used. The perpetuity is still of theoretical interest today. What is a perpetuity? A perpetuity is a security that offers to pay a fixed or variable coupon, at regular intervals, forever (in perpetuity). If the coupon is variable, then it is referred to as a floating rate perpetuity. Almost all perpetuities issued in recent years have been floating rate perpetuities, e.g., the floating rate perpetuities issued by financial institutions in the Euromarkets during the 1980s. The most well known perpetuity is a *consol*, originally issued by the British government in the eighteenth and early nineteenth century, which pays a fixed coupon. This perpetuity was so named because it originated from the consolidation of a number of different types of government debt issues, i.e., *consol* is a short form for consolidated debt issue. Consol issues traceable back to these early debt operations are still traded on the English exchanges.

The perpetuity is more than a historical curiosity. The pricing formula for this security is of theoretical value, if only to illustrate a geometric series. Consider deriving the pricing formula for such a security when coupons are fixed and paid annually:

$$P^{perp} = \sum_{t=1}^{\infty} \frac{C}{(1+y)^t} = \frac{C}{(1+y)} \left\{ 1 + \frac{1}{1+y} + \frac{1}{(1+y)^2} + \frac{1}{(1+y)^3} \dots \right\}$$

$$\text{Recall: } \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$\therefore P^{perp} = \frac{C}{(1+y)} \left\{ \frac{1}{1 - \frac{1}{(1+y)}} \right\} = \frac{C}{y}$$

It can be immediately verified that, when coupons are fixed and paid quarterly, the perpetuity has the same pricing formula. Because quarterly coupons are paid sooner than annual coupons, this result may seem odd. However, this result can be explained when it is observed that the annual and quarterly coupon perpetuities will sell for different prices. Hence, even though the pricing formula is the same, the yields will not be the same.

Another interesting result occurs for the default free floating rate perpetuity where the coupon is variable (floating) and equal to the current interest rate times the par value of the perpetuity, i.e., $C = yM$. It follows that this perpetuity will always sell at par because the coupon will adjust to keep the price of the perpetuity equal to par. This was the idea behind the floating rate perpetuities issued in the Euromarkets during the 1980s. Because financial institutions are regularly coming to the market to reissue short maturity debt, issuers could save on financing costs by offering a security that has a floating coupon. Similarly purchasers could save on commissions and other costs associated with rolling over short term debt issues. However, this analysis depends on the level of default risk staying relatively constant. In the face of actual default risk shocks, market makers were forced to absorb large amounts of these securities at falling prices. The losses incurred led to a collapse of the market for this type of security (Parente and Weintraub 1987).

One interesting extension of the fixed coupon perpetuity pricing formula occurs with the pricing of fixed coupon, fixed term annuities. The basic formula for pricing such an annual pay annuity can be stated as:

$$PV_A = \sum_{t=1}^T \frac{\$A}{(1+y)^t}$$

where $\$A$ is the annual coupon payment, y is the applicable interest rate and T is the term over which the annuity payment is received. This general pricing formula appears in numerous guises, such as in consumer loans for automobiles, house mortgages and the like. In these applications, the payment frequency is monthly. By observing that the cash flow stream from a fixed term annuity can be conceived as a perpetual annuity minus a perpetuity that starts at $T + 1$ (valued at T) the perpetuity formula can be used to reexpress this sum as a single closed form expression, i.e.:

$$PV_A = \frac{\$A}{y} - \left[\frac{\$A}{y} \frac{1}{(1+y)^T} \right] = \frac{\$A}{y} \left[1 - \frac{1}{(1+y)^T} \right] = \sum_{t=1}^T \frac{\$A}{(1+y)^t}$$

In effect, the price of a fixed coupon annuity is equal to the price of a perpetuity minus the present value of a perpetuity that starts at $T + 1$ (with price taken at T).

In contrast to perpetuities that are something of an oddity in modern fixed income markets, much of the modern focus and discussion of fixed income securities is concerned with bond valuation. Various quirks can arise in bond pricing problems. One useful example arises with using the same yield to maturity to value bonds with different coupon payment frequencies. For example, consider the following problem: a bond offers eight annual coupon payments of \$8 and will repay its face value of \$100 at the end of eight years. You observe that other similar bonds have yields to maturity of 10%. How much is this bond worth? If the coupons are paid semi-annually, how much is the bond worth? To solve this basic problem, let $C = \$8$, $M = \$100$, and the term to maturity (T) be $T = 8$:

$$P_B^A = \$8 \sum_{t=1}^8 \frac{1}{(1.1)^t} + \frac{\$100}{(1.1)^8} = \$8(5.335) + \$100(.467) = \$89.38$$

$$P_B^{SA} = \$4 \sum_{t=1}^{16} \frac{1}{(1.05)^t} + \frac{100}{(1.05)^{16}} = \$4(10.84) + \$45.80 = \$89.16$$

where P_B^A is the price of the annual coupon bond and P_B^{SA} is the price of the semi-annual bond.

The impact of increasing the coupon payments can be established with the following two problems: if these bonds have coupon payments of \$12 annually, how much is the bond worth? If the coupons are paid semi-annually how much is the bond worth? These questions can be solved as: $P_B^A = \$12(5.335) + \$46.70 = \$110.72$ and $P_B^{SA} = \$6(10.84) + \$45.80 = \$110.84$. To assess the impact of increasing the term to maturity, compare these bond prices with the 8% coupon prices calculated with 10 years to maturity. Do the same for the bonds with the 12% coupon. What do you observe about the relationships between the prices? Solving for the bond prices gives for $C = \$8$, $T = 10$, $y = .10$, $P_B^A = \$8(6.145) + \$38.60 = \$87.76$ and $P_B^{SA} = \$4(12.46) + \$37.70 = \$87.54$. Similarly, for $C = \$12$, $T = 10$, $y = .10$, $P_B^A = \$12(6.145) + \$38.60 = \$112.34$ and $P_B^{SA} = \$6(12.46) + \$37.70 = \$112.46$.

This series of bond valuations illustrates the pricing differences that arise for discount and premium bonds. This illustration requires some definitions to be introduced. A "straight bond" requires a stream of fixed coupon payments paid at regular intervals plus a "return of principal" at maturity that involves a payment on the maturity date equal to the stated *par value* (M) of the bond. If no coupons are offered the bond is said to be a *zero coupon* or *pure discount* bond. If the price of the bond $P_B > M$, then the bond is referred to as a *premium* bond. This occurs when the annual coupon payment C satisfies: $(C/M) > y$, i.e., the coupon rate exceeds the yield to maturity on the bond.

If the price of the bond $P_B < M$, then the bond is referred to as a *discount bond* and $(C/M) < y$. If $P_B = M$, the bond sells at its par value, then $(C/M) = y$ and the bond is referred to as a *par bond*.

In the bond valuation solutions, it can be observed that, for discount bonds with the same yield but different term to maturity, $P_B^{10} < P_B^8$. For premium bonds, with the same yield but different term to maturity, $P_B^{10} > P_B^8$. Shorter term bonds have higher prices when the bonds sell at a discount and longer term bonds sell at higher prices when the bonds sell at a premium. For discount bonds, with the same term to maturity and yield to maturity: $P_B^{SA} < P_B^A$. This result seems counter-intuitive because the semi-annual bond pays the coupon sooner than for the annual bond. For premium bonds, with the same term to maturity and yield to maturity: $P_B^{SA} > P_B^A$. All these results are somewhat misleading because with the same C and T a semi-annual coupon bond will always be preferred to an annual coupon bond, because a portion of the cash flows are received sooner. Hence, the semi-annual bond will sell for a higher price, and lower stated yield to maturity.

In an era of calculators, workstations and computerized calculation, the problem of directly calculating the yield to maturity has lost its sharp edge. The days of starting with an observed coupon, bond price and term to maturity, looking up present value factors and present value of annuity factors in tables and painstakingly interpolating to arrive at a yield to maturity are rapidly receding into the past. Instead of interest tables, introductory investment texts now include a CD with the appropriate yield calculation programs. Consider a question designed to provide a glimpse into times past. The objective is to arrive at a solution problem by hand: a bond offers semi-annual coupon payment of \$5 (i.e., it is a 10% coupon bond) and will repay its face value of \$100 at the end of twenty years. The bond is currently selling for \$84.95. What is the offered yield to maturity on the bond?

How to solve for the yield to maturity given the P_B ? As bonds are traded in terms of price, this is the typical calculation that has to be done in practice. Absent some type of financial calculator with the appropriate present value functions, it is necessary to solve by hand using the trial and error method, i.e., guess a yield and solve for the bond price. If the calculated price is above the observed price, then try a higher yield and recalculate the price. If the calculated price is below the observed price, then try a lower yield and recalculate the price. Continue this process until a yield is determined that has a desirable level of precision. In doing these calculations without present value and present value of annuity tables it is necessary to make use of the expression for the present value of an annuity given previously.

4.2 THE CONCEPT OF DURATION

The concept of duration

Duration is a fundamental concept in the analysis of fixed income securities, both for individual securities and for portfolios. Though duration has a number of different

applications, the basic concept can be motivated by considering the question: what is an appropriate measure for the sensitivity of the price of a fixed income security to a change in the interest rate? For purposes of illustration, this question can be evaluated by taking the derivative of the bond price with respect to the interest rate. More precisely, observing that $d(1 + y) = dy$:

$$\frac{dP_B}{d(1 + y)} = - \left\{ \sum_{t=1}^T \frac{tC}{(1 + y)^{t+1}} + \frac{TM}{(1 + y)^{T+1}} \right\}$$

$$\rightarrow - \frac{1 + y}{P_B} \frac{dP_B}{d(1 + y)} = \frac{\sum_{t=1}^T \frac{tC}{(1 + y)^t} + \frac{TM}{(1 + y)^T}}{P_B} = D^* = \text{Macaulay duration}$$

Elementary economics teaches that using the unadjusted derivative to measure the sensitivity of a variable X , say quantity demanded, with respect to the change in another variable Y , say the price of the commodity or income of the consumer, is ineffective because the starting level of X and Y will impact the result. Instead, the elasticity measure is used where the derivative is scaled by the level of X and Y . This is the case with *Macaulay duration*, which is the elasticity of the bond's price with respect to a change in interest rates. Though an elasticity number is usually dimensionless (dollars divided by dollars), due to the presence of compound interest it is acceptable to express duration in years, e.g., Hicks (1939, p. 187, n. 1). An important property of D^* occurs for a zero coupon bond ($C = 0$) where $D^* = T$, i.e., the duration of a zero coupon bond equals the term to maturity.

In practice, two closely related forms of duration are encountered, *Macaulay duration* and *modified duration*. The difference can be illustrated with the following question: given D^* , calculate the change in the price of a bond for a given change in interest rates. This leads to:

$$D^* = - \frac{1 + y}{P_B} \frac{dP_B}{d(1 + y)} \rightarrow -dP_B = \frac{D^*}{1 + y} P_B dy = DP_B dy$$

$$\text{where: } D = - \frac{1}{P_B} \frac{dP_B}{dy} = \frac{D^*}{1 + y}$$

The measure D is the modified duration. As discussed below, Macaulay duration is typically used in situations where the duration concept is being used as an adjusted term to maturity while the modified duration occurs in theoretical analysis of bond portfolio management applications.

Making use of the ratio expression for the present value of an annuity, some tedious algebra permits the duration for an annual coupon bond to be reexpressed as a number of different possible ratio expressions. Two such expressions are:

$$\begin{aligned}
 D^* &= \frac{\sum_{t=1}^T \frac{tC}{(1+y)^t} + \frac{TM}{(1+y)^T}}{P_B} = \frac{1+y}{y} - \frac{1+y+T\left(\frac{C}{M}-y\right)}{\frac{C}{M}\{(1+y)^T-1\}+y} \\
 &= \frac{1+y}{P_B} \left\{ \frac{C}{y} \left[\frac{1}{y} - \frac{1}{(1+y)^T y} - \frac{T}{(1+y)^{T+1}} \right] + \frac{TM}{(1+y)^{T+1}} \right\}
 \end{aligned}$$

The final expression follows from direct evaluation of the derivative. Such formulas are useful when programming duration into a spread sheet or hard coding into a statistical or mathematical programming package (see end of chapter questions for the par bond simplification).

The duration formula has to be adjusted when the coupon payment frequency is not annual. For example, the Macaulay duration of a semi-annual coupon bond is determined as:

$$\begin{aligned}
 \frac{dP_B}{d\left(1+\frac{y}{2}\right)} &= - \left\{ \sum_{t=1}^{2T} \frac{t(C/2)}{\left(1+\frac{y}{2}\right)^{t+1}} + \frac{2TM}{\left(1+\frac{y}{2}\right)^{2T+1}} \right\} \\
 \rightarrow - \frac{1+\frac{y}{2}}{P_B} \frac{dP_B}{d\left(1+\frac{y}{2}\right)} &= \frac{\sum_{t=1}^{2T} \frac{t(C/2)}{\left(1+\frac{y}{2}\right)^t} + \frac{2TM}{\left(1+\frac{y}{2}\right)^{2T}}}{P_B} = D^{*SA} = \text{Macaulay duration (in half years)}
 \end{aligned}$$

It follows that modified duration has to be adjusted appropriately, i.e., for semi-annual bonds $D^{SA} = (D^{*SA})/(1 + (y/2))$. Given that the conventional unit of measurement for duration is years, to be comparable to duration for annual coupon bonds both D^{SA} and D^{*SA} have to be divided by two to get a value that is measured in years and is comparable to duration for annual coupon bonds.

Some examples (taken from CFA Level I examinations) serve to illustrate the basic points: "An 8%, semi-annual coupon, 20 year corporate bond is priced to yield 9%. The Macaulay duration for this bond is 8.85 years. Given this information, the bond's modified duration is ..." (1992 exam). Recognizing that the Macaulay duration in half years has been divided by two to get a value measured in years, the solution follows appropriately: $(\text{Macaulay duration})/(1 + (y/2)) = D^*/(1 + (y/2)) = 8.85/1.045 = 8.47$. In words, modified duration gives the percentage change in the bond price for a given change in yield (not percentage change in yield as required for Macaulay duration). Similarly: "An 8%, 15 year bond has a yield-to-maturity of 10% and a modified duration of 8.05 years. If the market yield changes by 25 basis points, how much change will there be in the bond's price?" (1991 exam). The solution is determined as: $\% \Delta P_B = \text{modified duration} \times dy = 8.05 \times .25 = 2.01\%$.⁸

A final example question is: "A 6% coupon bond paying interest semi-annually has a modified duration of 10 years and sells for \$800, and is priced at a yield to maturity (YTM) of 8%. If the YTM increases to 9%, the predicted change in price, using the duration concept, decreases by ..." (1994 exam). The solution to this question follows from observing that modified duration for semi-annual coupon bonds is:

$$D = \text{Modified duration} = \left(1 + \frac{y}{2}\right)^{-1} \text{ Macaulay duration (in years)}$$

$$-dP = (D^*)(dy)(P) \rightarrow PVBP = \frac{D^* \times P}{10,000}$$

where $PVBP$ is the present (price) value of a basis point. It follows: $(10 \times 800 \times 100 \text{ basis points})/10,000 = \80 . This formula for calculating price volatility is referred to a *dollar duration* (Fabozzi 1989, p. 74). Why divide by 10,000? This converts to one basis point (bp) = .0001 from 1 which is the unit of measurement for bp in the calculation.

The duration concept applies not only to bonds but to any sequence of future cash flows. As long as a single interest rate is used to discount the cash flows, the calculation of a duration measure is not complicated. For example, consider the duration of a perpetuity:

$$\frac{dP^{perp}}{d(1+y)} = \frac{dP^{perp}}{dy} = -\frac{C}{y^2} \rightarrow D^* = \frac{C}{y^2} \frac{1+y}{\frac{C}{y}} = \frac{1+y}{y}$$

As an example, if $y = 10\%$, then the duration of the perpetuity is 11 years. Changing to $y = 5\%$ the duration increases to 21 years. Similarly, the duration of a fixed annual coupon annuity can be calculated as:

$$\frac{dPV^A}{d(1+y)} = \frac{d}{dy} \left[\frac{1}{y} - \frac{1}{y(1+y)^T} \right]$$

$$\rightarrow D^* = -\frac{1+y}{PV^A} \frac{dPV^A}{d(1+y)} = \frac{1+y}{y} - \frac{T}{(1+y)^T - 1}$$

As an example, if $T = 10$ years and $y = 8\%$ then an annual fixed coupon annuity will have a duration of 4.87 years.

Assuming the constant growth in dividends required for the Gordon stock pricing model to hold (see section 8.1), it is also possible to calculate the duration of the common stock price (P_S) with respect to changes in the expected return on the stock (k):

$$P_S(0) = \frac{Div(1)}{k-g} \rightarrow \frac{dP}{dk} = -\frac{Div(1)}{(k-g)^2} \rightarrow D^* = \frac{1+k}{k-g}$$

where $Div(1)$ is the dividend paid at $t = 1$. For example, if the expected return on the stock is $k = 12\%$ and the growth rate of dividends is $g = 8\%$ then the duration of the

stock taken with respect to a change in expected returns is 28 years. To be comparable to the duration for, say, a bond some assumptions need to be made about dk/dy . A simpler formula would apply for the case where the preferred stock is being valued with the perpetuity pricing formula.

Historical background: Macaulay, Hicks, Samuelson and Redington

Following Bierwag et al. (1983) and others, it is conventional to date the starting point for the intellectual history of the duration concept as Macaulay (1938). This starting point has validity in a number of ways. One obvious way is in terms of the introduction of a specific formula; Macaulay stated the analytical solution for duration both in terms of the sum of discounted cash flows and as a closed form ratio expression. Macaulay can also be considered as a starting point for indepth analysis of the types of questions that the duration concept is designed to address. Though some such questions were identified previously, the empirical implications of interest rate changes and yield curve behavior were of secondary importance to theoretical questions about the origins of interest or the impact of changing price levels on the rate of interest, e.g., Fisher (1896). Solutions to the types of questions being addressed by Macaulay had not been systematically proposed. As such, the seminal contributions of Macaulay (1938) extend well beyond the introduction of the analytical formula for duration.

Section 1.3 advanced the position that Finance is a human science. As such, there is an element of timelessness in the intellectual contributions of the past. Unlike the natural sciences where knowledge about the external world is cumulative, progressing inductively toward an increasingly greater knowledge of natural phenomenon, understanding in the human sciences is more elusive. When accurate accounting has been made of differences in the historical context, classics of the past still can provide insights that may be of value to contemporary observers. In the same fashion that, say, Keynes (1936) or Graham and Dodd (1934) still contain insights, Macaulay (1938) also has contemporary relevance. Like Keynes and Graham and Dodd, Macaulay was profoundly influenced by the economic and social events associated with the Great Depression. The limitations of theoretical models based on rational decision makers are explicitly recognized in many places, e.g., "the 'laws' of a *completely* 'rational' economy cannot be formulated." Yet, like Keynes and Graham and Dodd, Macaulay (1938, p. 10) does see value in such analysis, if the limitations are correctly interpreted: "the relations that would exist under specific instances of accurate forecasting of *particular aspects* of human nature are . . . often easily uncovered."

Macaulay (1938) clearly understood the need to provide a better measure of the "length" of a bond than the term to maturity. Bonds provide a sequence of coupon cash flows together with the return of principal at maturity. By considering only the term to maturity, bonds with different coupons but the same maturity date appear to have the same length even though the lower coupon bond has a larger portion of the cash flow being paid farther in the future. To address this shortcoming of the term to maturity, Macaulay proposed the weighted average term to maturity concept now justly referred to as Macaulay duration. The use of "duration" to refer to the weighted average term to

maturity can be traced to the verbal description that accompanies the mathematical derivation:

Now, if present value weighting be used, the "duration" of a bond is an average of the durations of the separate single payment loans into which the bond may be broken up. To calculate this average the durations of each individual single payment loan must be weighted in proportion to the size of the individual loan; in other words, by the ratio of the present value of the individual future payment to the sum of all the present values, which is, of course, the price paid for the bond.

The notion of the price of a bond as the sum of the prices of the zero coupon component payments is clearly recognized. The mathematical statement of the formula is unambiguous.

The contrast between Macaulay (1938) and Hicks (1939) can be used as a pedagogical illustration of the difference between Finance and Economics, even though at the time both Macaulay and Hicks identified themselves as "economists." Where Macaulay is concerned with evaluating properties of traded securities, i.e., long-term and short-term bonds, Hicks (1939, p. 3) aims to "reconsider the value theory of Pareto and then to apply this improved value theory to those dynamic problems of capital that Wicksell could not reach with the tools at his command."⁹ In effect, Hicks (1939, p. 5) aims to develop a theoretical apparatus: "This is a work on Theoretical Economics, considered as the logical analysis of an economic system of private enterprise, without any inclusion of reference to institutional controls." This exercise is confounded by Hicks' claim to be deeply influenced by Keynes (1936), stating that Hicks (1939) is a "pedestrian" effort by comparison. The concern with abstract notions such as "value" and "capital" seems far removed from Macaulay's concerns with the yields on bonds.

Despite the concern with abstract notions, the duration concept is clearly presented in Hicks (1939, p. 186), not in the specific context of bond valuation but, rather, in the more general context of valuing a capital asset:

The capital value of a stream of payments ($x_0, x_1, x_2, \dots, x_v$) is $x_0 + \beta x_1 + \beta^2 x_2 + \dots + \beta^v x_v$.
The elasticity of this capital value with respect to the discount ratio β is

$$\frac{\beta x_1 + 2\beta^2 x_2 + 3\beta^3 x_3 + \dots + v\beta^v x_v}{x_0 + \beta x_1 + \beta^2 x_2 + \beta^3 x_3 + \dots + \beta^v x_v}$$

(for the elasticity of a sum is the *average* of the elasticities of its parts). Now when we look at the form of this elasticity we see that it may be very properly described as the *Average Period* of the stream: for it is the *average length of time for which the various payments are deferred from the present, when the times of deferment are weighted by the discounted values of the payments*.

where $\beta = 1/(1 + r)$. Because Hicks is examining a capital asset, the future payments $\{x_i\}$ can take a wide range of values. For example, if the capital asset is a "wasting asset liable to give out at some future date" then the payments from the asset will be declining over time. Hicks is concerned with evaluating the impact of interest rate

changes on these variable payments relative to the impact on a "standard stream" where the payments are "constant in money terms from week to week." In order to account for the depreciation of a wasting asset, to be equal in present value to a standard stream the receipts from the wasting asset would have to be higher than the standard stream in the near term and be "re-lent" in order to have sufficient payments in the future: "the lower the rate of interest is, the greater the sum he will have to re-lend in order for the interest on it to make up for the expected failure of receipts from his wasting asset in the future" (p. 187).

Hicks (1939, p. 187) clearly evaluates the impact of an interest rate change on capital assets of unequal durations:

It follows from all this that if the average period of the stream of receipts is greater than the average stream with which we are comparing it, a fall in the rate of interest will raise the capital value of the receipts stream more than that of the standard stream, and will therefore increase income. But if the average period of the stream of receipts is less than that of the standard stream, it is a rise in the rate of interest which will increase income.

In interpreting Hicks, it is important to recognize that, instead of a bond as the basic cash flow pattern, Hicks is concerned with a fixed annuity, i.e., the standard stream. Hicks (1939, p. 188) evaluates the duration of a perpetuity:

What is in fact the average period of a stream of constant size and indefinite length, discounted throughout at the same rate of interest? It can easily be shown that it is equal to the reciprocal of the rate of interest, i.e., to the number of "years purchase." If the rate of interest is 5 per cent. per annum, the average period of a standard stream is 20 years.

In a footnote, Hicks solves for the duration of the perpetuity as $1/r$. Hence, despite defining the average period as an elasticity, i.e., Macaulay duration, Hicks actually has the modified duration in mind.

By comparison to the tomes of Macaulay (1938) and Hicks (1939), the short journal article of Samuelson (1945) is a pale effort. Upon closer inspection it is difficult to see why Bierwag et al. (1983) and Bierwag (1987) give it recognition similar to Hicks (1939) and Redington (1952). The primary contribution is that Samuelson (1945) takes a step towards the use of duration in fixed income portfolio management, moving away from the use of duration as an adjusted term to maturity measure of Macaulay and as an analytical tool in the theoretical abstractions of Hicks. The basic insight is to consider the impact of interest rate changes on the banking system by evaluating the relative change in value of asset and liabilities. This evaluation is accomplished by taking a derivative of the aggregated fixed income prices with respect to a change in interest rates. The implications of interest rate changes for insurance companies and universities with endowment funds are also considered. Recognizing what was later called the "duration gap," Samuelson (1945) recognizes that an increase in interest rates will impact the value of assets differently from that of liabilities when the "durations" are not equal.

Bierwag (1987, p. 58) claims that "Samuelson (1945) . . . independently invented the duration measure." This statement is difficult to sustain. Observing that there are no

references given in the article, Samuelson uses the term "weighted average time period" to refer to the duration concept. The derivation of the weighted average time period is given as (p. 19):

Let N_t = inpayment t years after the present, C_t = corresponding outpayments, V = present value, i = interest rate per annum averaged over time. Then:

$$V = \sum \frac{N_t}{(1+i)^t} - \sum \frac{C_t}{(1+i)^t}$$

$$\text{and } \frac{dV}{dt} = -\frac{\ln(1+i)}{(1+i)^2} \left\{ \sum \frac{tN_t}{(1+i)^{t-1}} - \sum \frac{tC_t}{(1+i)^{t-1}} \right\}$$

By rearranging terms we find that $(dV/dt) > 0, = 0, < 0$ depending upon whether $\bar{N} > \bar{C}$, $\bar{N} = \bar{C}$, $\bar{N} < \bar{C}$, where $\bar{N}$, $\bar{C}$ are respectively weighted average time periods of inpayments and outpayments, whose weights are proportional to discounted dollar amounts.

Though there are some interesting elements, this formulation cannot fairly be called an independent invention of the duration measure. The formula is not expressed as an elasticity and not identified as relating to adjusted term to maturity.

Bierwag et al. (1983, p. 16) correctly observe that Samuelson (1945, p. 19) reaches the conclusion: "increased interest rates will help any organization whose [weighted] average time period of disbursements is greater than the average time period of receipts." In modern terms, if the duration of liabilities is greater than the duration of assets, increasing interest rates will increase net worth. Yet, Samuelson (p. 25) goes on to conclude: "the banking system as a whole is immeasurably helped rather than hindered by an increase in interest rates. Indeed, it receives much greater benefit than either universities or insurance companies, and commercial banks would profit more than savings banks." The basis for this conclusion is that "the average time period of its inpayments is less than that of its outpayments." This is contrary to the received view that financial institutions such as banks face a duration gap where the duration of assets, e.g., mortgages, exceeds the duration of liabilities, e.g., checking deposits.

In contrast to Samuelson (1945), by explicitly introducing and developing the concept of "immunization" Redington (1952) does represent a significant advancement in the development of the duration concept, though Redington works only with the derivatives and does not recognize the connection to an elasticity measure. Redington (1952, pp. 288-9) introduces immunization as a fundamental improvement on the concept of "matching" where: "The word 'matching' implies the distribution of assets to make them, as far as possible, equally as vulnerable as the liabilities to those influences which affect both. In its widest sense this principle includes such important aspects as the matching of assets and liabilities in currencies." Given this, Redington makes the following observation:

The word "matching" has such a wide and general connotation that it is necessary to adopt a new label with a more precise significance. For this purpose, I use the word "immunization" to signify the investment of the assets in such a way that the existing business is

immune to general change in the rate of interest. The definition is not exact, but it should not mislead. On the basis of this definition immunization is to be regarded as a particular form of matching.

From this point Redington explicitly assumes a "uniform rate of interest whatever the term," i.e., a flat yield curve, and then proceeds to apply a Taylor series expansion to derive the conditions on assets and liabilities required to achieve immunization.

As discussed in section 5.1, Redington is able to theoretically derive the two rules for classical immunization: match the duration of assets and liabilities; ensure that the convexity of assets is greater than liabilities. For this contribution, Redington has been justly recognized. However, development of the classical immunization rules occupies only a small part of Redington (1952). The bulk of the article is concerned with practical application of the rules. In this discussion, Redington recognizes numerous limitations that represent a contribution over and above the theoretical analysis. For example, Redington recognizes the following complications: difficulties in determining the yields on assets and liabilities; the presence of options in both the assets and liabilities; indeterminacy in cash flow estimation where the assets contain equities, properties and certain types of mortgages; the need to actively manage asset portfolios; and the presence of surplus funds.

The uses and calculation of duration

Duration is one of those fascinating concepts that have applications that extend well beyond the basic mechanics of the calculation. The discussion of the historical background reveals at least three different uses that can be identified: a measure of the adjusted or weighted average term to maturity (Macaulay); an elasticity measure of the sensitivity of a sequence of fixed income cash flows to changes in the interest rate (Hicks); and, as reflected in Redington's rules for classical immunization, as a tool for the management of fixed income portfolios. As an adjusted term to maturity, inspection of the duration formula reveals that, for a zero coupon bond, $D^* = T$. Hence, any calculated duration value reexpresses the "term to maturity" of a cash flow pattern in terms of the term to maturity of a zero coupon bond. This raises the practical question of how much the stated term to maturity will differ from the duration. Numerous examples will be given in section 5.1. At this point, all that is being illustrated is some basic calculations.

Observing that $D^* = T$ for a zero coupon bond, it is apparent that as C increases the duration will fall. This raises the question about how much the duration falls as the coupon rises. To see an example of this effect, calculate the durations of a 3 year, 4% annual coupon bond, a 3 year 8% annual coupon bond and a 3 year 10% annual coupon bond - all priced to yield 8% to maturity. These durations are:

$y = 8\%$	$C = \$4$	$C = \$8$	$C = \$10$
Duration (D^*)	2.879	2.783	2.742
P_B	89.6916	100	105.154

The duration values are not much different than the zero coupon duration ($= 3$) because the total value of the bonds has relatively little coupon income compared to the principal plus coupon value that is paid at $t = 3$. For example, when $C = \$4$ only $\$8$ (not adjusted for discounting) is paid prior to maturity while $\$104$ is paid at maturity. The impact of increasing the number of coupon payments is evident from setting $T = 30$ and $C = \$8$, then the duration is 12.16 years.

As will be apparent in section 5.2, duration is a point elasticity measure. It is well known that point elasticities can be quite inaccurate for large changes in the exogenous variables. To see this, consider how much the value of a 3 year, 4% coupon bond changes if interest rates increase 400 basis points from 8% to 12%. What would the duration formula predict? The relevant calculations reveal:

$$\begin{aligned} y = 12\% \quad \text{Duration } (D^*) &= 2.87211 & \Delta P &= (D^*)(\Delta y / (1 + y_0)) * P_0 \\ P_B[y = .08] &= 80.7853 & &= 9.56438 \end{aligned}$$

The duration estimate can be compared with the actual $\Delta P = 89.6916 - 80.78653 = 8.90507$. Even for the case of short term bonds, where the impact on price of a change in interest rates is not large, duration gives a relatively inaccurate estimate. Recognizing that the bond price function can be expanded in a Taylor series, to correct for this inaccuracy higher order terms in the Taylor series, i.e., the convexity term, can be included. The conclusion required at this point is that, for large changes in yield, the duration estimate deviates significantly from actual price change.

Duration measures the impact of a change in the interest rate on the price of the bond. When interest rates change there are two inversely related impacts on the bond value. Consider the case of an increase (decrease) in interest rates. In this case, the principal value plus last coupon will fall (rise) in price due to the impact of compounding. This is the *principal risk*. However, a higher (lower) interest rate means that coupons paid prior to the maturity can be reinvested at a higher (lower) interest rate. This is *reinvestment risk*. When used as a tool in balance sheet immunization, duration provides a method of determining when the positive (negative) reinvestment risk just balances against the negative (positive) principal risk. Such duration matching considerations arise in other contexts, such as a bond trader making a yield curve play by, say, shorting an intermediate-term bond and buying a short term and long term bond. Duration provides a guide to the relative amounts of the short term and long term bonds that have to be purchased to just offset the interest rate risk of the intermediate-term bond.

Though most of the discussion to this point has been concerned with the duration of individual bonds, duration also plays an important role in bond portfolio management. The extension from portfolios with single bond positions to portfolios with many bonds is straightforward. Consider a portfolio containing three bonds with prices P_1, P_2, P_3 . These prices are actual prices making the total value of the portfolio (V): $V = P_1 + P_2 + P_3$. It follows:

$$\begin{aligned} \frac{dV}{dt} &= \frac{dP_1}{dt} + \frac{dP_2}{dt} + \frac{dP_3}{dt} \rightarrow \frac{1}{V} \frac{dV}{dt} = \frac{P_1}{V} \frac{1}{P_1} \frac{dP_1}{dt} + \frac{P_2}{V} \frac{1}{P_2} \frac{dP_2}{dt} + \frac{P_3}{V} \frac{1}{P_3} \frac{dP_3}{dt} \\ D_V &= w_1 D_1 + w_2 D_2 + w_3 D_3 \end{aligned}$$

where the value weights w_i are the same as those defined in the mean-variance portfolio model in section 3.1. For example, if the portfolio is composed of equal amounts of the three 3 year bonds described above:

$$D_p = \sum_{i=1}^3 w_i D_i \quad \text{where} \quad w_i = \frac{\$P_i}{\sum_{i=1}^3 P_i}$$

The duration of portfolio = $(1/3)2.87918 + (1/3)2.78326 + (1/3)2.74236 = 2.8016$.

4.3 THE TERM STRUCTURE OF INTEREST RATES

Calculation of implied zero coupon interest rates

The *term structure of interest rates* is concerned with the empirical relationship between term to maturity and the implied zero coupon rates calculated from the default free coupon bond market.¹⁰ Because implied zero rates are not directly observed, it is necessary to estimate these variables from observed coupon bond prices. This involves bootstrapping spot interest rates from semi-annual coupon bond prices. The bootstrapping technique, e.g., Fabozzi (2000), Sundaresan (2002), is an iterative process for calculating implied zero coupon interest rates (spot interest rates) from observed coupon bond rates. The process requires the observed yields for coupon bonds of each relevant term to maturity along the yield curve. In practice, spot rates would typically be extracted from the yield curve for federal government bonds, Treasury bonds in the US or government of Canada bonds in Canada. Because these types of bonds pay semi-annual coupons, precision requires that the bootstrap be executed at semi-annual intervals.

For purposes of illustrating the bootstrap technique, assume that the relevant bonds are sold at par and pay coupons semi-annually. Further assume that the observed 6 month yield is 8.87%. For the US implied zero curve, this rate would be obtained by taking the observed 6 month Treasury bill discount rate and converting to a true yield basis. Because Tbills do not pay coupons this means that the quoted yield is for a 6 month zero coupon bond. If the observed yield on a 1 year semi-annual coupon bond is assumed to be 9.04, for a \$100 par value bond this implies a semi-annual coupon payment of 4.52. Given this, the iteration for solving a sequence of implied zero coupon rates begins by discounting the first semi-annual coupon payment at the 6 month, zero coupon rate and solving for the implied 1 year zero coupon rate. For a bond sold at par this requires that:

$$100 = \frac{4.52}{1 + \frac{.0887}{2}} + \frac{104.52}{\left(1 + \frac{z_1}{2}\right)^2}$$

where z_1 is the implied 1 year zero coupon rate, which can be calculated as 0.090438.

Having solved for z_1 , the next step in the iteration involves using z_1 to solve the implied zero coupon rate, $z_{1.5}$, using a 1.5 year par coupon bond. If the observed rate on 1.5 year coupon bonds is 9.155, then this implies a semi-annual coupon payment on a \$100 par bond of 4.5775. This leads to:

$$100 = \frac{4.5775}{1 + \frac{0.0887}{2}} + \frac{4.5775}{\left(1 + \frac{z_1}{2}\right)^2} + \frac{104.5775}{\left(1 + \frac{z_{1.5}}{2}\right)^3}$$

Substituting the value for z_1 determined previously and solving gives $z_{1.5} = 0.091629$. The next step in the iteration involves solving for z_2 . Taking the observed 2 year yield to be 9.2% produces:

$$100 = \frac{4.6}{1 + \frac{z_2}{2}} + \frac{4.6}{\left(1 + \frac{z_1}{2}\right)^2} + \frac{4.6}{\left(1 + \frac{z_{1.5}}{2}\right)^3} + \frac{104.6}{\left(1 + \frac{z_2}{2}\right)^4}$$

This formula can be used to solve for z_2 , using the previously computed values for z_1 and $z_{1.5}$. This iterative process continues until the zero coupon rate for the desired term to maturity is calculated. The relevant zero coupon rate can be used to do calculations involving implied zero rates, e.g., solving for implied forward interest rates.

In certain cases, exactly precise implied zeros are not required. If this is the case, then it is possible to achieve computational simplifications by proceeding under a number of assumptions. In particular it is possible to reduce the number of computations by a factor of two by assuming that the observed government bond prices are for annual coupon bonds. Another simplification can be achieved by taking the nearest available bond instead of estimating a par bond yield curve. To see how this simplified calculation process works, consider the example in Poitras (2002, p. 250) that involves solving for the Canadian spot interest rates from quotes obtained from the *Globe and Mail* for August 28, 1994. This involves picking the following bills/bonds from the available maturities:

- 1 year Tbill $z_1 = .0720$
- 6.5% 1 Aug. 1996 $P_2 = 97.505$ ($y_2 = .07887$)
- 7.5% 1 Jul. 1997 $P_3 = 98.225$ ($y_3 = .08197$)
- 6.5% 1 Sep. 1998 $P_4 = 93.350$ ($y_4 = .08474$)
- 7.75% 1 Sep. 1999 $P_5 = 96.800$ ($y_5 = .08542$)

These bonds were selected because they were closest to the required maturity dates.

Two possible bootstrap solution techniques are available: the direct approach and the par bond approach. The direct approach involves using the observed price and coupon to solve for the spot interest rate. Solving for z_2 :

$$97.505 = 6.5/(1 + z_1) + 106.5/(1 + z_2)^2$$

Using this method, $z_2 = .0792$. As given above, the par bond approach uses the result that when the stated yield to maturity equals the C/M then the bond sells at par:

$$100 = 7.887/(1 + z_1) + 107.887/(1 + z_2)^2$$

Using this method $z_2 = .0791428$. The difference of .6 of a basis point is due to a combination of the assumption that the bond pays annual coupons, i.e., a semi-annual yield is used as an annual yield, and to the difference between the actual maturity date (August 1) and the required maturity date (August 28). Calculating the spot rates out to 5 years, it is evident that the differences involved are generally small:

$$\text{Par bond: } z_3 = .082423 \quad z_4 = .0854697 \quad z_5 = .0861374$$

$$\text{Price/coupon: } z_3 = .08232 \quad z_4 = .08595 \quad z_5 = .08630$$

Even for the 5 year implied zero, the difference is only 1.7 basis points. Observe that when the yield curve slopes up, as in this case, the implied zero curve will be above the yield curve. Similarly, when the yield curve slopes down, the implied zero curve will be below the yield curve. When the yield curve is flat, then both curves will be equal. This relationship follows mathematically from observing that the yield to maturity acts as a form of geometric average of the spot rates.

Traditional theories of the term structure

As noted, the term structure of interest rates refers to the functional relationship between the term to maturity and the associated implied zero coupon interest rates. This is a theoretical development on the yield curve that refers to the functional relationship between term to maturity and yield to maturity. Though the yield curve can be readily determined from inspection of bond price/yield quotes available every day in the financial press, the yield curve suffers from the defect that the term to maturity is an inaccurate measure of the true "length" of the cash flow from a bond. As observed in the discussion of duration, differences in coupons for bonds with the same term to maturity means that the time pattern of the cash flows for these bonds will differ. Because a greater proportion of the cash flow will occur later in the future, a bond with a lower coupon will behave like a longer term bond compared to a bond with a higher coupon. This problem is corrected by considering the term structure of interest rates. Changes in the shape, slope and level of the term structure are a fundamental source of risk in the valuation of individual bonds and in the management of bond portfolios, e.g., Crack and Nawalkha (2000).

Explaining the empirical behavior of the term structure of interest rates at a given point in time and over time is a fundamental problem in fixed income security analysis. Analysis of this problem has a long history. Traditional explanations use the interpretation of implied forward rates as the defining elements to distinguish different explanations. These explanations can be roughly divided into three separate categories: the unbiased expectations hypothesis; the liquidity premia or risk premia hypothesis; and

the market segmentation or preferred habitat hypothesis. Under the unbiased expectations hypothesis of the term structure, implied forward rates are the market's prediction about what interest rates will be in future periods. Under the liquidity premia hypothesis, implied forward rates are upwardly biased predictors of future interest rates. Under the market segmentation hypothesis, it is not possible to formulate predictions about future interest rates from the shape of the observed term structure of interest rates.

Given the importance of the implied forward rate to the theoretical analysis of term structure behaviour, it is somewhat surprising that a standard notation for forward rates is not available. Complete notation requires three variables: the time the forward interest rate starts, the time the forward interest rate ends and the date the yield curve which is used to derive the forward interest rate is observed. As bonds are typically used to derive the implied forward rates, time is measured in years. Different sources use notation that only takes account of some of these three variables while others, e.g., van Horne (2001) use a complete notation such as ${}_{t+n}r_{jt}$ where $t + n$ is the start date, j is the term to maturity of the implied security and t is the date the term structure is observed. In contrast, Bodie, et al. (p. 589) use an incomplete notation: f_t where t refers to the start date, and it is assumed that all the terms to maturity for the implied securities are 1 year and that the date for observing the term structure is t . This is very limited notation.

It is assumed in developing the notation used here that the term structure is always observed at $t = 0$, i.e., in the notation $f_{i,j}$, this is the implied forward interest rate observed at $t = 0$, starting at time i and ending at time j . It is assumed in developing the notion used here that the term structure is always observed at time $t = 0$, so the time date is dropped from the notation. In this case, j minus i gives the term to maturity of the implied security. For example: $f_{1,2}$ is the implied forward interest rate starting at year 1 and ending at year 2; $f_{5,10}$ is the implied forward interest rate starting at year 5 and ending at year 10. This method of subscripting is also applied to future spot interest rates, i.e., the notation applied to future spot interest rates is $z_{i,j}$, the spot interest rate for a zero coupon bond which starts at time i and ends at time j . It is important to recognize that, in the following discussion, all expectations are assumed to be conditional and taken on information available at $t = 0$.

The three general groups of theories about term structure behaviour are concerned with different explanations of the relationship between the current term structure and future term structure level, slope and shape. These theories of the term structure of interest rates are motivated using the relationship between implied forward rates and expected spot interest rates. Because implied forward rates depend on the shape of the term structure, the various theories of the term structure seek to make predictions about the behaviour of future interest rates based on the level, slope and shape of the observed term structure of interest rates. The different theories each have a different interpretation about the relationship between the implied forward interest rate and predictions about future spot interest rates.

UNBIASED EXPECTATIONS HYPOTHESIS

The sharpest group of theories fall into the general category of the *unbiased expectations hypothesis* (UEH). Under the UEH, implied forward rates are unbiased predictors of

future spot rates: $f_{i,j} = E[z_{i,j}]$. To identify the relationship between the current term structure and predictions of future interest rates under the UEH, assume that interest rates are expected to be unchanged, then: $E[z_{1,2}] = z_1$. From the two portfolio (rollover vs. buy and hold) example given above, it follows that in this case:

$$(1 + z_1)(1 + E[z_{1,2}]) = (1 + z_1)^2 = (1 + z_2)^2$$

Hence, under the UEH assuming interest rates will not change implies $z_1 = z_2$ and the observed yield curve will be flat. In other words, under the UEH a flat yield curve at $t = 0$ implies that future spot interest rates are expected to be unchanged. If this is not true, then under the UEH investors will sell the portfolio with the lower expected return and purchase the higher expected return portfolio. This will produce an adjustment in the prices for the 1 year and 2 year securities to the point where the yield curve shape reflects the underlying market expectations.

Assuming that interest rates are expected to rise, then: $E[z_{1,2}] > z_1$. From the two portfolio example, it follows that because:

$$(1 + z_1)(1 + E[z_{1,2}]) = (1 + z_2)^2$$

It must be that $z_1 < z_2$ and the observed ($t = 0$) yield curve will be upward sloping, i.e., under the UEH an upward sloping yield curve implies that future spot interest rates are expected to be higher than current rates. Similarly, when interest rates are expected to fall, $E[z_{1,2}] < z_1$ and the $t = 0$ yield curve will be downward sloping. UEH is attractive because it gives sharp predictions but it is not clear that these predictions are always accurate.

The UEH argument becomes more complicated if a three period investment horizon is considered. In this case, the two portfolio problem is expanded to compare a three period buy and hold strategy with a strategy of buying a 1 year and rolling forward either into another one year (followed by another one year) or into a 2 year. Consider the equality of the buy and hold with rolling three 1 years:

$$\begin{aligned}(1 + z_3)^3 &= E[(1 + z_1)(1 + z_{1,2})(1 + z_{2,3})] \\ &= E[1 + z_1 + z_{1,2} + z_{2,3} + z_1 z_{1,2} + z_1 z_{2,3} + z_{1,2} z_{2,3} + z_1 z_{1,2} z_{2,3}]\end{aligned}$$

In this case, if rates are expected to be unchanged, $E[z_{1,2}] = z_1$, $E[z_{2,3}] = z_1$, it does *not* follow that the UEH implies the current yield curve is flat.

To see this, refer back to the two period case:

$$\begin{aligned}(1 + z_2)^2 &= 1 + 2z_2 + z_2^2 = E[(1 + z_1)(1 + z_{1,2})] \\ &= E[1 + z_1 + z_{1,2} + z_1 z_{1,2}] = 1 + z_1 + E[z_{1,2}] + z_1 E[z_{1,2}]\end{aligned}$$

It follows that if rates are expected to be unchanged, $E[z_{1,2}] = z_1$, then the current yield curve will be flat ($E[z_{1,2}] = z_1 = z_2$) and the implied forward rate will equal the market

expectation of the future spot rate. This result does not follow for the three period (and greater) case because:

$$E[z_{1,2}z_{2,3}] \neq E[z_{1,2}]E[z_{2,3}] \rightarrow E[z_{1,2}z_{2,3}] = \text{cov}[z_{1,2}, z_{2,3}] + E[z_{1,2}]E[z_{2,3}]$$

Hence, unless implied zeros are serially uncorrelated ($\text{cov}[z_{1,2}, z_{2,3}] = 0$) then there will be an advantage to the rollover strategy compared to the buy and hold strategy.

Faced with a flat yield curve, the presence of serial correlation in future implied zero rates permits the rollover strategy to achieve, on average, higher returns compared to the buy and hold strategy. In the three period case, this follows because after one period the rollover strategy can observe the direction of rates and, if rates move adversely, then the rollover can be made into a 2 year security, avoiding the potential loss associated with the future rollover into a 1 year security. Similarly, if rates move favorably, then it is possible to rollover into another 1 year seeking to gain from a further favorable movement in rates that is captured by the final rollover into a 1 year security. Though the analysis is more complicated, this argument generalizes to longer holding periods and yield curves that slope upward or downward. In effect, there is a form of "liquidity premium" associated with serial correlation in spot rate changes. Hence, even if rates are expected to be unchanged, the UEH could imply that this is consistent with an upward sloping yield curve because short rates would have to be lower than long rates to compensate for the higher potential expected return associated with the rollover strategy compared to buy and hold.

The complication in specifying the precise specification and associated implications of the UEH have been explored by Ingersoll (1987, ch. 18) where three distinct forms of the UEH are identified: the local expectations hypothesis; the yield to maturity expectations hypothesis; and the return to maturity expectations hypothesis. The return to maturity version applies to the UEH where accurate accounting is given to the possibility of positive covariance between future spot rates. The yield to maturity version converts the spot rates to yields and still accounts for the positive covariance. The local expectations hypothesis retains the condition that the t to $t + 1$ implied forward rate $f_{t,t+1} = E[z_{t,t+1}]$. Ingersoll expends considerable effort to demonstrate that the local expectations hypothesis is the only version of the UEH that is consistent with absence of arbitrage. As demonstrated in chapters 5 and 6 and other sources, e.g., Sundaresan (2002, p. 230) this result is employed in binomial and other models of term structure behavior that, in turn, are used to value fixed income contingent claims and other securities.

LIQUIDITY PREMIUM HYPOTHESIS

The potential for serial correlation in implied zero rates is not the only possible theoretical argument for a liquidity premium in implied zero rates. For example, the need to hold precautionary cash balances could produce a higher demand for short term securities relative to long term securities that would appear as a liquidity premium in short rates. An important motivation for liquidity premia is risk aversion. As a consequence

of this motivation, the next group of explanations for term structure behavior, the liquidity premia hypothesis (LPH), is sometimes referred to as the risk premia hypothesis (RPH). The LPH maintains that, because the buy and hold portfolio is less liquid (or, under RPH, has greater price risk) than the rollover portfolio, then the return on the buy and hold portfolio has to be higher in order to compensate, i.e., if rates are expected to be unchanged:

$$(1 + z_1)(1 + E[z_{1,2}]) < (1 + z_2)^2$$

It follows that, under LPH, the implied forward rate will no longer be an unbiased predictor of future interest rates:

$$1 + f_{i,j} > 1 + E[z_{i,j}]$$

In other words, the $f_{i,j}$ are *upward biased predictors* of future interest rates.

This upward bias associated with implied forward rates can be used to define the *liquidity premia* (LP):

$$1 + f_{i,j} = 1 + E[z_{i,j}] + LP_{i,j} \quad \text{or} \quad f_{i,j} - LP_{i,j} = E[z_{i,j}]$$

Under the LPH, an important feature of the liquidity premia is the *monotonicity property*:

$$0 < LP_{1,2} < LP_{1,3} < LP_{1,4} < \dots$$

In other words, the longer the term to maturity of the implied security, the higher will be the liquidity premia.

One drawback of the LPH is the lack of sharpness for making interest rate predictions from the shape of term structure. Assume that interest rates are expected to be unchanged, then: $E[z_{1,2}] = z_1$. From the two portfolio example, it follows that in this case: $(1 + z_1)^2 < (1 + z_2)^2$. This result is due to the presence of the liquidity premia. Hence, $z_1 < z_2$ and the observed yield curve will be upward sloping when rates are expected to be unchanged. This result makes it more difficult to make predictions about future interest rates if LPH is correct. Under the LPH a flat or downward sloping yield curve implies that future spot interest rates are expected to fall. However, when the yield curve is *upward sloping* the LPH is unable to make precise predictions. On balance, under LPH there is a wide range of yield curve shapes for which no prediction is possible, the prediction depends on the size of the liquidity premia. Empirical attempts to estimate the liquidity premia are mixed. At best, the premia appear to be *time varying*.

MARKET SEGMENTATION AND PREFERRED HABITAT HYPOTHESES

The UEH made sharp predictions about the relationship between implied forward rates and expected implied zero rates. The LPH made less sharp predictions but still imposed

monotonicity restrictions on the liquidity premia. The last group of theories is characterized by making either a weak or no connection between implied forward rates and future implied zero rates. There are subtle differences between the two main theories in the last group. One theory is the *preferred habitat hypothesis*. The basic idea of this hypothesis is that investors prefer to invest in certain maturity segments of the yield curve but will switch to other maturities if the premia are high enough. This hypothesis has two implications: there are still liquidity premia; and, the liquidity premia no longer obey the monotonicity property, i.e., liquidity premia do not necessarily increase in size with term to maturity. In order to make inferences about the connection between yield curve shape and market expectations about future interest rate changes, some versions of the preferred habitat hypothesis incorporate other variables to explain liquidity premia behavior.

In contrast to the preferred habitat hypothesis, the *market segmentation hypothesis* maintains that investors are rigidly restricted in the maturities in which to invest. Maturity switching costs are very high, significantly higher than in the preferred habitat hypothesis, leaving yield curve shape to be fully determined by the supply and demand within the various maturity segments along the yield curve. This approach is commonly found in the financial press where considerable attention is given to, say, the impact on the yield curve of the specific tranche sizes at the US Treasury auction. Both of the preferred habitat and market segmentation hypotheses are unable to formulate predictions about future interest rates from the shape of the observed term structure of interest rates. Both of these theories explain, either in whole or in part, specific yield curve shapes by referencing the relative supply and demand for securities along the yield curve.

Estimating term structure models

Term structure estimation appears in a number of distinct forms. One form deals with the problem of getting the best fit equation, at a given point in time, to the observed set of bond yields. This estimation problem is aimed at providing a complete estimated set of par bond yields that can be used to determine implied zeros for each cash flow payment date. Given a best fit equation, the relevant estimated bond yields (or prices) can be determined from the fitted equation and used to solve for the implied zeros using the method described above. Alternatively, following McCulloch (1971), Linton et al. (2001) and others, it is possible to estimate the *discount function* directly and use this estimated function to directly derive the implied zero coupon interest rates. These implied zeros can then be used to calculate implied forward rates and to provide an estimated yield curve.

The general procedure suggested by McCulloch (1971) and adapted and extended by others can be motivated by observing that the discount function at time t , $DF(t)$, can be approximated by a smoothly declining polynomial function. The discount function is defined by the bond pricing function:

$$P_B = \sum_{t=1}^T \frac{C}{(1+z_t)^t} + \frac{M}{(1+z_T)^T} = \sum_{t=1}^T C DF(t) + M DF(T)$$

of this motivation, the next group of explanations for term structure behavior, the liquidity premia hypothesis (LPH), is sometimes referred to as the risk premia hypothesis (RPH). The LPH maintains that, because the buy and hold portfolio is less liquid (or, under RPH, has greater price risk) than the rollover portfolio, then the return on the buy and hold portfolio has to be higher in order to compensate, i.e., if rates are expected to be unchanged:

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$$P_B = \sum_{t=1}^T \frac{C}{(1+z_t)^t} + \frac{M}{(1+z_T)^T} = \sum_{t=1}^T C DF(t) + M DF(T)$$

While this specification is for a discrete $DF(t)$, in general the function is continuous and the discrete form is only used for pedagogical and practical purposes. For simplicity of exposition, assume that $DF(t)$ is a quadratic function:

$$DF(t) = d_0 + d_1t + d_2t^2$$

Substituting in the bond price function gives:

$$P_B = \left(\sum_{t=1}^T C(d_0 + d_1t + d_2t^2) \right) + M(d_0 + d_1T + d_2T^2)$$

Reexpressing this in terms of the cash flows occurring at each period of time ($CF(t)$) produces:

$$\begin{aligned} P_B &= \sum_{t=1}^T CF(t) (d_0 + d_1t + d_2t^2) \\ &= d_0 \left(\sum_{t=1}^T CF(t) \right) + d_1 \left(\sum_{t=1}^T t CF(t) \right) + d_2 \left(\sum_{t=1}^T t^2 CF(t) \right) \end{aligned}$$

This provides the specification of a regression equation that can be estimated for a sample of bonds. Given the estimated coefficients, the discount function at any point in time can be determined by evaluating the estimated $DF(t)$ at that point, e.g., say $t = 2.5$ years then $DF(2.5) = d_0 + d_1(2.5) + d_2(2.5)^2$. An implied zero rate is obtained by inverting specific values of the discount function. In recent studies the polynomial function used is of the spline form, e.g., Fisher et al. (1995), that can be conceived as a piecemeal sequence of polynomial functions connected at appropriately specified knot points. However, despite the more sophisticated estimation techniques, the general procedure described is representative of the methodologies used to estimate the discount function.

Another type of term structure estimation deals with the problem of fitting a model that best describes the evolution of the term structure over time. This estimation problem is significantly more complicated than the problem of fitting the discount function and has a number of different applications. Included in these applications are the pricing of fixed income securities with embedded options, e.g., callable bonds and mortgage backed securities, and the pricing of interest rate derivatives. An important feature of theoretical bond pricing models is the method of converting a stochastic model of the interest rate into a prediction about the evolution of the term structure. This can be illustrated by considering a *one factor model* of the term structure. In this case a stochastic model is assumed for the short term zero coupon interest rate. For simplicity, say the relevant rate is a one year rate, r_t . In addition to the observed one year rate that can be taken as given, $r_1 = z_1$, the fitted stochastic model for this interest rate is used to generate a sequence of future one year interest rates, $r_{1,2}, r_{2,3}, r_{3,4}, r_{4,5} \dots$. These interest rates can be used to discount the contingent cash flows from securities with embedded options.

Chapter 6 devotes considerable attention to the valuation of fixed income securities with embedded options and the specification of various one factor models is discussed there. What remains to be discussed at this point is the *calibration* problem. In particular, under the UEH, a sequence of future interest rates can be interpreted as implied forward rates for one year securities. Assuming that the UEH is true, then the stochastic sequence of future interest rates can be used to recreate the current term structure, e.g., in a Monte Carlo framework the following relationships hold along any given path:¹¹

$$(1 + z_2)^2 = (1 + z_1)(1 + f_{1,2}) = (1 + z_1)(1 + r_{1,2})$$

$$(1 + z_3)^3 = (1 + z_1)(1 + f_{1,2})(1 + f_{2,3}) = (1 + z_1)(1 + r_{1,2})(1 + r_{2,3})$$

$$(1 + z_4)^4 = (1 + z_1)(1 + f_{1,2})(1 + f_{2,3})(1 + f_{3,4}) = (1 + z_1)(1 + r_{1,2})(1 + r_{2,3})(1 + r_{3,4})$$

Because the stochastic model can be used to generate any number of equally likely future paths for short term interest rates, the equally weighted average of the implied zeros calculated using the above recursion is required to equal the observed term structure.

The calibration problem occurs because it is usually the case that the stochastic model is not capable of recreating the current term structure using the recursion procedure described above. As a consequence, the future sequence of stochastic rates, $r_{1,2}$, $r_{2,3}$, $r_{3,4}$, $r_{4,5}$, ..., has to be "calibrated" by adding (or subtracting) some value sufficient to ensure that the current term structure is obtained. While this might seem somewhat suspicious, the need to calibrate is not surprising. The parameters of the stochastic model are estimated from past data and, as a consequence, are like an average of the stochastic behavior of the short rate over the sample period. Because the shape of the term structure changes over time, there is likely to be some deviation of the current term structure from the average of past values. The calibration problem arises because there is no theoretical direction given about how calibration is to be achieved. This raises questions about the validity of valuations based on arbitrary calibration methods.

To address the calibration problem a number of methods for estimating the term structure, e.g., Ho and Lee (1986), Black, Derman and Toy (1990) and Heath, Jarrow and Morton (1992), invert the process of model specification. The conventional process is to specify a stochastic model of interest rates, estimate the parameters of the model and then compute the implied zeros and the prices of the zero coupon bonds. The implied zeros are then compared with the implied zeros calculated from the observed Treasury yield curve and the parameters are calibrated to enable the model to fit actual prices. The calibrated procedures use the observed market data as a starting point and then determine the stochastic evolution of interest rates from that starting point. The various calibrated models that are available differ in the procedure by which this is done. Ho and Lee (1986), for example, model the evolution of the whole term structure using a binomial process, imposing arbitrage restrictions on admissible changes. Black, Derman and Toy (1990) model the evolution of spot rates while Heath, Jarrow and Morton (1992) model the evolution of forward rates.

In addition to estimating discount functions and stochastic models, term structure estimation also includes studies of the relationship between term structure shape and future economic activity, e.g., Harvey (1997), Plosser and Rouwenhorst (1994). This type

of term structure estimation raises a number of econometric issues about the specification of the model generating the term structure, e.g., Is a VAR in levels appropriate? Is there cointegration? How are structural breaks to be incorporated? Leaving these issues aside, there is reasonably strong statistical evidence of a positive relationship between interest rate spreads and future growth rates in real GDP. In effect, the slope of the term structure is a leading indicator of economic growth, widening prior to expansions and contracting prior to downturns. This empirical result is consistent with mean reversion in the slope of the term structure.

4.4 BASICS OF CREDIT RISK AND DEFAULT RISK

The legal aspects of default

For analytical purposes, there are considerable advantages to abstracting from credit risk and embedded option features. Yet, the bulk of fixed income securities do not conform to this abstraction.¹² Even for bonds that would seem to conform to the abstraction of no credit risk, such as US Treasuries, there is still an element of credit risk that has to be addressed. For example, on rare occasions the US Congress has threatened to withhold legislation to increase the debt ceiling raising the possibility that the US Treasury would be unable to make scheduled interest or principal payments on outstanding debt issues, e.g., Nippani et al. (2001). Following Fabozzi (2002, ch. 7), credit risk can be decomposed into three parts: default risk; credit spread risk; and downgrade risk. For bonds, default risk relates to the possibility that the issuer will not fulfill the obligations set out in the bond indenture. This could be due to the failure to make a coupon payment or to return the principal value at maturity or to meet some other provision in the indenture, such as a covenant on net asset value. Such events have a number of possible implications.

Because the possibility of default varies across bond issuers, this will be reflected in the quoted prices and yields for different bond issues. The difference between yields for issues with different credit ratings is the *credit spread*. Due to a variety of factors, the credit spread changes over time. For example, in periods of contracting economic activity credit spreads will typically widen to reflect the pressure on firm cash flows needed to make debt payments. Credit spread risk relates to these changes in the difference between yields for issues with different credit ratings. To facilitate this assessment there are a number of services that provide ratings to the bond market. Related to credit spread risk is the concept of downgrade risk. While credit spread risk depends on general conditions such as aggregate economic activity, downgrade risk is more firm specific. For example, it is possible for, say, a telecom firm such as Worldcom to experience a credit downgrade from a rating agency at a time when the credit risk for telecom firms was narrowing relative to US Treasuries. Another example would be a firm that experienced an unfavorable result in litigation, e.g., Dow-Corning or Johns Mansville. Given this, it is expected that the factors causing a widening of credit spreads will also contribute to an increase in downgrades for individual firms.

Credit spread risk and downgrade risk are extensions of the issue of default risk. Credit spreads widen and narrow because of changes in the perception that default will occur. Downgrading of a firm's debt by a credit rating agency occurs because there is the perception that default is more imminent. Yet, while driven by economic factors, default is a legal event. As such, it is necessary to detail the relevant laws and legal remedies associated with default on a debt obligation.¹³ A fixed income security is a debt instrument that is defined by a contract. For a corporate bond, this contract is the bond indenture. For publicly traded issues falling within the scope of the regulations of the Securities and Exchange Commission, such indentures must conform to the Trust Indenture Act (1939) that require appointment of a trustee to protect the interests of bondholders set out in the indenture. In addition, as publicly traded securities, the debt instruments must satisfy the relevant SEC rules regarding registration, resale of securities and the like. Debt instruments have a number of SEC Rules of specific relevance, such as Rule 415 on shelf registrations and Rule 144A on resale of privately placed securities.

Though there is considerable variation in the specifics of indentures, the American Bar Association (ABA) does provide examples of model contracts for specific bond indentures. For example, there are 15 articles, plus preamble, for the model debenture indenture and 16 articles, plus preamble, for the model mortgage bond indenture. Given that there is variation between specific indentures and the model indenture outlined by the ABA, the indenture contents do have a common set of essential characteristics. The preamble will typically make reference to the basis under which the indenture was issued, e.g., "the articles of incorporation permit the board of directors to authorize up to \$xxx million of senior subordinated debentures." There will also be articles associated with the definition of terms, methods of notifying trustees, the specific wording on the security certificate, the denominations, the record dates and so on. While such articles are important from a legal standpoint, it is the articles relating to remedies and covenants that are of greatest interest to the assessment of default risk. Articles relating to mergers and consequences of changes in the status of assets are also important, as are preambles and articles that describe the property securing a secured debt issue.

The indenture sets out the conditions required to initiate a default proceeding. Conventionally, once a credit event sufficient to trigger a default has occurred, e.g., failure to make a scheduled coupon or principal payment, the trustee (or a holder of 25% or more of the outstanding debt) can demand payment in full of any outstanding obligations. Whether the trustee initiates such an action is not straight forward. Because the trustee has a fiduciary obligation to act in the best interest of the bondholders, the trustee may deem it more prudent to enter into negotiations with the issuer to attempt to get a resolution to the indenture violation. If the debtor is unable or unwilling to resolve the indenture violation, a US corporate issuer would likely respond by filing for bankruptcy under the relevant chapter of the Bankruptcy Reform Act (1978). Similarly, if the trustee wants to proceed with a default action, a bankruptcy filing can be made by a creditor as well as a debtor. The Bankruptcy Reform Act has 15 chapters dealing with different types of bankruptcy. For filings resulting from a corporate debt issuer violating an indenture, the relevant chapter governing the bankruptcy filing would be either Chapter 7, governing liquidation, or Chapter 11, governing reorganization.

The problems confronting a trustee having to decide the correct course in dealing with an issuer in default on an indenture provision are legally complex. Once the bankruptcy process is initiated for a corporation in default, a number of provisions of the bankruptcy code come into effect that create considerable uncertainty about the final disposition. In particular, a bankruptcy filing introduces the bankruptcy court into the decision making process. The law requires that there be an active role for judicial supervision and oversight. Decisions of the court are binding on the parties involved. Another provision of the bankruptcy code involves the imposition of a standstill agreement on all creditors. This is required to provide for an orderly process of liquidation or reorganization, preventing actions such as senior creditors seizing assets that are essential to the viability of the firm as a going concern or allowing the firm to make disbursements to certain creditors at the expense of others. One final provision of importance is the provision that the debtor retain control of firm operations during the period that the bankruptcy filing is being decided, unless a specific directive is ordered by the court such as the appointment of a receiver.

In theory, a bankruptcy proceeds under the *absolute priority rule*. This rule dictates that senior creditors are to be paid in full before junior creditors receive any payments. For a number of reasons, the absolute priority rule may not be followed. One of these reasons has to do with the judicially supervised negotiation process between the debtor and the creditors that commences with the bankruptcy filing. Under Chapter 7 filings, the negotiating process is usually not overly complicated and the division of assets conforms relatively closely to the absolute priority rule. However, under Chapter 11 filings, the negotiating process can be onerous. The process is complicated by two provisions: during the reorganization the court may permit the firm to undertake new financing that is given a senior claim to existing claims; and, the reorganization plan does not require unanimous consent, only a majority of all creditors holding two-thirds of the outstanding value in each debt classification. This legal structure makes a bankruptcy filing under Chapter 11 somewhat uncertain for debt holders.

Not surprisingly, there is considerable evidence that the distributions in Chapter 11 bankruptcies do not typically adhere to the absolute priority rule, e.g., Fabozzi et al. (1993), Weiss (1990). Due to all the legal complexities and other uncertainties, there are real incentives for the trustee and large individual creditors to avoid formal bankruptcy proceedings and to attempt a resolution outside the court process. In addition to monitoring bankruptcy filings, the major credit rating agencies also issue credit event releases that recognize various types of financial distress and the associated resolution. All this makes the securities of distressed firms, whether in bankruptcy or working toward a resolution outside the court process, a potentially fruitful area for identifying abnormal returns using the techniques of security analysis, e.g., John (1993).

Assessment of credit risk and default risk

Information on credit risk and default risk comes from two sources: ratings agencies and market prices. Because of the large number of traded fixed income securities, firms have emerged that specialize in providing credit ratings for a wide range of different

debt issues. In the US, these firms are Standard and Poor's Corporation, Moody's Investors Services Inc. and Fitch Ratings (created by the merger of Fitch IBCA with Duff and Phelps Credit Rating Co.). It is often the case that firms pay the rating agencies to have their debt issues rated. In addition to ratings agency information, the larger investment banking firms and institutional investors will also have resources dedicated to doing credit analysis on specific firms and sectors that are of interest. There are also firms that specialize in ratings for specific industries such as Demotech, Inc. that assesses insurance companies and HMO's. As illustrated in table 4.6, there is a close

Table 4.6 Summary of corporate bond rating systems and symbols

Fitch	Moody's	S&P	Summary description
Investment grade – high credit worthiness			
AAA	Aaa	AAA	Gilt edge, prime, maximum safety
AA+	Aa1	AA+	High grade, high credit quality
AA	Aa2	AA	
AA–	Aa3	AA–	
A+	A1	A+	Upper-medium grade
A	A2	A	
A–	A3	A–	
BBB+	Baa1	BBB+	Lower-medium grade
BBB	Baa2	BBB	
BBB–	Baa3	BBB–	
Speculative – lower credit worthiness			
BB+	Ba1	BB+	Low-grade, speculative
BB	Ba2	BB	
BB–	Ba3	BB–	
B+	B1	B	Highly speculative
B	B2		
B–	B3		
Predominately speculative, substantial risk, or in default			
CCC+	Caa	CCC+	Substantial risk, in poor standing
CCC		CCC	
CC		CC	
C	C	C	May be in default, very speculative
		C1	Extremely speculative
			Income bonds – no interest being paid
DDD			Default
DD			
D		D	

Source: Rating agency websites

correspondence in the symbolic method of specifying ratings used by the different ratings agencies. However, this does not mean that the same methodologies are used by different ratings agencies to arrive at a credit rating for a specific firm.

Inspection of table 4.6 reveals three general categories of debt ratings: investment grade, rated BBB- (Baa3) and above; speculative, below investment grade, rated B (B-, B3) to BB- (Ba1); and, purely speculative, D to CCC+ (Caa). The ratings category that a debt issue falls into can have an important impact on the pricing of the issue in the market. In particular, various institutional borrowers, such as pension funds and insurance companies, are restricted from holding issues that are classified as being below investment grade. When a debt issue is downgraded to a rating below investment grade this can have a significant impact on the price of the issue. Similarly, certain types of bond funds, e.g., high yield funds, invest primarily in issues that are below investment grade. The implication is that lower rated issues typically have to offer substantially higher yields in order to attract the types of investors that purchase debt with below investment grade ratings.

Though there is a close correspondence between the credit rating issued by the rating services and, presumably, by the in-house credit analysts, the ratings do not translate precisely into pricing of debt issues. For example, it is possible for a bond with a lower credit rating to sell at a lower yield than a comparable bond with a higher credit rating. It is possible for this to happen because the relative credit ratings are not judged to be accurate by the market. However, it is more likely that this type of pricing occurs because the market price of an issue depends both on the credit rating and the potential *recovery rate*.¹⁴ The recovery rate is the payout on the debt issue that takes place when default occurs. For example, the recovery rate for a corporate mortgage bond will depend on the value of the collateral securing the issue. If the collateral is sufficient to repay the principal value of the borrowing, a default by the issuer will not have severe consequences. Similarly, it is possible that, though default may be less likely, the recovery rate may also be lower in the event that default does occur, e.g., due to the asset structure of the company.

For a portfolio of bonds, it is possible to calculate the "default loss rate" as the product of the default rate and the recovery rate, e.g., Fabozzi (2002, p. 202). For example, if 10% of the issues in a portfolio defaulted and a weighted average of 50% of the value of the investment was recovered then the default loss rate was 5%. Ignoring costs associated with the disruption of cash flow associated with having to wait until the defaulted issue is paid out to bondholders, if the portfolio was initially purchased with a promised yield of more than 5% above Treasuries then the investor would, *ex post*, have done better than investing in Treasuries. As such, analysis of fixed income securities subject to credit risk requires both the default rate and the recovery rate to be assessed to determine an appropriate valuation. Table 4.7 provides evidence from Altman and Bana (2003) on both default rates and recovery rates for US high yield corporate bonds for 1978–2002. Accurate interpretation of table 4.7 requires understanding of how default rates are calculated.

The numerous studies of default rates use a variety of calculation methods. For example, Asquith et al. (1989) report that about one in three high yield bond issues default. This is a cumulative default rate that is calculated by using the total outstanding

Table 4.7 Default rates and losses,^a 1978–2002

Year	Par value outstanding ^a (\$ millions)	Par value of default (\$ millions)	Default rate (%)	Weighted price after default	Weighted coupon (%)	Default loss (%)
2002	757,000	96,858	12.80	25.3	9.37	10.15 ^b
2001	649,000	63,609	9.80	25.5	9.18	7.76
2000	597,200	30,295	5.07	26.4	8.54	3.95
1998	465,500	7,464	1.60	35.9	9.46	1.10
1996	271,000	3,336	1.23	51.9	8.92	0.65
1994	235,000	3,418	1.45	39.4	10.25	0.96
1992	163,000	5,545	3.40	50.1	12.32	1.91
1990	181,000	18,354	10.14	23.4	12.94	8.42
1988	148,187	3,944	2.66	43.6	11.91	1.66
1986	90,243	3,156	3.50	34.5	10.61	2.48
1984	40,939	344	0.84	48.6	12.23	0.48
1982	18,109	577	3.19	38.6	9.61	2.11
1980	14,935	244	1.50	21.1	8.43	1.25
1978	8,946	119	1.33	60.0	8.38	0.59
Arithmetic average ^c 1978–2001:			3.62	\$43	11.14	2.19
Weighted average ^c 1978–2001:			5.49			3.16

^a Excludes defaulted issues.

^b Default 1 loss rate adjusted for fallen angels is 7.58%.

^c Averages are based on the full annual sample while the table only reports detailed biannual values.

Source: Altman and Bana (2003)

amount of high yield debt issued in an initial year and dividing this value into the cumulative value of outstanding issues that defaulted for all issues for that year. In contrast, table 4.7 reports the default rate by dividing the par value of outstanding bonds that default in a given year by the total par value outstanding issues for that year. This provides a year-by-year estimate. For example, table 4.7 reports that the default rate on high yield bonds in 1996 was 1.23%. Until the wave of defaults in 2002, the highest level of default reported was for 1991 where a default rate of 10.27% is reported. Results in Altman and Bana (2003) are not confined to this method of estimating default rates, a wealth of other information is provided including defaults by original rating, years to default from original issue date, and defaults by industry.

Table 4.7 also provides information on recovery rates. This is calculated as the weighted average price after default, where the weights are the fraction of the par value of defaults in the year for that issue. Being all less than \$30 per \$100 par value, the recovery rates for 1999–2002 are at the bottom end of recovery rate values over the last two decades. The strong economic fundamentals of 1992–97 permitted recovery rates that were usually in excess of \$50 per \$100. Table 4.8 provides further useful information on recovery rates by decomposing the results by type of issue. It is significant that

Table 4.8 Weighted average recovery rates on defaulted debt by seniority per \$100 face amount, 1978-2002

Default year	Senior secured		Senior unsecured		Senior subordinated		Subordinated		Discount and zero coupons		All seniorities	
	No.	\$	No.	\$	No.	\$	No.	\$	No.	\$	No.	\$
2002	37	52.81	254	21.82	21	32.79	0	0.00	28	26.47	340	25.32
2001	9	40.95	187	28.84	4	18.37	0	0.00	37	15.05	281	25.48
1998	6	70.38	21	39.57	6	17.54	0	0.00	1	17.00	34	37.27
1996	4	59.08	4	50.11	9	48.99	4	44.23	3	11.99	24	51.91
1994	5	48.66	8	51.14	5	19.81	3	37.04	1	5.00	22	39.44
1992	15	59.85	8	35.61	17	58.20	22	49.13	5	19.82	67	50.03
1990	12	32.18	31	29.02	38	25.0	24	18.83	11	15.63	116	24.66
1988	13	67.96	19	41.99	10	30.70	20	35.27			62	43.45
1986	8	48.32	11	37.72	7	35.20	30	33.39			56	36.60
1984	4	53.42	1	50.50	2	65.8	7	44.68			14	50.62
1982			16	39.31			4	32.91			20	38.03
1980			2	26.71			2	16.63			4	21.67
1978			1	60.00							1	60.00
Total/average	158	52.32	764	35.18	365	29.66	247	31.03	125	21.05	1,659	33.16
Median		55.75		41.99		31.91		31.00		17.50		39.44

Source: Altman and Bana (2003). Total/average values given are for the full sample while detailed information is only listed biannually. Total/average and median values are for all years from 1978-2002

senior secured debt does not provide full protection, with a low recovery rate of \$26.90 in 1999 (not reported), below that of the recovery rate of senior unsecured debt in the same year. The implication is that the priority claim of security is less important than it would seem. Another interesting result is the absence of any defaults on subordinated debentures in 2001 and 2002. This is likely the result of less credit worthy companies being generally unable to issue this type of security.

The empirical evidence on credit risk and default risk

Even before Macaulay (1938), there were credit rating services and empirical studies of the relationship between crediting ratings and bond performance. Macaulay (1938, p. 58) captures the essence of the problem involved in making a credit rating:

In grading bonds at any particular date, we are concerned with how good the bonds *were*, not with how good they *turned out to be*. It is, of course, true that, in a metaphysical sense, how good they turned out to be was how good they really were. But prices and yields can be *directly* affected only by *forecasts* of the future, never by the *facts* of the future. It is, therefore, to forecasts that we must restrict our concept of goodness. It is true that to grade bonds on any basis of how good they *seemed* to be, or even *should* have seemed to be, is to lean upon a flimsy reed. But there is clearly nothing else to do.

Having come to this conclusion, we are faced with the question, should the grading be based on actual or on ideal forecasting, on how good the bonds seemed to be or on how good they *should* have seemed to be. At once we notice that any "should have seemed" grading is tarred with the same stick as grading based on what the future eventually revealed. It is almost always clear, *after the event*, that, though the future is essentially unknowable, a more shrewd and intelligent analysis of the facts that *were* available should have prevented much of the bad forecasting that actually occurred.

Macaulay (1938, pp. 59–60) provides the following assessment of the performance of credit ratings:

the history of bond prices demonstrates conclusively that . . . bonds are usually graded very incorrectly by the market – and the statisticians. Collapse of the credit of a corporation is seldom seen far in advance; on the other hand, innumerable bonds that pay on time all coupons and the face of the bond are graded low throughout their existence. This is, of course, exactly what might be expected. As the future cannot be known, bonds must be graded on a probability basis and, unless they are of ultra-superior quality, the information available for grading them on such a basis is almost always quite inadequate.

Macaulay (1938) was largely concerned with developing indexes for interest rates. This is an exercise that has to deal directly with the problem of credit assessment in order to evaluate which securities to include in the indexes. As such, Macaulay was a leading authority in the assessment of credit risk.

Macaulay (1938, p. 58) dealt explicitly with the difficulty of developing a credit rating methodology:

this difficulty cannot be overcome by arbitrarily limiting the range of facts upon which forecasts "should have" been based. The essential element in any "should have" system of grading must clearly be that no pertinent and important consideration shall be neglected. To assume, for example, for purposes of yield comparisons, bonds can be more correctly graded by some simple mathematical formula whose variables are all derivable from either present or past financial reports of the debtor corporation than they are in fact graded in the open market is to exhibit an optimism that is difficult if not impossible to defend.

Interpretation of Macaulay's jaded views on credit evaluation needs to be tempered by the historical context. For example, Macaulay was faced with a more difficult problem of assessing credit risk than contemporary researchers, if only because US Treasury securities were not available as a benchmark. This was due to the use of US Treasury securities in the currency system, both before and after the creation of the Federal Reserve system, leading to yields that "were naturally much lower than if the bonds had been valued for their interest payments alone." Macaulay was also faced with the difficulty of obtaining accounting and other information about the firms being rated.

In general, modern credit rating firms do an impressive job of collecting and processing information in order to arrive at a credit rating for an individual firm. However, Macaulay's observation that "innumerable bonds that pay on time all coupons and the face of the bond are graded low throughout their existence" still applies today. Table 4.9 provides information from S&P on credit ratings drift over a one to ten year horizon. (Moody's does similar analysis.) The information is presented in the form of a transition matrix that gives the probabilities that a firm rated in one credit category will be in that category or another category a number of years in the future. For example, consider the five year transition matrix for an initially AAA rated firm. Only 60.6% of these firms were still AAA five years hence, while 0.11% had gone into default. In contrast, almost 20% of BBB rated firms and almost 40% of CCC rated had gone into default in five years. The importance of the NR (not rated) category in table creates some difficulty in interpretation.

Table 4.9 Selected transition matrices based on S&P rated issues

Average one-year transition rates

Initial rating	Ratings at end of first year (%)								
	AAA	AA	A	BBB	BB	B	CCC	D	N.R.
AAA	90.34	5.62	0.39	0.08	0.03	0.00	0.00	0.00	3.54
AA	0.64	88.78	6.72	0.47	0.06	0.09	0.02	0.01	3.21
A	0.07	2.16	87.94	4.97	0.47	0.19	0.01	0.04	4.16
BBB	0.03	0.24	4.56	84.26	4.19	0.76	0.15	0.22	5.59
BB	0.03	0.06	0.40	6.09	76.09	6.82	0.96	0.98	8.58
B	0.00	0.09	0.29	0.41	5.11	74.62	3.43	5.30	10.76
CCC	0.13	0.00	0.26	0.77	1.66	8.93	53.19	21.94	13.14

Table 4.9 (Cont'd)

Average two-year transition rates

Ratings at end of second year (%)

Initial rating	AAA	AA	A	BBB	BB	B	CCC	D	N.R.
AAA	81.51	10.14	1.05	0.18	0.06	0.03	0.00	0.00	7.04
AA	1.18	79.41	11.87	1.07	0.16	0.18	0.01	0.04	6.09
A	0.10	3.86	77.57	8.60	1.07	0.41	0.03	0.11	8.26
BBB	0.10	0.47	8.44	70.48	6.80	1.46	0.33	0.48	11.44
BB	0.03	0.11	0.87	10.79	56.21	10.11	1.61	2.95	17.32
B	0.00	0.14	0.54	0.92	8.8	53.85	3.83	11.01	20.84
CCC	0.14	0.00	0.57	1.72	2.15	11.61	32.09	28.37	23.35

Average five-year transition rates

Ratings at end of fifth year (%)

Initial rating	AAA	AA	A	BBB	BB	B	CCC	D	N.R.
AAA	60.60	17.32	3.41	0.93	0.14	0.11	0.00	0.11	17.39
AA	2.32	57.01	22.06	3.26	0.53	0.44	0.08	0.24	14.08
A	0.16	6.4	56.74	13.79	2.30	1.12	0.17	0.50	18.87
BBB	0.23	1.25	14.52	44.64	8.47	2.38	0.58	1.76	26.18
BB	0.05	0.30	2.35	15.16	24.23	9.37	1.45	9.14	37.94
B	0.00	0.15	0.87	2.53	10.41	19.42	2.05	19.98	44.59
CCC	0.18	0.00	0.35	2.65	3.36	6.54	7.77	39.22	39.93

Average ten-year transition rates

Ratings at end of tenth year (%)

Initial rating	AAA	AA	A	BBB	BB	B	CCC	D	N.R.
AAA	37.30	20.22	7.84	3.02	0.12	0.00	0.00	0.64	30.85
AA	2.88	33.41	28.09	6.49	0.84	0.24	0.11	0.95	27.00
A	0.37	6.29	37.47	15.23	3.34	1.16	0.11	1.75	34.29
BBB	0.35	1.77	15.71	26.17	6.48	1.57	0.20	4.30	43.45
BB	0.16	0.08	3.44	11.77	7.55	3.32	0.44	18.14	55.12
B	0.00	0.10	0.85	3.07	4.77	3.92	0.38	29.48	57.43
CCC	0.26	0.00	0.26	1.05	3.67	1.83	0.00	46.86	46.07

Source: Table 16 in Leo Brand and Reza Bahar, "Corporate defaults: will things get worse before they get better?" Special Report, Standard & Poor's Corporation

Table 4.10 Defaults by original rating (investment grades vs. non-investment grade, by year)

	Defaulted issues ^a	% originally rated investment grade	% originally rated non-investment grade
2002	321	39	61
2001	258	14	86
2000	142	16	84
1998	39	31	69
1996	24	13	88
1994	16	0	100
1992	59	25	75
1990	117	16	84
1988	64	42	58
1986	55	15	85
1984	14	21	79
1982	20	55	45
1980	4	25	75
1978	1	100	0
Total	1,511	23	77

^a Where an original rating from either S&P or Moody's was available.

Source: Altman and Bana (2003). Total values given are for the full sample while detailed information is only listed biannually

Another approach to the assessment of default is provided in table 4.10 that reports on the percentage of defaults, by year, classified according to the original rating. The 41% of defaulting issues in 2002 that were originally rated as investment grade is almost disturbing. If included in table 4.9, this would also have an impact on the estimates of the credit ratings transition matrix.

Examining the evidence in table 4.11 reveals that, while the bulk of defaults occur within 5 years of the initial issue date, in years when default rates are high such as 2001–2, issues that have had a longer life also default. In periods of strong economic activity, e.g., 1996–8, few issues default that have been in the market for more than five years. In assessing credit ratings, it is important to recognize that the pricing implications of credit ratings relate to the credit spreads in yields.

Figure 4.1 captures the time series properties of investment quality bonds. Yet, the relationship between yields on such bonds needs to be carefully compared to yields on bonds that are considered to be below investment grade. The difficulty with making such a comparison is that the dispersion of yields from bond to bond is less systematic due to factors such as industry and firm characteristics. With this in mind, table 4.12 provides credit spreads for both investment and non-investment grade bonds for issuers that are banks. For a number of reasons, e.g., the regulatory structure, this group is

Table 4.11 Distribution of years to default from original issuance date (by year of default, 1990–2002)

Years to default	1990		1992		1993/4		1996	
	No. of issues	% of total	No. of issues	% of total	No. of issues	% of total	No. of issues	% of total
1	3	3	0	0	3	8	2	8
2	25	23	0	0	6	16	3	13
3	23	21	7	13	5	14	3	13
4	18	17	10	19	2	5	8	33
5	23	21	8	15	4	11	1	4
6	5	5	12	22	2	6	5	21
7	5	5	5	9	7	19	0	0
8	4	4	4	7	0	0	0	0
9	1	1	0	0	0	0	0	0
10	1	1	8	15	2	5	2	8
Total	108	100	54	100	37	100	24	100
Years to default	1998		2000		2001		2002	
	No. of issues	% of total	No. of issues	% of total	No. of issues	% of total	No. of issues	% of total
1	2	6	19	10	40	12	29	8
2	5	15	51	28	69	21	51	15
3	10	30	56	31	87	26	61	18
4	3	9	14	8	65	19	56	16
5	10	30	13	7	27	8	45	13
6	2	6	5	3	14	4	21	6
7	1	3	12	7	21	6	8	2
8	0	0	4	2	5	1	7	2
9	0	0	3	2	4	1	12	3
10	0	0	6	3	3	1	54	16
Total	33	100	183	100	335	100	344	100

Source: Altman and Bana (2003)

more homogeneous than, say, oil and gas drillers. Examination of table 4.12 dramatically illustrates an important characteristic of credit spreads: the sizable premium that appears when a bond is graded below investment grade BBB-. The empirical rationale for such a premium is not supported in the default by rating data or in the ratings drift statistics. Even though the observation date for table 4.12 is associated with relatively wide credit spreads, this result is robust across time.

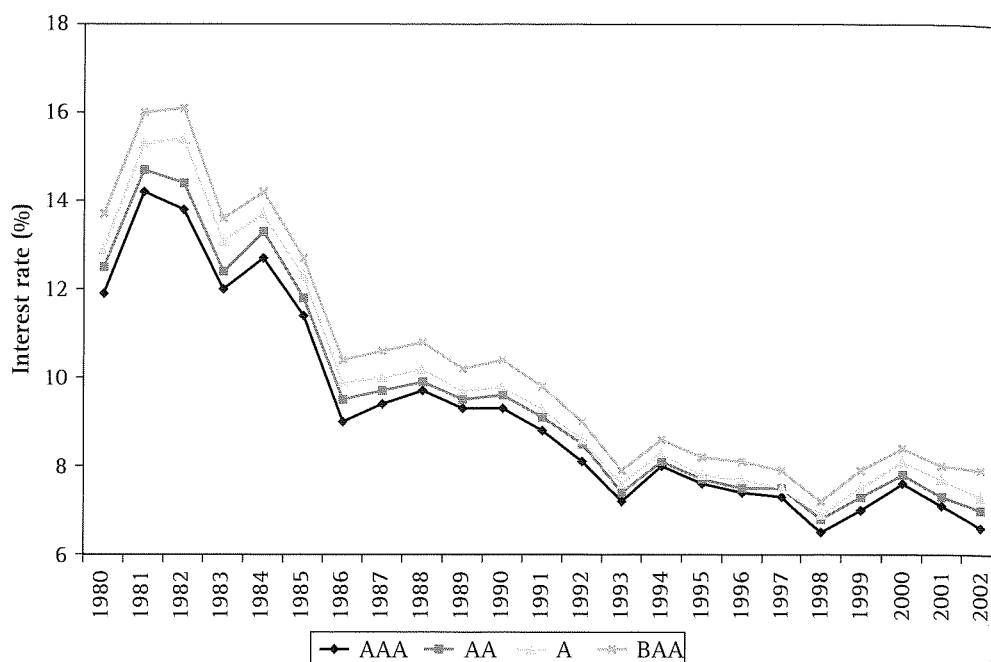


Figure 4.1 Interest spreads for investment grade bonds, 1980–2002

Table 4.12 Reuters corporate spreads for banks for February 10, 2003 (in basis points)

Ratings	1 year	2 year	3 year	5 year	7 year	10 year	30 year
Aaa/AAA	27	36	47	61	74	88	107
Aa1/AA+	33	49	53	61	84	99	118
Aa2/AA	35	54	56	75	87	101	121
Aa3/AA–	37	57	58	80	91	105	130
A1/A+	61	73	77	95	107	123	145
A2/A	64	76	79	97	108	125	148
A3/A–	68	79	82	101	115	128	150
Baa1/BBB+	81	98	104	124	153	180	202
Baa2/BBB	84	106	112	130	158	183	207
Baa3/BBB–	91	111	117	135	163	190	212
Ba1/BB+	605	615	625	635	655	675	695
Ba2/BB	615	625	635	645	665	685	705
Ba3/BB–	625	635	645	655	675	695	715
B1/B+	775	785	795	825	865	905	955
B2/B	785	795	805	835	875	915	965
B3/B–	795	805	815	845	885	925	975
Caa/CCC	1,195	1,205	1,215	1,240	1,270	1,330	1,280

Source: www.bondsonline.com (Bond Market Association website)

Questions



1 Your father is about to retire. His firm has given him the option of retiring with a lump sum of \$50,000 or an annuity of \$8,000 for 10 years. Which is worth more if the discount rate is: (a) 6%; (b) 18%?

2 You are in the market for a house. Your effective all-in market borrowing rate for a house mortgage from a bank is 10%. One of the houses you are considering purchasing has an assumable \$200,000, 5 year mortgage at 8%, with a 20 year amortization. The asking price on the house is \$400,000. What is the value of the concessionary financing for this house?

3 The following is a list of prices for (riskless) zero-coupon bonds of various maturities: 1 year, \$97.10; 2 year, \$92.50; 5 year, \$74.70; 10 year, \$50.80; 30 year, \$13.10. Calculate: the yields to maturity of each bond; and, all possible implied forward rates.

4 Briefly explain the relationship between expectations of future interest rates and implied forward rates calculated from the current term structure of interest rates in the expectations, liquidity preference and market segmentation hypotheses.

5 A bond offers semi-annual coupon payments of \$3 (i.e, it is a 6% coupon bond) and will repay its face value of \$100 at the end of five years. You observe that other similar bonds have yields to maturity of 8%. How much is this bond worth? If the coupons are paid quarterly, how much is the bond worth?

6 Using the information in table 1.9, calculate the implied zero coupon rates for the US Treasury curve. Compare these rates to the observed interest rates for US Treasury strips. Using the information from table 4.5 plot the yield curve for government of Canada issues. Calculate the associated spot rate curve and plot the spot rate curve against the yield curve. Does the spot rate curve lie above or below the yield curve? Does your answer change if the yield curve is inverted?

7 Demonstrate that the duration formula for an annual coupon par bond is equal to:

$$\frac{1+y}{y} \left[1 - \frac{1}{(1+y)^T} \right].$$

NOTES

- 1 There are a number of excellent sources that cover in considerable detail the basics and beyond for fixed income analysis. Essential sources include Fabozzi (2000, 2002), Garbade (1996) and Sundaresan (2002). In addition to the generalist practitioner journals, such as the *Financial Analysts Journal* and the *Journal of Portfolio Management*, there are also some excellent journals dedicated to this topic such as the *Journal of Fixed Income*.
- 2 For more detail on the specific holdings in the Government Account Series see the *Treasury Bulletin*, table FD-3.
- 3 Due to complications associated with stripping callable bonds, since the introduction of the STRIPS program in 1985 the US Treasury has not issued bonds with call features. Prior to this date, the 30 year US Treasury bond issues had 25 years until the call date, i.e., the bonds were callable during the last five year period at the end of the term to maturity. As with corporate bonds, this feature was incorporated in order to facilitate refunding of the issue. As there was 25 years to the call date at primary issue, the cost of the call feature had a negligible impact on the new issue price. Circa. 2003, there are about 15 callable Treasury issues outstanding.
- 4 Though the three types of closed-end bond funds differ in the tax treatment of the dividend income distribution, capital gains or losses are taxed the same as for other securities.
- 5 Unlike open-end funds, closed-end funds do not continuously offer their shares for sale. A fixed number of shares are sold at one time (in the initial public offering), after which the shares trade on a secondary market, such as the New York Stock Exchange (NYSE) or the American Stock Exchange (AMEX). The price of closed-end bond fund shares that trade on a secondary market after their initial public offering is determined by the market and may be greater or less than the shares' net asset value. Akhigbe and Madura (2001) examine the performance of seasoned offerings of closed-end funds.
- 6 For the specific dates examined, both these sources report that implied zero and strip rates are approximately equal at short maturities with implied zeros being slightly below strip rates at intermediate maturities and strip rates being well below the implied zero rate at long maturities. (See end of chapter exercises.)
- 7 If the settlement date is the same as the date the money market security is purchased, then $tsm = T - t$. However, as there is a usually a settlement lag, this is not usually the case. For longer term securities it is possible to ignore the settlement lag without having a significant impact on pricing. In addition, the impact of the settlement lag can be captured in the calculation of accrued interest, an element of the price that is being ignored for pedagogical reasons. However, the short term to maturity for money market securities and the absence of accrued interest makes the settlement lag an important element. The tsm notation is designed to recognize this impact.
- 8 There are a number of possible complications which can be introduced such as: the coupon payments (and compounding) occurring more than once a year; and, the dollar price change being required instead of % price change. Both these changes occurred in the 1994 exam.
- 9 References to Hicks (1939) are to the second edition of 1946. Changes between the first and second editions were largely aimed at correcting errors in the derivation of the consumer's choice problem and adding a brief discussion of the connection to Samuelson's dynamic treatment of the problems examined in Hicks (1939).
- 10 The "term structure of interest rates" terminology can also be used to refer to implied zero coupon rates derived from bonds with default risk, e.g., Bierwag et al. (1992) refer to the duration of bonds from different term structures where the term structures differ due to default risk. However, where further qualification is not given, the term structure of interest rates refers to default free securities.
- 11 In a binomial model these relationships would be somewhat more complicated as the evolution

of zero coupon interest rates is determined by taking the expected value of the interest rates at the two future nodes, discounted back at the interest rate for the previous node. This point is discussed in section 5.2.

- 12 The discussion in this section relates to bonds but the basic concepts apply to all fixed income securities. For example, there is default risk on a mortgage that will be reflected in, say, the requirement that mortgage insurance be purchased at the time the mortgage is issued. Loans at commercial banks will feature different rates of interest to reflect the different assessed credit ratings for borrowers.

- 13 Though default on federal, state and local government debt is possible, the legal implications are more complicated. Though only the case of corporate default is examined in detail, the basic principles of the issuer seeking protection under the bankruptcy laws and the purchaser seeking remedies under the terms of the debt contract still applies.

- 14 Other important factors that can influence the yield spread between two different debt issues are: the liquidity; differences in covenants and special provisions such as callability; and, the tax status of the coupon and principal payments.