

Chapter Summary

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chapter 6

Bonds with Embedded Options

6.1 TYPES OF BONDS WITH EMBEDDED OPTIONS

Basics of bonds with embedded option features

The basic discounted cash flow analysis used to value fixed income securities centers on bonds that are default and option free and that provide a fixed coupon and return of principal at maturity. Such securities are referred to as (default and option free) *bullet securities* or *straight bonds* (debt). The analytical advantages of working with straight bonds are apparent from the discussion in chapters 4 and 5. Though a sizeable fraction of the outstanding fixed income security issues can be more or less considered as straight debt, there is a larger fraction that does not qualify due to the presence of embedded options or default risk. Recognizing that default can be viewed as a type of option, the class of bonds with option features extends well beyond those issues with explicit features such as callability or convertibility. In a sense, even straight debt can be analyzed as having an option associated with the reinvestment of coupons. In addition to being useful in valuing non-straight debt, the techniques that are employed to value bonds with option features can also be used in the design of investment strategies for fixed income portfolios.

The price of a bond with an embedded option can be decomposed into two parts, the price of the straight bond without the option plus or minus the price of the embedded option. If the option is a *seller's option* then the option will make the bond less attractive to the purchaser and, as a result, the price of the option will be subtracted from the straight bond price. Examples of securities with seller's options include callable bonds and mortgage backed securities. It is also possible to view default as a type of seller's option. If the option is a *buyer's option* then the option will make the bond more attractive to the purchaser and the option price will be added to the straight bond price. Examples of securities with buyer's options include convertible bonds, putable bonds and extendible bonds. In most cases, it is not possible to directly unbundle the price of the straight bond and the price of the option, i.e., these values are not observed. All that is observed is the traded price of the bond with the embedded option attached.

If the value of the embedded option cannot be unbundled from the straight bond component of the observed bond price, this creates significant difficulties for the discounted cash flow valuation methods used to analyze the fixed income securities examined in chapters 4 and 5. Even if the option value can be unbundled, some method for determining the option value is still required in order to assess the impact of potential changes in, say, interest rate levels on the value of the option. This chapter is concerned with developing the different approaches to valuing bonds with embedded options. The valuation problem can be approached using either continuous time or discrete time methods. The objective is often to provide a method of adjusting the calculated yield to maturity to incorporate the embedded option feature. This is the objective of option adjusted spread analysis. Credit spread analysis seeks to explain the behavior of the difference between the yields on a default and option free bond and on a bond subject to credit risk.

The discounted cash flow valuation techniques used in chapters 4 and 5 did not incorporate the possibility of contingencies in the future cash flows arising from embedded options. Various analytical techniques are available to value fixed income securities with embedded options. Based on the impressive analytical advances associated with option pricing theory, e.g., Black-Scholes (1973), it would seem natural to apply continuous time methods to specify closed form solutions for the embedded options. For a number of reasons, this is not easily accomplished if only because interest rate derivatives pose complications that do not arise in valuing, say, options on common stock or currency. In addition, it is not straightforward to specify the riskless hedge portfolio for embedded options that is needed to specify the relevant PDE. Even when the PDE can be specified, specific features of the embedded option can make it difficult to derive a closed form solution and numerical methods are required in order to determine a security value.¹ In turn, application of numerical methods may produce "pricing errors that are too large for practical applications" (e.g., Buttler 1995). In some cases, all that can be practically adopted is portfolio management techniques such as the use of Greeks for general functional forms.

Recognizing that there may be potential complications and limitations, when closed form solutions derived from PDEs are not available to value bonds with embedded options, it is still possible to arrive at a solution by adopting other techniques of contingent claims analysis to value the embedded options. Where PDEs are available but closed forms cannot be determined, solutions can be obtained by application of numerical methods, e.g., Dunn and McConnell (1981) for GNMA's and Buttler (1995) for callable bonds. In addition, it is also possible to use discrete time methods such as binomial trees and Monte Carlo simulation. Whether such methods produce credible solutions depends, in part, on the user of the solutions. For example, a market maker required to determine a continuous bid/offer spread on a security has a higher need for precision than a bond portfolio manager trying to determine whether to buy bond A or bond B. Much of the theoretical analysis for bonds with embedded options takes the perspective of a market maker, where a high degree of pricing precision is required. For purposes of security analysis and investment strategy, the techniques are more useful as a guide to decision making than for determining "precise" values.

Callable bonds, sinking funds and convertible bonds

To the uninitiated, it is confusing to find that the "callability" option for a callable bond is quite different than the "call" option written on a stock or fixed income security. This is because callability is a seller's option while the call option is a buyer's option. Once it is recognized that the callability feature is a call option for the seller much of the confusion is removed. However, it is difficult to avoid the habit of taking the buyer's perspective. From this perspective, the conventional (buyer's) call option traded on, say, the Chicago Board Options Exchange is similar to the conversion feature embedded in convertible bonds, with the caveat that the currency used to exercise a convertible bond is usually the par value of the bond. In addition, there is considerable variation in the types of call provisions that are attached to bond issues. In some cases, such as the call provisions that were attached to US Treasury long bonds prior to 1986, the issuer's exercise decision was driven by considerations other than the level of interest rates, i.e., the US Treasury was more concerned with the impact of calls on the orderly marketing of issues than with saving on interest costs, per se (see box 6.1).

A pure "callable bond" combines a straight bond with the provision that the bond issuer can redeem/buy back/"call" the issue prior to the stated maturity date. In keeping with the terminology used in options markets, three general types of call provisions are possible: American, Bermudan and European. These references are only terminological and are not intended to represent anything about the geographical distribution for the use of a particular feature. An *American callable bond* can be redeemed at any time prior to maturity, subject to a possible initial *lockout period* where the bond is not callable; a *Bermudan (semi-American) callable bond* can only be redeemed on specific dates prior to maturity; a *European callable bond* can only be redeemed on one specific date. The date(s) on which the bond can be called is (are) the call date(s). An American callable bond that does not have any period for which the bond is not callable is *currently callable*. The price that the issuer pays to redeem the issue is the call price. For American and Bermudan callable bonds, the call price will typically vary across call dates. Call exercises for each type of bond are subject to a notice period, usually from 30 to 60 days.

A variety of different methods are used to set call prices. One common method is to set call prices starting at par plus coupon scaling down to par at some later date. Bonds called between coupon payment dates are adjusted for accrued interest. Wilson and Fabozzi (1996, p. 116) provide the example (box 6.2) of a callable 30 year issue of 10% sinking fund debentures made by Anheuser-Busch in mid-1988.

In addition to being currently callable, this issue is also subject to a sinking fund provision that may have different redemption prices. Given the wide possible variation in the terms of callable bond issues, Wilson and Fabozzi (1996, p. 112) correctly caution: "The importance of knowing the terms of bond issues, especially those relating to redemption, cannot be overstressed."

Rationality requires that, at primary issue, the bond issuer receives less for a bond that is callable than for a bond that is not callable. The issuer is willing to take less for the issue in order to obtain the benefits associated with the right to redeem the issue

Box 6.1 Graham and Dodd on bond call provisions

Graham, Dodd and Cottle (1962, pp. 753–4) provide a historical perspective on the use of call provisions. Beyond this statement, there is little discussion of call provisions.

The call provision of bonds

Nearly all corporate bonds issued for some decades past have a call provision which enables the company to pay them off before maturity at a stipulated price, which often declines as maturity approaches. (Most preferred stocks carry a comparable provision, but the call price rarely if ever changes with time.) Comparatively little attention was paid to the significance of these call features until the sharp drop in interest rates after 1933 – and the low rates continuing for more than twenty years thereafter – brought on an almost universal retirement of callable bond issues and their replacement by new ones carrying lower coupons. The effect of this provision was then seen to be the depriving of bondholders of most of the price benefit that should accompany a drop in interest rates, although they had no comparable option to safeguard them against a fall in the price of their bonds when interest rates advance. Otherwise stated, the call provision compels bondholders to accept a cut in their interest return when the general interest rate drops appreciably, but does not give them any increased income when the rate advances.

We have illustrated this point elsewhere by the following typical example: In 1928, American Gas & Electric Co. sold an issue of 100-year 5 percent debentures to the public at 101, yielding 4.95 percent. Four years later, in the depths of the depression, the debentures sold as low as 60, with a yield of 8.25 percent. This reflected the impact of unusually unfavorable economic and market conditions on a good-quality investment bond. Conversely, and under favorable circumstances, the interest rate applicable to bonds of this quality fell under 3 percent. This *should* have meant an advance in the price of this 5 percent issue to 160 or better. But at this point, in March, 1946, the call feature was availed of and the company redeemed the issue at only 106.

Since 1954 bondholders have experienced a third unfavorable experience of this sort – this time occasioned by a sharp rise in the basic interest rate. The low coupon bonds issued in the long period of low-interest rates sustained a severe decline. In the case of American Gas & Electric, its $3\frac{3}{8}$ percent debentures issued above par in 1952 sold as low as 80 in 1961. The purchasers of Consolidated Edison of New York $2\frac{3}{4}$'s at $102\frac{7}{8}$ in 1947 had the more disconcerting experience of seeing their price drop below 70 in 1959.

In our view the call feature in the typical bond contract is a thinly disguised instance of “heads I win, tails you lose.” Its results may be summarized as follows: When interest rates go down the call feature forces the investor to lose the income he counted upon; when the interest rate goes up the investor loses in the principal value of his bond.

The SEC has taken the stand that a call provision is a necessary feature in public utility bond issues (under their jurisdiction) to safeguard the company against paying higher interest on its old bonds than the going rate in future years. This view is logical as far as it goes, but it completely ignores the contrary and equally legitimate interests of the bond investor. Perhaps the correct solution of this problem would be to make the interest payments on long-term bonds vary with changes in the basic rate; if the company wishes to retain a right to call the bonds – presumably for reasons of convenience – then a corresponding right should be given the holders to tender their bonds to the company at discount equal to the premium that the company would have to pay on call. A provision to vary the coupon with interest-rate conditions would bear some resemblance to that found in a number of foreign bonds which protect the holder against a fall in the gold or commodity value of the currency. However, our tentative suggestion would operate in both directions, to safeguard both parties to the bargain.

The triple “whipsawing” that institutional bondholders have submitted to since 1929 has led them to demand more protection as far as call features are concerned. Many, perhaps most, tax-free and industrial issues in recent years have been made noncallable for a substantial period after issuance. *Example:* In 1961 the State of California issued \$100 million of bonds at various coupon rates and maturing serially from 1963 through 1987. This issue was made callable only between 1983 and 1987.

A more sophisticated arrangement permits an early call provided that interest rates have not fallen appreciably – which means that the bondholder would be able to replace his called bonds at an equivalent coupon rate. *Example:* In 1962 Ohio Oil Co. sold at 100 a large issue of 4 percent debentures due 1987. For the first five years they were made nonredeemable “through a funding operation at an interest cost to the company of less than 4.36 percent.” They could be redeemed in whole or in part out of “spare cash” at prices ranging from 104.6 in 1962 to 103.9 in 1966. Beginning in 1967 the company will be free to redeem bonds with any funds at prices gradually declining toward par in 1986. The analyst should be cognizant of the implications of the various types of provisions, and pay due heed to them in making his bond selections.

early. Typically, the desired benefit is the ability to take advantage of lower interest rates in the future. These lower rates would usually be due to a change in the general level of interest rates but could also be due to a narrowing of credit spreads for lower credit issuers or to an improvement in the credit status of the firm. In some cases, the call provision may be included to facilitate the implementation of a sinking fund provision. In other cases, such as the call provision on “old” Treasury long bonds, the benefits were associated with orderly refunding of the issue. In order to reduce the cost of the call feature, the issuer may introduce a range of additional features. For example, there may be a lockout period governing first possible call date. There may also be protection against redemptions done with refundings. In contrast, there may also be

Box 6.2 Redemption Schedule for Anheuser-Busch Companies, Inc., 10% Sinking Fund Debentures issued July 1, 1988 and due July 1, 2018

Redemption

The debentures will be redeemable at the option of the company at any time in whole or in part, upon not fewer than 30 nor more than 60 days' notice, at the following redemption prices (which are expressed in percentages of principal amount) in each case together with accrued interest to the date fixed for redemption:

If redeemed during the 12 months beginning July 1,

%		%		%	
1988	110.0	1995	106.5	2002	103.0
1989	109.5	1996	106.0	2003	102.5
1990	109.0	1997	105.5	2004	102.0
1991	108.5	1998	105.0	2005	101.5
1992	108.0	1999	104.5	2006	101.0
1993	107.5	2000	104.0	2007	100.5
1994	107.0	2001	103.5	2008 and thereafter	100.0

provided, however, that prior to July 1, 1998, the Company may not redeem any of the Debentures pursuant to such option, directly or indirectly, from or in anticipation of the proceeds of the issuance of any indebtedness for money borrowed having an interest cost of less than 10% per annum.

Source: Prospectus dated June 23, 1988.

features that increase the call probability, such as restrictive covenants designed to reduce default risk.

Refunding is the process of replacing a secondary bond issue by making a primary bond issue. Entities with a permanent stock of debt, such as the US Treasury, pay for maturing issues by refunding with (rolling over into) new issues. For corporate issuers, refunding takes on additional importance because most redemption of callable issues is accomplished with refunding. Protection against refunding is significant because it makes redemption more difficult. Ten year refunding protection is a common provision on long term corporate bond. However, call protection is stronger in protecting redemption from any source of funds, not just new bond issues. An important legal example occurred with the Archer-Daniels-Midland (ADM) issue of \$125 million in 30 year, 16% sinking fund debentures in May 1981. This issue was currently callable but had 10 year refunding protection. In the following two years ADM made a series of lower cost bond issues, as well as issuing over \$140 million in common equity. The redemption of the

issue in August 1983 led to a lawsuit claiming that the refunding protection was violated. The courts sided with ADM.²

The decision for an issuing corporation about whether and when to redeem an outstanding bond issue is complicated. In turn, this complicates the problem of valuing a callable bond. Assuming that the redemption is paid for using a refunding, the process involves a corporation issuing a notice to call an issue, with payment usually 30 to 60 days later. This payment will be made using a new bond issue with a lower interest rate that may involve a lockout period. Because this process involves considerable fixed costs, it is not usually feasible or profitable for the corporation to undertake another redemption until rates fall significantly and a significant amount of time has gone by. Hence, in making a redemption decision a corporation has to identify the interest rate level at which it is optimal to exercise. In addition, there may be other factors that have to be taken into account than the interest rate level, e.g., there may be restrictive covenants in the debt issue that the corporation may want to extinguish by redeeming the issue.

In perfect capital markets with a zero notice period, the issuer will exercise the call on a nonconvertible callable bond as soon as possible following the market price of the bond reaching the call price (Ingersoll 1977; Brennan and Schwartz 1977). However, markets are not perfect and notice periods are not zero and, as a consequence, the call policies of issuers do not always follow the perfect capital markets result. In contrast to empirical studies of optimal exercise for callable convertible bonds, there are relatively few empirical studies on nonconvertible callable bonds. The most detailed study, King and Mauer (2000), examined over 1,600 US issues (1,126 utilities, 495 industrials and 21 transportation covering 530 different firms) over a 20 year period from 1975 to 1994. For these issues, King and Mauer find that 86% of issues were called long after the bond price first reaches the call price, with an average delay of 27 months; 56% of the delayed call issues traded at least 2% above the call price for 3 or more consecutive months and 25% traded above the call price for 12 or more consecutive months.

Prior to King and Mauer, Vu (1986) had presented an unusual empirical result based on a smaller sample of 102 issues: three-quarters of the nonconvertible callable bonds had a bond price *below* the call price immediately before the call announcement date. This result is unusual because calling a bond in this case would involve paying an exercise price that is higher than the observed price of the bond in the secondary market. This would indicate that the optimal call policy was influenced by factors other than interest cost minimization, e.g., the desire to eliminate the restrictive covenants contained in bond indentures. This result was not supported by King and Mauer where 81% of issues had a bond price above the call price immediately before the call announcement date. In addition, King and Mauer found that the call delay was shorter in periods of low interest rate volatility and steeper term structures. Firms that were larger and had a higher interest cost saving from calling the issue also tended to have short call delays. Market prices for the different issues appeared to reflect the call policies of the firms.

In addition to explicit call provisions, the bond indenture may also contain non-call exception features (e.g., Blanc and Gordon 1999). The four most common such provisions are: sinking funds; maintenance and replacement funds; release and substitution

provisions; and, eminent domain provisions. The use of the terminology "sinking fund" can be traced back to the revolution in English government finance in the late seventeenth and early eighteenth century, e.g., Poitras (2000, ch. 11). The original idea was to create a fund that, over time, would be sufficient to pay off a specific government debt issue. The cash flow into the fund would be tied to a dedicated source of funding associated with the revenue from a specific tax. While "sinking funds" for government debt financing have long since disappeared, the notion is still important for corporate bond issues. The objective of the sinking fund provision is usually to raise the primary issue price by reducing the default risk. Though a number of variations are available, in general a sinking fund is an indenture provision that provides for the regular retirement of a portion of a debt issue. Sinking fund payments are made by the issuing corporation to the bond indenture trustees.

Sinking fund payments are usually made annually. On long term issues, the sinking fund payments may not begin until after the refunding protection expires, usually after ten years. While it is possible for a sinking fund provision to be satisfied by the issuing company depositing funds or property into a "fund" to be controlled by the trustees for the purpose of retiring the issue at maturity, it is more common for the issuer to redeem a specific amount of the debt issue at periodic intervals. In some *non-specific sinking funds*, a number of the outstanding bond issues may fall within the scope of the sinking fund. Such funds permit the sinking fund requirement to be "funneled" onto one issue, permitting the issuer to select the issue for redemption that provides the greatest cost savings. The funneling provision often appears on sinking funds for high yield issues.

How the sinking fund for a specific issue is handled varies from indenture to indenture. For example, redemption for sinking fund purposes may be at a different price than if the whole issue is called for refunding purposes. Par value is a common sinking fund redemption price. Issues that are called for sinking fund purposes are often selected by lot, affecting the expected maturity of a sinking fund issue for a bondholder. It is usually the case that, instead of calling a specific amount of the outstanding bonds, the issuing corporation is permitted to repurchase a sufficient amount of bonds in the open market and deliver those bonds to satisfy the sinking fund provision. This would be advantageous for the issuer when the issue was trading at a lower price than the sinking fund redemption price. Because the issuer would be purchasing a significant amount of bonds, this would be favorable for purchasers due to enhanced issue liquidity. However, if the issue is trading above the sinking fund redemption price, this would have the same type of negative impact as a call provision.

In addition to call provisions and non-call exceptions, the bond indenture may also contain a conversion provision. Pure "convertible" bonds can be defined as a combination of a straight bond and a (buyer's) option to convert the bond into the common stock of the issuing corporation. Occasionally, the conversion option permits the bond to be converted into other securities of the firm than the common stock. Other variations on the conversion option are usually given a different name. For example, an "exchangeable security" permits conversion of the bond into the common stock of another corporation. Such a feature is not typical. A more common "conversion" feature is the puttable bond that permits the bond to be redeemed (converted to cash) at the option of the bondholder, e.g., Crabbe and Nikoulis (1997). The price of pure convertible

bonds can be modeled as equal to the straight bond price plus the price of the buyer's conversion option. (Because exercise involves an increase in the number of common shares, the embedded conversion option is a form of warrant.) As such, pure convertible bonds are good candidates for applying the continuous time valuation methods developed for exchange traded call options, e.g., Black (1976). Unfortunately, convertible bond valuation is usually complicated by the market practice of issuing callable convertibles.

Though there may be a lockout period, over time most convertible bond indentures also have had a call provision. The presence of these two unobserved embedded options poses severe analytical problems. One option, exercisable by the bondholder, depends mostly on the common stock price and the other option, exercisable by the issuer, depends on a range of factors including the level of interest rates. Even under the simplifying assumptions that the conversion option depends solely on the common stock price/conversion price relationship and that the callable feature depends solely on the level of interest rates, the valuation problem is quite intractable due to the presence of two random variables combined with the potential for both the issuer and bondholder to exercise an embedded option. In practice, there is empirical evidence that the prices of many convertible bonds are substantially underpriced, relative to the sum of the straight bond price plus the embedded conversion option, e.g., Greiner et al. (2002), Athanassokos and Carayannopoulos (2000). As this underpricing can occur for bonds trading significantly below par value (call price), the underpricing is at least partly due to factors other than the presence of the call feature.

Many of the empirical studies of convertible bonds are concerned with identifying the firm's optimal call strategy for callable convertibles, e.g., Ederington and Goh (2001). The objective is typically to explain the observed fall in the common stock price that usually occurs immediately following a conversion forcing call of a convertible bond. There has also been disagreement over whether the firm delays the call decision beyond the optimal call date. It is difficult to examine these studies without acquiring scepticism about the progress of "knowledge" in modern Finance. Much like the perfect markets result for a callable nonconvertible bond, the optimal call policy for a callable convertible bond with a zero notice period is to call the bond at the first date that conversion can be forced on the bondholder, e.g., Brennan and Schwartz (1977), Ingersoll (1977a). By forcing conversion, the value of the bondholders' conversion option, a liability of the firm, is minimized. In practice, Asquith (1995) and others demonstrated that convertible bonds are not called until the conversion premium is well above the minimum conversion value. Asquith, for example, estimates an average premium of the stock price over the conversion price on the call date of 26%.

Confronted with the empirical evidence about call delays and a falling stock price around the convertible call date, proponents of modern Finance, e.g., Harris and Raviv (1985), were able to exploit the strong theoretical preference for signaling models during the 1980s to provide a theoretical rationale for these two "stylized" facts. Calling a convertible issue was a "signal" from the firm that equity was overvalued. Firms that did not want to send this signal would not call the convertible resulting in a long delay of the convertible call decision. While there was initially "strong evidence" advanced in support of this hypothesis, the gradual demise of signaling theory during the 1990s has been accompanied by an accumulation of empirical evidence against the theoretical

signaling explanation of convertible call policy. For example, Mazzeo and Moore (1992) and Ederington and Goh (2001) demonstrate that the drop in the stock price on the convertible call date is transitory, likely associated with the sudden increase in the floating supply of common stock. On average, the price decline is usually reversed within two to three weeks.

As for the delay in the convertible call decision, Butler (2002) argues that the perceived "delay" can be explained by properly modeling the call conversion decision with a non-zero notice period. Recognizing that typical notice periods are between 15–60 days, significant delay beyond the first possible conversion date is sensible because the call decision involves an exchange of the convertible bond, i.e., straight bond plus embedded warrant, for the common stock plus a put option. In effect, during a non-zero notice period the bondholder has the right to put the common stock back to the firm in exchange for the call price on the underlying debt issue. As such, observed call premiums are consistent with the firm seeking to minimize the value of the put by waiting until the put is sufficiently out of the money that it has zero or near zero value. The precise premium will depend on the conventional factors governing the put price, e.g., volatility of the stock, length of the notice period, spread between the call price and the price of the straight bond, the expected dilution of the common stock due to conversion.

The objective of the "delay" in convertible call conversion is to wait until the put value is sufficiently out of the money that the value is near zero. In addition to the basic problem of valuing the put, there are also other factors to consider that are associated with the costs of the put being in the money on the expiration date. The objective of a convertible call is to force the bondholder to convert the debt into common stock. A "busted call" occurs when the firm incorrectly judges the call decision and the put is in the money on the expiration date. This results in the bond issue being put back to the firm for cash. In such cases, the firm may have added expenses, if not difficulty, in obtaining on short notice the funds needed to refund the (unanticipated) bonds that are put back. The firm may also have to incur additional costs associated with further security issues to achieve the desired capital structure. The importance of the put option associated with convertible calls is recognized in the structure of recent convertible issues where zero coupons and short (1 year) put options have been added to the conventional callable convertible design (Dialynas et al. 2001, Springsteel 2002).

The convertible bond presents an attractive vehicle for the hedge funds, investment bankers and financial engineers seeking to profit from the modern renaissance in derivative securities (Poitras 2002, ch. 1). Pricing depends on the specific features of the bonds being examined and "structured" variations on convertible bonds, e.g., zero coupon-short put-callable-convertibles, can be complicated to value and market. A new terminology, Co-Co (contingent convertible), has been introduced to aid in the marketing effort. Such securities present excellent opportunities for the convertibles division of the large investment banks, such as Merrill Lynch, Goldman Sachs, Citigroup/Salomon and JP Morgan, to rationalize the big fees, usually 2% of proceeds, that are charged to firms seeking to "tailor" financing requirements to purchaser demands for specific security designs. (See box 6.3 for a description of a recent convertible deal.) For example, the leading US investment bank for convertibles issuance, Merrill Lynch, derived 9% of first half 2002 total investment banking income from convertibles issuance.

Box 6.3 The convertible bond

The following description of a convertible bond deal done by Silicon Integrated in 2002 illustrates the degree of complexity that such deals have attained:

"ING completed an \$80m convertible bond issue for Silicon Integrated Systems Corp, Taiwan's second largest designer of computer chipsets, on Wednesday during London trading hours. Silicon Integrated has been waiting all this calendar year to sell an ADR issue through its chosen lead manager, Bear Stearns. A window of opportunity to raise new funds through a straight equity placement did not appear, and therefore the company opted to brave the inclement market conditions to sell bonds that it wants to ensure will convert into equity.

The \$80m issue plus \$20m greenshoe has a five year maturity and a put at the end of year two to yield 3.704%, equivalent to 100bp above US Treasuries. The conversion premium is just 8.2%. The zero coupon notes are 18 months to call and then subject to call at 125% of the accreted value. There are annual conversion premium downward re-fixes subject to an aggregate 80% floor. The bond also features a onetime special re-fix by which the issuer can effectively guarantee conversion by pushing the notes deep into the money. This allows the company to re-fix the conversion premium at 85% of the market share price.

ING declared itself pleased with the distribution and the structure. The bonds were sold with a theoretical value of 112.4%, based on a credit spread of Libor plus 350bp, a zero coupon, a stock borrow cost of 5% and a volatility assumption of 35%. The bond floor emerged at 94.4%, enough to encourage outright buyers to take up 50% of the placement and asset swappers to take a quarter of the book. Hedge funds took the remaining 25% of the notes. The lead said the book was made up of 40 accounts, half of which were European."

Source: "Silicon Integrated sells \$80m CB despite stormy markets," Euroweek, July 12, 2002.

The observed mispricing of convertible bond prices has fostered the emergence of a large number of hedge funds aimed at "arbitraging" the mispricing that is commonly observed in convertible bond prices. For example, market observers estimate that hedge funds bought over 50% of new convertible issues during 2000–2002. Given that US convertibles issuance in 2001 was a record \$105 billion, the market activities of hedge funds are considerable. As Evans (2002) observes: "Convertibles are a perfect product for sophisticated, liquidity-oriented hedge funds. They can trade the volatility, strip the products into their constituent debt and equity parts, buy and sell credit protection, asset swap the paper and exploit pricing arbitrages in the secondary market." Though convertible arbitrage strategies have been used for decades, e.g., Evans (1965), the general absence of pure arbitrage opportunities has meant that modern hedge fund strategies have had to extend into strategies that have a higher level of inherent risk. As discussed below, the Greeks provide an important risk management technique that is

used by hedge funds to control the risk of convertible bond strategies, e.g., Krishnan and Mains (2002).

Convertible bonds are not the only equity-linked security available. A significant variation on equity-linked security design was the large scale Japanese warrant bond issues of the 1980s, e.g., Vaidya et al. (1995). At primary issue, a warrant bond is a combination of a straight bond and a detachable in the money warrant. A warrant bond issue explicitly recognizes that, from put-call parity, it is possible to unbundle the price of common stock into the sum of a straight bond and an in-the-money detachable warrant. More precisely, put-call parity for a non-dividend paying stock requires (e.g., Poitras 2002, ch. 7):

$$S(t) + PP[S, t; X, T, r, \sigma] = W[S, t; X, T, r, \sigma] + Xe^{-rt^*}$$

where W is the warrant price, t^* is the the number of years remaining until the expiration date of the put and warrant, r is the continuous time interest rate, σ is the volatility of the continuous rate of return on the stock, X is the exercise price and PP is the put price. (For simplicity a zero coupon continuously compounded debt issue has been assumed.) Observing that the put and warrant have the same expiration date and exercise price and are written on the same stock, it follows that when the warrant is deep in the money the put will be deep out of the money, i.e., the put price will be nearly zero.

Traditionally, convertible bonds were structured such that the embedded warrant was priced out of the money at primary issue. This makes the convertible similar to a path independent insured stock position (e.g., Poitras 2002, ch. 9), where the exercise price of the put is significantly above the current stock price. In addition, convertible bonds were long-dated with terms to maturity of 15 years and longer. Hence, at primary issue, a convertible bond was similar to a straight bond with an equity-linked "sweetener" that is only significant some considerable time in the future. (More recently, the structure of convertible bond issues has been more flexible and a variety of different structures are observed.) A warrant bond changes the traditional convertible bond structure in two important ways. First, by having the warrant deep in the money at primary issue, making the put price near zero, the warrant bond is closer to an equity issue than a debt issue. Second, by making the warrant detachable, a warrant bond issue enhances the marketability of the issue to a wider number of potential purchasers.

Unlike a traditional convertible bond that is similar to a straight bond issue, a warrant bond is a substitute for an equity issue. There are a number of advantages of using a warrant bond as a substitute for direct equity financing. Given that the warrant value could be anywhere from 15–35% of the initial value, the issuer gains by being able to raise debt with lower coupons. In turn, future warrant exercise would provide funds needed to repay the issue at maturity. Purchasers would have the advantage of being able to unbundle the components of an equity issue, selling off either the warrant or debt component if desired. For purchasers holding both parts, the bonds would provide a stable cash flow over the life of the bond and the return of principal would provide the funds needed to purchase the equity. Though this form of funding is still popular in east Asian markets, the collapse of the Japanese equity market following the large warrant bond issues of the 1980s has acted to stem the demand for such issues from international investors.

*Mortgage backed securities*³

One of the most remarkable features of the US capital markets is the process of funding mortgages. In Canada, for example, mortgages are funded primarily by financial institutions such as the chartered banks. When used to raise funds to finance a residential house purchase, the Canadian mortgage loan contract features an amortization period that is typically well in excess of the term of the loan. A conventional residential fixed rate mortgage would have a one to five year term with significant prepayment penalties. At the end of the term of the mortgage loan, the unpaid balance on the mortgage would be refinanced by creating a new fixed rate mortgage that reflects market interest rates prevailing at that time the mortgage is rewritten. At this time the borrower has the right to shift mortgage to another lender, using the funds raised to pay out the unpaid balance on the maturing mortgage without penalty. At the time the mortgage is rewritten the borrower has the ability to adjust the amortization period, select a different term and select other features such as a floating rate. In addition to the security of the property, these mortgage loans are also backed by the full faith and credit obligation of the borrower.

A borrower in the Canadian mortgage market can only dream of obtaining a 30 year fixed rate mortgage with minimal prepayment penalties and featuring a competitive rate of interest. The availability of this type of mortgage in the US can be attributed to the system of funding mortgages through the issuance of mortgage backed securities.⁴ This institutional history of this system can be traced back to 1938 and the creation of the Federal National Mortgage Association (FNMA). This institution was created to purchase Federal Housing Administration (FHA) and Veterans Administration (VA) mortgages from lenders, providing a secure source of funds for that type of borrowing. The FNMA was restructured in 1968 into two separate entities: the Government National Mortgage Association (GNMA), an agency of the federal government situated within the Department of Housing and Urban Development; and the "new" FNMA that was rechartered by Congress as a private corporation responsible for making a secondary market in mortgages not insured by FHA or VA. Two years later, in 1970, the Federal Home Loan Mortgage Corporation (FHLMC) was established under a government charter, with ownership residing in the 12 Federal Home Loan Banks.

The key innovation in mortgage funding came with the introduction by GNMA of the mortgage pass-through security in 1970. The FHLMC started issuing pass-throughs in 1971 and, in 1981, the FNMA followed suit.⁵ The basic idea behind a pass-through is to securitize the cash flows from a pool of mortgages. The mortgages are originated by financial institutions and then sold to the agency issuing the pass-through. These *mortgage originators* are primarily commercial banks, thrift institutions, mortgage brokers, insurance companies and pension funds. Cash flows from the mortgages are "passed through" to the purchasers of the security issue after the payment of a nominal servicing fee to the entity engaged in the relevant activities of collecting and processing the mortgage payments, initiating default proceedings where appropriate and the like. These activities can be handled by the originating institution or by some other *mortgage servicer*. Depending on the characteristics of the mortgage pool, there may be various types of *mortgage insurance* that are required. This insurance can include policies directly on individual mortgages, e.g., due to a high loan-to-value ratio, or credit risk

policies written on the whole pool. Pools in specific regions may also be required to have specific peril insurance, e.g., tornado insurance in Florida.

Because the cash flows on a pass-through depend on the payments from the underlying pool of mortgages, a pass-through security is qualitatively different from a straight bond. The holder of a pass-through receives on a monthly basis the pro rata share of the scheduled interest and principal payments on the underlying mortgage pool. In addition, the holder also receives any unscheduled principal payments that result from prepayments and defaults. While the scheduled principal and interest payments are predictable, the unscheduled pre-payments are not, making the cash flows from a pass-through somewhat unpredictable. To increase the marketability of mortgage backed securities, starting in 1983 the FHLMC introduced the *collateralized mortgage obligation* (CMO) that rebundles the cash flow from a pass-through into tranches of different securities with different maturity dates.⁶ One example of the advantages of a CMO is that financial institutions can purchase a security with a maturity date that better matches the maturity structure of deposits. Tax changes in the 1986 Tax Reform Act permitted CMOs to be issued as Real Estate Mortgage Investment Conduits (REMICs). Recognizing that almost all CMO issues are now in the form of REMICs, market convention is to use the terms REMIC and CMO interchangeably.

Taken together, pass-throughs and CMOs represent the class of mortgage backed securities (MBS). Though the bulk of MBS are issued by GNMA (Ginnie Mae), FNMA (Fannie Mae) and FHLMC (Freddie Mac), there is also some issuance of nonagency MBS, including MBS used to finance commercial mortgages, e.g., Fabozzi (2001, chs 27–8). Recognizing that it is difficult to obtain precise information on the size of the non-agency MBS market, the size of the market for agency MBS is given in table 6.1. Though the relationship between CMOs and pass-throughs varies over time, about 40% of pass-throughs are used to collateralize CMOs. For example, by mid-year 2002 there had been \$241 billion in CMOs issued (Bond Market Association, 2002). Table 6.1 reveals that the market for MBS expanded rapidly during the 1980s. This was largely due to the ongoing collapse of the thrift institutions as a major source of long-term mortgage financing. The amount of MBS issued in any year is dependent on the demand for the underlying mortgages. Periods of declining interest rates, e.g., 2001–2, led to a large number of mortgage refinancings and higher levels of MBS issuance.

A mortgage pass-through is a key innovation from many perspectives. Even though there has been a secondary market for individual mortgages that goes back centuries, the introduction of MBS increases the marketability of secondary mortgages, significantly reduces the credit risk by introducing agency guarantees and alters the cash flow profile of the underlying security by aggregating the individual mortgage cash flows into a pool. The process of aggregating the cash flows from individual mortgages into the cash flow from a pool of mortgages is an essential element in valuing MBS. The basic cash flow characteristics of an individual mortgage are well known. A conventional fixed rate mortgage is self-amortizing, requiring a sequence of (usually) equal monthly payments made over the term of the mortgage. Each payment can be divided between interest and principal, with the size of principal paydown relative to interest increasing over the life of the mortgage until the full mortgage balance is paid off on the last scheduled payment date. Other types of mortgage structures, e.g., adjustable rate, two-step and balloon, are possible.

Table 6.1 Issuance of agency mortgage backed securities, 1980–2002^a (\$ billions)

Date	GNMA	FNMA	FHLMC	Total
1980	20.6	—	2.5	23.1
1981	14.3	0.7	3.5	18.5
1982	16.0	14.0	24.2	54.2
1983	50.7	13.3	21.4	85.4
1984	28.1	13.5	20.5	62.1
1985	46.0	23.6	41.5	111.1
1986	101.4	60.6	102.4	264.4
1987	94.9	63.2	75.0	233.1
1988	55.2	54.9	39.8	149.9
1989	57.1	69.8	73.5	200.4
1990	64.4	96.7	73.8	234.9
1991	62.6	112.9	92.5	268.0
1992	81.9	194.0	179.2	455.2
1993	138.0	221.4	208.7	568.1
1994	111.2	130.5	117.1	358.8
1995	72.9	110.4	85.9	269.2
1996	100.9	149.9	119.7	370.5
1997	104.3	149.4	114.3	368.0
1998	150.2	326.1	250.6	726.9
1999	151.5	300.7	233.0	685.2
2000	103.7	211.7	166.9	482.4
2001	174.6	528.4	389.6	1,092.6
2002 ^a	174.0	723.3	547.1	1,444.4
2003	220.0	1,198.6	713.3	2,131.9

^a As of September 30, 2002

Sources: GNMA, FNMA, FHLMC, Bond Market Association

A mortgage presents two types of risks that can impact the cash flow: prepayment risk and default risk. For a MBS, default risk is controlled by the use of mortgage insurance and agency guarantees. However, though principal is not at risk, a default will typically trigger an eventual prepayment of principal. As such, default risk on a MBS can be treated as a component of prepayment risk. The mortgages that form the pool for a MBS usually permit homeowners to prepay the outstanding mortgage balance in whole or in part at any time prior to maturity without penalty. The precise form of the prepayment provisions is governed by state and federal law and depend on the type of mortgage, e.g., adjustable rate mortgages are governed by federal law with fixed rate mortgages falling under state law. A small number of states do not permit prepayment penalties with some other states restricting the type of penalty. Another feature impacting prepayment risk is whether the mortgages in the pool are assumable. Mortgages in FNMA and FHLMC MBS pools are generally not assumable with GNMA pools being assumable.

Because the realized cash flows from a MBS depend on the mortgage prepayments, it is necessary to estimate prepayment flows in order to value these securities. Various methods and models are available, e.g., Deng et al. (2000). Though firms that are market makers in the MBS market have in-house prepayment models, the industry benchmark

Table 6.2 Mortgage pass-through cash flow, at 100% PSA

Month	Start balance	PSA mortal.	Mortgage payment	Principal	Interest	Prepayment	Service fee	Cash flow	End balance
1	100,000	0.000166	841	49	792	17	42	816	99,934
2	99,934	0.000333	841	50	791	33	42	832	99,851
3	99,851	0.000501	840	50	790	50	42	849	99,751
4	99,751	0.000669	840	50	790	67	42	865	99,634
5	99,634	0.000837	839	51	789	83	42	881	99,500
6	99,500	0.001005	839	51	788	100	41	897	99,349
7	99,349	0.001174	838	51	787	117	41	913	99,181
8	99,181	0.001343	837	52	785	133	41	929	98,996
18	96,584	0.003050	819	55	765	294	40	1,074	96,235
19	96,235	0.003223	817	55	762	310	40	1,087	95,870
20	95,870	0.003396	814	55	759	325	40	1,100	95,489
21	95,489	0.003569	812	56	756	341	40	1,113	95,093
31	90,860	0.005143	777	58	719	467	38	1,206	90,336
32	90,336	0.005143	773	58	715	464	38	1,200	89,814
33	89,814	0.005143	769	58	711	462	37	1,193	89,294
34	89,294	0.005143	765	58	707	459	37	1,187	88,777
99	60,354	0.005143	547	69	478	310	25	832	59,975
100	59,975	0.005143	544	70	475	308	25	827	59,597
209	27,372	0.005143	310	94	217	140	11	439	27,138
210	27,138	0.005143	309	94	215	139	11	436	26,905
211	26,905	0.005143	307	94	213	138	11	434	26,673
359	283	0.005143	143	141	2	1	0	144	141
360	141	0.005143	142	141	1	0	0	142	0

Where: Start balance is mortgage balance at the beginning of month; PSA mortal. is the mortality rate estimated for that month using 100% of PSA; Mortgage payment is the monthly mortgage payment = principal + interest; Prepayment is the estimated prepayment = (Start balance) (PSA mortal.); Service fee is the assumed contractually specified service fee; Cash flow is the payment made to holder of the pass through; and, End balance = Start balance - Principal - Prepayment.

Original mortgage balance: \$100,000; mortgage rate: 9.5%; term of mortgage: 30 years (360 months); servicing fee: 0.5%; PSA = 100%.

was developed by the Public Securities Association (incorporated 1976, name changed to Bond Market Association in 1997). This benchmark is referred to as 100% PSA. With the demise of the PSA as the association name, the acronym now refers to Prepayment Speed Assumptions. Based on empirical experience with FHA mortgages, 100% PSA assumes a prepayment rate that increases each month for the first 30 months and then is level thereafter. Tables 6.2 and 6.3, adapted from Fabozzi (1989, pp. 246-8), illustrate the impact of assuming 100% PSA and 150% PSA on the cash flows of a pass-through. The prepayment amount for any month is determined by multiplying the prepayment mortality factor from the PSA by the outstanding mortgage principal amount. Increasing from 100% to 150% PSA involves multiplying the 100% PSA mortality factor by 1.5.

Table 6.3 Mortgage pass-through cash flow, at 150% PSA

Month	Start balance	PSA mortal.	Mortgage payment	Principal	Interest	Prepayment	Service fee	Cash flow	End balance
1	100,000	0.000250	841	49	792	25	42	824	99,926
2	99,926	0.000501	841	50	791	50	42	849	99,826
3	99,826	0.000753	840	50	790	75	42	874	99,701
4	99,701	0.001005	840	50	789	100	42	898	99,551
5	99,551	0.001258	839	51	788	125	41	923	99,375
6	99,375	0.001512	838	51	787	150	41	947	99,174
7	99,174	0.001767	836	51	785	175	41	970	98,947
8	98,947	0.002022	835	52	783	200	41	994	98,695
18	95,324	0.004615	809	54	755	440	40	1,209	94,830
19	94,830	0.004878	805	54	751	462	40	1,228	94,313
20	94,313	0.005143	801	54	747	485	39	1,247	93,774
21	93,774	0.005407	797	55	742	507	39	1,266	93,213
31	87,224	0.007828	746	55	691	682	36	1,392	86,487
32	86,487	0.007828	740	55	685	677	36	1,381	85,755
33	85,755	0.007828	734	55	679	671	36	1,369	85,028
34	85,028	0.007828	728	55	673	665	35	1,358	84,308
99	48,211	0.007828	437	55	382	377	20	794	47,779
100	47,779	0.007828	434	55	378	374	20	787	47,350
209	16,241	0.007828	184	56	129	127	7	304	16,059
210	16,059	0.007828	183	56	127	125	7	301	15,878
211	15,878	0.007828	181	56	126	124	7	299	15,699
359	112	0.007828	57	56	1	0	0	57	56
360	56	0.007828	56	56	0	0	0	56	0

Where: Start balance is mortgage balance at the beginning of month; PSA mortal. is the mortality rate estimated for that month using 150% of PSA; Mortgage payment is the monthly mortgage payment = principal + interest; Prepayment is the estimated prepayment = (Start balance) (PSA mortal.); Service fee is the assumed contractually specified service fee; Cash flow is the payment made to holder of the pass through; and, End balance = Start balance - Principal - Prepayment.

Original mortgage balance: \$100,000; mortgage rate: 9.5%; term of mortgage: 30 years (360 months); servicing fee: 0.5%; PSA = 100%.

6.2 GREEKS FOR BONDS WITH EMBEDDED OPTIONS*

What are the Greeks?*

The "Greeks" refer to the names given to derivatives of option prices and other value functions, e.g., Poitras (2002, ch. 9). The most important of these names are *delta* (Δ), *gamma* (Γ), *theta* (θ), *rho* (ρ) and *vega*, hence the reference to Greeks, i.e., the various derivatives are referred to using Greek letters. In particular, the first derivative of

the price of a call option (C) written on a stock, taken with respect to the stock price ($\partial C / \partial S$), is the "delta" of that call option (Δ_c). Similarly, the delta of a put option on the common stock is the first derivative of the put price with respect to a change in the stock price (Δ_p). A partial derivative is taken because the call (put) option price is a function of other variables, such as time. In general, a "delta" refers to the first derivative of a price or value function taken with respect to the price of the underlying asset or commodity (the state variable price). The delta for an option written on a fixed income security would be the first derivative of the call option price with respect to the price of the fixed income security. For a bond with an embedded option, the delta of the option is typically the first derivative taken with respect to the price of a *riskless* bond.

Delta and the other Greeks were initially developed for use in the analysis of options, e.g., Cox and Rubinstein (1985). The Greeks are particularly useful in risk management. Delta, for example, measures the change in the value of a position with respect to a change in the state variable. The extension to more recent risk management techniques, such as value at risk, is straightforward, e.g., Jorion (2001). In almost all cases, the relevant risk is the change in a "price," e.g., the change in stock prices, the change in commodity prices, even the change in exchange rates. However, for assessing the risk of fixed income securities, the state variable "price" is an illusive concept. For this reason, risk management for fixed income securities has developed notions such as duration and convexity. In particular, duration measures the change in the value of a fixed income position with respect to a change in "interest rates." Though it is possible to conceive of reexpressing duration in terms of the change in, say, a riskless fixed maturity zero coupon bond, this would add more confusion to an already imprecise concept (as discussed in section 5.3). Instead, it is more practical to detail how the Greeks relate to the conventional notions of duration and convexity.

When referring to options on fixed income securities, whether the option is embedded or traded separately, the Greek concepts make sense. For example, the delta of an exchange traded bond option measures the change of the option price with respect to a change in the price of the underlying bond. When referring to a bond with an embedded option, including bonds subject to default risk, it is also possible to talk about the delta in terms of the change in the price of the embedded riskless straight bond. Letting P_{OB} be the price of the bond with the embedded option and P_B be the riskless straight bond price component of the embedded option price ($P_{OB} = P_B \pm$ Embedded option price), it follows that:

$$\Delta_{OB} = \frac{dP_{OB}}{dP_B} = \frac{P_{OB}}{P_{OB}} \frac{P_B}{P_B} \frac{y}{y} \frac{dy}{dy} \frac{dP_{OB}}{dP_B} = \frac{P_{OB}}{P_B} \frac{D_{OB}}{D_B}$$

The delta of the embedded option bond equals the ratio of the price of the embedded option bond to the straight bond component multiplied by the ratio of the duration of the embedded option bond to the duration of the straight bond component. To interpret the delta of the embedded option bond consider the case where the embedded option has no value. In this case, $P_{OB} = P_B$ and the durations will also be equal. It follows that the delta will be one. Similarly, consider a currently callable bond where the callable bond price is above the call price. In this case, $P_{OB} < P_B$ and the duration of the

embedded option bond will be significantly shorter than the duration of the straight bond because of the likelihood of an early call date. It follows that the delta of the embedded option bond will be much less than one, approaching zero as the probability of immediate call approaches one.

Given this, it is possible to provide another interpretation of the delta of an embedded option bond. Let $P_{OB} = P_B \pm P_O$, where P_O is the price of the embedded option. This specification follows because the embedded option price is subtracted from the straight bond price for a seller's (issuer's) option and added to the price for a buyer's option. It follows that:

$$\Delta_{OB} = \frac{dP_{OB}}{dP_B} = 1 \pm \frac{dP_O}{dP_B} = 1 \pm \Delta_O$$

In effect, when the embedded option is deep out of the money then the delta of the embedded option bond will be close to one and the price will change like a straight bond price. As the option goes into the money the delta of the embedded option bond will deviate from one. Recalling that the delta of the embedded option is a non-linear function of the state variable, this property is carried over to the change in the delta of the embedded option bond. One of the insights of option pricing theory is that it is possible to provide a functional form for the option delta.

Discussion of the other Greeks follows appropriately. In option pricing theory, the second derivative of a call option written on a stock, taken with respect to the stock price ($\partial^2 C / \partial S^2$) is the gamma of the call option on a stock. In general, gamma measures the changes in delta with respect to the change in the state variable. This Greek is important in risk management for portfolios where the objective is to attain a delta target, e.g., delta equal to zero. Because option deltas change as the state variable changes, it may be necessary to dynamically trade the position to achieve the delta target. Gamma gives information about the rebalancing frequency. Other derivatives follow appropriately: theta is the time derivative; "vega" is the volatility derivative; and rho is the riskless interest rate derivative.⁷ In theory, each of these concepts can be extended to embedded option bonds (theta for straight bonds has already been examined in section 5.3). However, the precise method of doing this for vega or rho is not obvious. For example, vega could be assessed by assuming a specific stochastic process that, in turn, could be used to specify zero coupon bond prices (see end of chapter 5 questions) and embedded option prices. The method for doing this would, at best, be complicated.

The Greeks play a useful role in the analysis of portfolios that contain options. If a closed form option pricing formula is available, then the Greeks can be solved directly. While identifying a closed form is relatively straightforward for options on commodities and equities where there is a single price for the underlying state variable, e.g., stock prices or exchange rates, there are considerable problems for options written on fixed income securities (e.g., Hull 2003, pp. 530–1). To see this, consider the Black (1976) model for an option on a Eurodollar futures contract. This model requires the state variable to be lognormally distributed. This assumption is inappropriate for fixed income prices because par value is an upper bound on the price

determined by the inability of interest rates to be negative. Following Stoll and Whalley (1993, pp. 369-73), this problem can be circumvented by choosing yields as the state variable. In this case, lognormality is consistent with the yield not being negative. By setting a short time to expiration and using observed implied volatilities, the possibility of Eurodollar futures prices going to zero as the yield increases without limit is effectively avoided.

Given this, the payoff for the call option on the expiration date can now be expressed as: $C_e(T) = \max[0, (100 - eu(T)) - X] = \max[0, (100 - X) - eu(T)]$ where C_e is the price of the Eurodollar call option, eu is the add-on interest rate on the Eurodollar deposit to be delivered, T is the expiration date and X is the exercise price. This is the first quirk associated with the Eurodollar interest rate option contract: the payoff on a call option has the same form as a payoff on a commodity put option, where the interest rate $eu(T)$ replaces $S(T)$ and the "exercise price" is expressed as $(100 - X)$ instead of X . Observing that $eu(t) = (100 - EU(t))$, where $EU(t)$ is the observed Eurodollar futures price, it is now possible to substitute directly into the Black (1976) formula for a European *put* option on a commodity futures contract to obtain the option pricing formula for a European *call* on a Eurodollar interest rate futures contract:⁸

$$C_e[EU, T; X, r, \sigma_e] = \exp^{-rt^*} [(100 - X)N[-d_2] - (100 - EU(t))N[-d_1]]$$

where:

$$d_1 = \frac{\ln \left[\frac{100 - EU(t)}{100 - X} \right] + .5\sigma_e^2 t^*}{\sigma_e \sqrt{t^*}} \quad d_2 = d_1 - \sigma_e \sqrt{t^*}$$

In this formulation σ_e is the volatility of the Eurodollar add-on interest rate, t^* is the fraction of the year remaining to maturity, r is the "riskless" interest rate and $N[d]$ is the value of the cumulative normal distribution function evaluated at d (see table 6.4).

As an example of applying the formula consider the Eurodollar futures and options quotes in table 6.5. Assume that the "price" for the nearby March Eurodollar futures contract is $EU(t) = 98.69$ which translates to an annualized interest rate on the deliverable three month Eurodollar of $100 - 98.69 = 1.31\%$. The method of quoting Eurodollar futures options is somewhat different. For example, on February 4, 2002 the March delivery call with a ($X =$) 98.75 exercise price is .27 or 27 basis points. To calculate the price of this option observe that the \$1 million par value 3 month Eurodollar deposit underlying the futures contract has a value of \$25 per basis point. Hence, ignoring transactions costs a call for one Eurodollar futures with a 98.75 exercise price is selling for $(.27)(\$25) = \6.75 . Observing that the price quotes are for February 4 and the option will expire on March 17, the fraction of the year remaining to maturity is $t^* = (41/365) = .112$. To use the option pricing formula for this option now requires an estimate for the volatility σ_e and the riskless rate r . Given that the call option price is observed, it follows that the formula provides a joint estimate of implied volatility and the riskless rate.

Table 6.4 Cumulative normal distribution values

Values of $N(d)$ for selected values of d					
d	$N(d)$	d	$N(d)$	d	$N(d)$
-2.95	0.0016	-1.00	0.1587	1.00	0.8413
-2.90	0.0019	-0.95	0.1711	1.05	0.8531
-2.85	0.0022	-0.90	0.1841	1.10	0.8643
-2.80	0.0026	-0.85	0.1977	1.15	0.8749
-2.75	0.0030	-0.80	0.2119	1.20	0.8849
-2.70	0.0035	-0.75	0.2266	1.25	0.8944
-2.65	0.0040	-0.70	0.2420	1.30	0.9032
-2.60	0.0047	-0.65	0.2578	1.35	0.9115
-2.55	0.0054	-0.60	0.2743	1.40	0.9192
-2.50	0.0062	-0.55	0.2912	1.45	0.9265
-2.45	0.0071	-0.50	0.3085	1.50	0.9332
-2.40	0.0082	-0.45	0.3264	1.55	0.9394
-2.35	0.0094	-0.40	0.3446	1.60	0.9452
-2.30	0.0107	-0.35	0.3632	1.65	0.9505
-2.25	0.0122	-0.30	0.3821	1.70	0.9554
-2.20	0.0139	-0.25	0.4013	1.75	0.9599
-2.15	0.0158	-0.20	0.4207	1.80	0.9641
-2.10	0.0179	-0.15	0.4404	1.85	0.9678
-2.05	0.0202	-0.10	0.4602	1.90	0.9713
-2.00	0.0228	-0.05	0.4801	1.95	0.9744
-1.95	0.0256	0.00	0.5000	2.00	0.9773
-1.90	0.0287	0.05	0.5199	2.05	0.9798
-1.85	0.0322	0.10	0.5398	2.10	0.9821
-1.80	0.0359	0.15	0.5596	2.15	0.9842
-1.75	0.0401	0.20	0.5793	2.20	0.9861
-1.70	0.0446	0.25	0.5987	2.25	0.9878
-1.65	0.0495	0.30	0.6179	2.30	0.9893
-1.60	0.0548	0.35	0.6368	2.35	0.9906
-1.55	0.0606	0.40	0.6554	2.40	0.9918
-1.50	0.0668	0.45	0.6736	2.45	0.9929
-1.45	0.0735	0.50	0.6915	2.50	0.9938
-1.40	0.0808	0.55	0.7088	2.55	0.9946
-1.35	0.0885	0.60	0.7257	2.60	0.9953
-1.30	0.0968	0.65	0.7422	2.65	0.9960
-1.25	0.1057	0.70	0.7580	2.70	0.9965
-1.20	0.1151	0.75	0.7734	2.75	0.9970
-1.15	0.1251	0.80	0.7881	2.80	0.9974
-1.10	0.1357	0.85	0.8023	2.85	0.9978
-1.05	0.1469	0.90	0.8159	2.90	0.9981
		0.95	0.8289	2.95	0.9984

2004

Interest Rate Futures

Treasury Bonds (CBT)-\$100,000; pts 32nds of 100%									
Mar	109-20	110-16	109-17	110-09	24	116-23	101-05	430,808	
June	108-23	109-01	108-13	108-28	24	116-15	104-00	10,195	
Est vol 114,656; vol Tue 134,747; open int 464,142, +4,406.									
Treasury Notes (CBT)-\$100,000; pts 32nds of 100%									
Mar	112-01	12-235	111-30	112-20	22.0	116-10	106-29	928,020	
June	—	—	—	111-04	22.0	111-15	107-13	290	
Est vol 288,353; vol Tue 303,885; open int 941,864, -2,107.									
10 Yr. Agency Notes (CBT)-\$100,000; pts 32nds of 100%									
Mar	—	—	—	na	—	—	—	0	
Est vol 0; vol Tue 0; open int na, *.									
5 Yr. Treasury Notes (CBT)-\$100,000; pts 32nds of 100%									
Mar	111-10	111-25	11-075	111-23	16.5	19-215	09-145	795,042	
Est vol 125,149; vol Tue 148,419; open int 820,085, -7,560.									
2 Yr. Treasury Notes (CBT)-\$200,000; pts 32nds of 100%									
Dec	07-202	107-21	107-18	107-21	6.7	07-287	106-06	10,470	
Est vol 11,115; vol Tue 19,587; open int 160,113, +87.									
30 Day Federal Funds (CBT)-\$5,000,000; 100 - daily avg.									
Dec	99.010	99.015	99.005	99.005	—	99.230	98.400	45,786	
Ja04	99.00	99.00	99.00	99.00	—	99.24	98.66	51,238	
Feb	98.99	99.00	98.99	98.99	0.01	99.22	98.70	77,544	
Mar	98.99	98.99	98.98	98.98	0.01	99.16	98.74	51,695	
Apr	98.97	98.97	98.96	98.97	0.01	99.17	98.96	63,198	
May	98.88	98.89	98.88	98.88	0.01	99.79	98.40	27,107	
June	98.87	98.87	98.86	98.87	0.02	98.96	98.38	15,844	
July	98.75	98.77	98.75	98.77	0.04	98.89	98.20	14,668	
Est vol 9,191; vol Tue 15,247; open int 349,596, +1,170.									
10 Yr. Interest Rate Swaps (CBT)-\$100,000; pts 32nds of 100%									
Mar	109-19	110-07	109-19	110-07	21	111-00	107-20	36,929	
Est vol 75; vol Tue 2,590; open int 36,930, +1,676.									
10 Yr. Muni Note Index (CBT)-\$1,000 × index									
Mar	102-16	103-01	102-16	103-01	16	103-14	99-21	1,916	
Est vol 498; vol Tue 170; open int 1,916, +4.									
Index: Close na; Yield na.									

	OPEN	HIGH	LOW	SETTLE	CHG	YIELD	CHG	OPEN INT
13 Week Treasury Bills (CME)-\$1,000,000; pts of 100%								
Jan	—	—	—	n.a.	—	—	—	0
Est vol n.a.; vol Tue 0; open int 0, unch.								
1 Month Libor (CME)-\$3,000,000; pts of 100%								
Jan	98.88	98.88	98.88	98.88	—	1.12	—	18,951
Feb	98.87	98.87	98.87	98.87	0.01	1.13	-0.01	31,195
Est vol 1,601; vol Tue 3,205; open int 74,691, +1,179.								
Eurodollar (CME)-\$1,000,000; pts of 100%								
Jan	98.83	98.83	98.82	98.82	—	1.18	—	103,770
Feb	98.81	98.81	98.79	98.81	0.01	1.19	-0.01	22,987
Mar	98.76	98.77	98.75	98.76	0.01	1.24	-0.01	810,547
Apr	98.72	98.72	98.72	98.72	0.03	1.28	-0.03	13,917
May	98.62	98.65	98.62	98.65	0.06	1.35	-0.06	9,886
June	98.51	98.57	98.51	98.56	0.07	1.44	-0.07	696,129
Sept	98.16	98.24	98.15	98.22	0.09	1.78	-0.09	576,374
Dec	97.71	97.83	97.69	97.81	0.13	2.19	-0.13	495,833
Mr05	97.27	97.41	97.28	97.40	0.12	2.60	-0.12	348,628
June	96.88	96.98	96.87	96.97	0.12	3.03	-0.12	278,748
Sept	96.52	96.61	96.49	96.60	0.13	3.40	-0.13	215,912
Dec	96.22	96.32	96.19	96.30	0.12	3.70	-0.12	176,090
Mr06	95.98	96.10	95.97	96.08	0.11	3.92	-0.11	141,182
June	95.77	95.86	95.76	95.85	0.10	4.15	-0.10	124,164
Sept	95.62	95.66	95.61	95.65	0.09	4.35	-0.09	102,376
Dec	95.37	95.46	95.37	95.45	0.09	4.55	-0.09	101,407
Mr07	95.23	95.29	95.21	95.29	0.08	4.71	-0.08	74,970
June	95.11	95.12	95.08	95.12	0.07	4.88	-0.07	65,690
Sept	94.90	94.96	94.90	94.97	0.07	5.03	-0.07	62,672
Dec	94.77	94.82	94.77	94.82	0.05	5.18	-0.05	53,056
Mr08	94.70	94.71	94.65	94.71	0.06	5.29	-0.06	46,171
June	94.59	94.61	94.54	94.61	0.06	5.39	-0.06	46,961
Sept	94.49	94.52	94.44	94.51	0.06	5.49	-0.06	29,345
Dec	94.40	94.42	94.40	94.41	0.08	5.59	-0.08	17,419
Est vol 236,941; vol Tue 435,250; open int 4,685,668, +1,398.								

Interest Rate Options

T-Bonds (CBT)

\$100,000; points and 64ths of 100%						
Price	Jan	Feb	Mar	Jan	Feb	Mar
108	2-18	2-58	3-36	0-01	0-40	1-17
109	1-19	2-14	2-59	0-01	0-60	1-41
110	0-28	1-40	2-23	0-10	1-22	2-04
111	0-04	1-09	1-56	0-50	1-55	2-38
112	0-01	0-49	1-29	1-47	2-30	3-10
113	0-01	0-31	1-07	-	3-13	3-52
Est vol 38,507;						
Tn vol 19,412 calls 25,929 puts						
Op int Tues 395,173 calls 296,135 puts						

T-Notes (CBT)

Price	Jan	Feb	Mar	Jan	Feb	Mar
111	1-41	2-07	2-37	0-01	0-31	0-61
112	0-43	1-23	1-60	0-03	0-52	1-20
113	0-05	0-57	1-26	0-29	1-17	1-50
114	0-01	0-32	0-62	1-25	1-56	2-22
115	0-01	0-16	0-42	2-24	2-40	—
116	0-01	0-07	0-26	—	—	3-50

Est vol 102,810 Tu 36 calls 71,317 puts
 Op int Tues 767,793 calls 883,751 puts

5 Yr Treas Notes (CBT)

Price	Jan	Feb	Mar	Jan	Feb	Mar
11,050	1-14	1-35	1-52	0-01	0-21	0-39
11,100	0-47	1-11	1-32	0-01	0-29	0-50
11,150	0-18	0-54	1-12	0-04	0-40	0-62
11,200	0-04	0-38	0-59	0-22	0-56	1-13
11,250	0-01	0-26	0-46	—	—	—
11,300	0-01	0-16	0-35	—	—	—

Est vol 22,793 Tu 11,042 calls 28,596 puts
 Op int Tues 123,621 calls 354,590 puts

Eurodollar (CME)

Eurodollar (CME)						
\$ million; pts. of 100%						
Price	Jan	Feb	Mar	Jan	Feb	Mar
9,825	5.15	—	5.22	0.00	0.05	0.07
9,850	2.70	—	2.80	0.05	0.10	0.15
9,875	0.45	0.60	0.70	0.30	0.45	0.55
9,900	0.00	—	0.10	—	—	2.45
9,925	—	—	0.02	—	—	4.87
9,950	—	—	0.00	—	—	7.35

1 Yr. Mid-Curve Eurodrlr (CME)

1,000,000 contract units; pts. of 100%						
Price	Jan	Feb	Mar	Jan	Feb	Mar
9,700	4.35	5.10	5.60	0.35	1.10	1.60
9,725	2.50	3.37	3.85	1.00	1.87	2.35
9,750	1.15	2.00	2.52	2.15	—	3.52
9,775	0.40	1.00	1.45	—	—	4.95
9,800	0.15	—	0.72	—	—	6.70
9,825	—	—	—	—	—	8.82

Est vol 5,952 Tu 17,956 calls 13,780 puts
 Op int Tues 729,969 calls 570,966 puts

As a pedagogical exercise, set $r = 1\%$ and $\sigma_e = 1.866\%$. Using these values to calculate the call option price reveals:

$$d_1 = \frac{\ln[(100 - 98.69)/(100 - 98.75)] + .5(1.866)^2.112}{1.866\sqrt{.112}} = .3873 \rightarrow N[-d_1] = .35$$

$$d_2 = .3873 - 1.866\sqrt{.112} = -.2372 \rightarrow N[-d_2] = .585$$

$$C_e = \exp^{-.01(.112)}[(100 - 98.75)]N[-d_2] - (100 - 98.69)N[-d_1] = .2724$$

This is consistent with the observed call option price of 27 basis points. The use of yields as the state variable is well suited to Eurodollar futures options where the “futures price” $EU(t) = 100 - eu(t)$. Specification of the Black model for an option on a bond futures contract, where the futures and options quotes relate to bond prices, again raises the complications associated with assuming lognormality for a fixed income security price. In addition, there are complications associated with valuing changes in the term structure affecting the bond and bond option value.

In order to develop a more sensible bond option pricing model the discussion in section 5.2 about different stochastic processes for the interest rate now assumes relevance. For example, the Vasicek model can be used to solve for zero coupon bond prices that, in turn, can be used to solve the option pricing formula for a zero coupon bond. Because the underlying process is OU, this will result in a different specification for the volatility. Hull (2003, pp. 545–6) demonstrates that the Ho-Lee model leads to a zero coupon bond pricing formula that is “essentially the same as Black’s model” with the proviso that the zero coupon bond prices are determined by the absence of arbitrage formula determined from the Ho-Lee model (see end of chapter 5 questions). Given the option pricing formulas for zero coupon bonds, coupon bearing bond options can be determined by treating each coupon payment as a zero coupon bond and summing across options for all payment periods. In any event, it is possible, in some fashion, to determine the Greeks for various fixed income options from the available valuation formulas or, where these are too complicated, using numerical procedures.

*Duration, convexity and the Greeks for callable bonds**

To derive the Greeks for a default free callable bond it is necessary to observe that the call feature is a seller’s option. This leads to:

$$P_{CB} = P_B - C_B$$

where P_{CB} is the observed price of the callable bond, P_B is the unobserved price of the straight bond, and C_B is the unobserved price of the “callable option” feature.⁹ The value of C_B is subtracted because a callable option is a seller’s option. The delta of the callable option plays a key role in the interpretation of the duration for the callable bond price. To see this, consider the modified duration for the callable bond:

$$\frac{1}{P_{CB}} \frac{dP_{CB}}{dy} = \frac{1}{P_{CB}} \left[\frac{dP_B}{dy} - \frac{dC_B}{dy} \right] = \frac{P_B}{P_{CB}} \left[\frac{1}{P_B} \frac{dP_B}{dy} - \frac{1}{P_B} \frac{dC_B}{dy} \right]$$

It follows that the duration of a callable bond is related to the duration of the embedded straight bond and the embedded option delta.

To demonstrate that these two durations are connected by the delta of the callable option, consider the following manipulation:

$$\frac{1}{P_B} \frac{dC_B}{dy} = \frac{1}{P_B} \frac{dC_B}{dP_B} \frac{dP_B}{dy} = \Delta_C \Delta_B$$

where D_B is the modified duration of the embedded straight bond and $\Delta_C = (dC_B/dP_B)$ is the delta of the callable option. The introduction of the chain rule is applicable because the call price depends on the straight bond price which, in turn, is a function of the yield, i.e., the callable option is only an implicit function of the yield to maturity. The delta is evaluated with respect to the straight bond price because changes in the callable option price depend on changes in the straight bond price, with the callable bond price being a function of the straight bond and callable option prices.

Given this, the duration for the callable bond follows appropriately:

$$\frac{1}{P_{CB}} \frac{dP_{CB}}{dy} = D_{CB} = D_B(1 - \Delta_C) \frac{P_B}{P_{CB}}$$

This functional relationship between the modified durations of the callable and straight bonds is illustrated in figure 6.1. This figure is only indicative as the price function for the callable bond will be altered somewhat if different assumptions are made about variables such as the call price, call date and interest rate volatility. Following the discussion in section 5.3, the duration can be heuristically interpreted as the slope of the bond price function, $P[y]$. Recognizing that figure 6.1 is for a callable bond that is approaching the call date, at price levels well below (yield levels well in excess of) the call exercise price, the callable and straight bond price functions are identical ($P_{CB} = P_B$). In the formula for the modified duration, this implies that $\Delta_C = 0$ at that yield level. This follows because the callable option is deep out-of-the-money in this region implying a zero or near zero price for the embedded option.

Figure 6.1 indicates that as the price level rises (yields fall) to a level approaching the call exercise price, the callable and straight bond price functions start to diverge. This is an implication of the callable option getting closer to the money and, as a result, the delta of the option takes on a significant positive value. As the bond price rises above the call exercise price, the callable option goes into the money and the delta of the option increases above the at-the-money level of approximately one-half. As the option goes deep in the money, the delta goes to one and the duration of the callable bond goes to zero. This implies that the callable bond will have a price that is bounded above. This illustrates an important feature of callable bonds: the upward price increase of a callable bond due to falling yields is bounded above. This upper bound will be tighter for shorter call dates and higher levels of yield volatility.

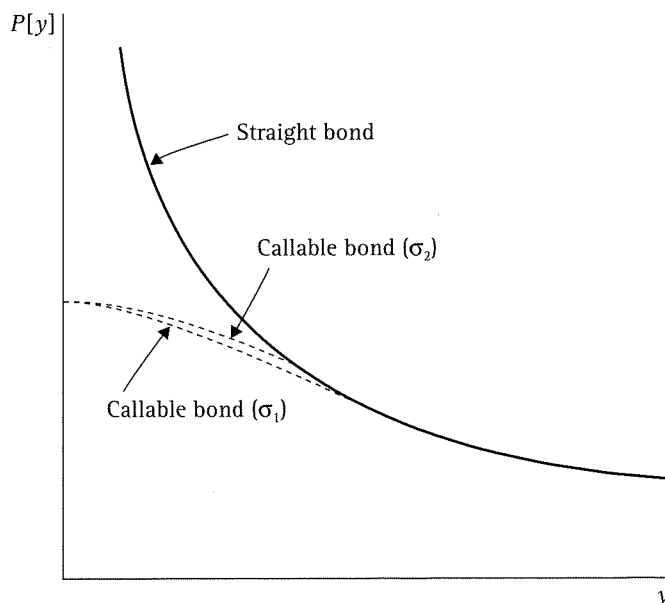


Figure 6.1 Price-yield relationship for a callable bond and a straight bond where $\sigma_2 > \sigma_1$

In much the same fashion that the duration of a callable bond is related to the delta of the embedded callable option, the convexity of the callable bond is related to the gamma of the embedded callable option. However, the relationship is analytically more complicated than for the duration-delta relationship. To see this consider the second derivative of the callable bond price:

$$\begin{aligned} \frac{dP_{CB}}{dy} &= \frac{dP_B}{dy} - \frac{dC_B}{dP_B} \frac{dP_B}{dy} \\ \rightarrow \frac{d^2P_{CB}}{dy^2} &= \frac{d^2P_B}{dy^2} - \left[\frac{dP_B}{dy} \frac{d^2C_B}{dP_B^2} \frac{dP_B}{dy} + \frac{dC_B}{dP_B} \frac{d^2P_B}{dy^2} \right] \end{aligned}$$

It follows that:

$$\frac{1}{P_{CB}} \frac{d^2P_{CB}}{dy^2} = \text{CON}_{CB} = \frac{P_B}{P_{CB}} \left[\frac{1}{P_B} \frac{d^2P_B}{dy^2} - \left(\frac{1}{P_B} \frac{dP_B}{dy} \frac{d^2C_B}{dP_B^2} \frac{dP_B}{dy} + \frac{1}{P_B} \frac{dC_B}{dP_B} \frac{d^2P_B}{dy^2} \right) \right]$$

Observing that:

$$\frac{1}{P_B} \frac{d^2P_B}{dy^2} = \text{CON}_B \quad \frac{d^2C_B}{dP_B^2} = \Gamma_C$$

The formula for the convexity of a callable bond can be stated as:

$$CON_{CB} = \frac{P_B}{P_{CB}} [CON_B(1 - \Delta_C) - P_B \Gamma_C(D_B)^2]$$

This results reveals that, unlike a straight bond that has $CON_B > 0$, a callable bond can have convexity that is negative over some range of bond prices.

When will convexity be negative? This can be seen by again examining figure 6.1 (see also end of chapter questions). As with the result for callable bond duration, for bond prices well below (yields above), the callable option will be deep out of the money and both the delta and gamma will be zero or near zero. In this case, $CON_{CB} = CON_B$. The negative convexity emerges as the bond price increases (yield decreases) from a very low (high) level. Starting from, say 20% yield, the durations of both the callable and straight bond will increase as the yields fall (positive convexity). However as the yield approaches the range where the call provision becomes effective, the duration of the straight bond will continue to increase but the duration of the callable bond will start to fall. Recalling that duration involves the value of the first derivative multiplied by minus one, the property that the duration falls as yields fall is the meaning of negativity convexity. As the callability option goes farther and farther into the money, the impact of yield changes on the duration of the callable bond will dissipate, implying that convexity will start to increase from a negative value back to zero.

*Duration, convexity and the Greeks for other embedded option bonds**

Callable bonds provide a relatively clean solution to duration, convexity and the Greeks for bonds with embedded options. Some other types of embedded option bonds, e.g., bonds with put provisions, are also relatively clean. However, embedded option bonds such as MBS and bonds with default risk are more complicated. This happens because the mechanism for interest rate changes to impact bond prices requires some type of valuation model to be specified. For example, an MBS security requires the specification of a prepayment model. The relationship between prepayments and interest rates is decidedly more complicated than in the case of the callability provision. Because a conventional convertible bond is, by construction, a defaultable security, the case of convertible bonds is also complicated as it requires both the default risk and the conversion option to be modeled. While the conversion option can be modeled using option pricing techniques, the default risk component requires a valuation model for the market value of the firm. The process of modeling default risk is discussed in section 6.4.

To derive the Greeks for a default free puttable bond it is necessary to observe that the put feature is a buyer's option. This leads to:

$$P_{PB} = P_B + PP_B$$

where P_{PB} is the observed price of the puttable bond, P_B is the unobserved price of the straight bond, and PP_B is the unobserved price of the "puttable option" feature.¹⁰ The value of PP_B is added because a puttable option is a buyer's option. As before, the delta of the puttable option plays a key role in interpreting the duration of the puttable bond price. To see this observe the modified duration for a puttable bond:

$$\frac{1}{P_{PB}} \frac{dP_{PB}}{dy} = \frac{1}{P_{PB}} \left[\frac{dP_B}{dy} + \frac{dPP_B}{dy} \right] = \frac{P_B}{P_{PB}} \left[\frac{1}{P_B} \frac{dP_B}{dy} - \frac{1}{P_B} \frac{dPP_B}{dy} \right]$$

It follows that the duration of a puttable bond is related to the duration of the embedded straight bond and the embedded option delta.

To demonstrate that these two durations are connected by the delta of the puttable option, consider the following manipulation:

$$\frac{1}{P_B} \frac{dPP_B}{dy} = \frac{1}{P_B} \frac{dPP_B}{dP_B} \frac{dP_B}{dy} = \Delta_P D_B$$

where D_B is the modified duration of the embedded straight bond and $\Delta_P = (dPP_B/dP_B)$ is the delta of the puttable option. The introduction of the chain rule is applicable because the put price depends on the straight bond price which, in turn, is a function of the yield, i.e., the puttable option is only an implicit function of the yield to maturity. The delta is evaluated with respect to the straight bond price because changes in the puttable option price depend on changes in the straight bond price, with the puttable bond price being a function of the straight bond and puttable option prices.

Given this, the duration for the puttable bond follows appropriately:

$$\frac{1}{P_{PB}} \frac{dP_{PB}}{dy} = D_{PB} = D_B(1 + \Delta_P) \frac{P_B}{P_{PB}}$$

This functional relationship between the modified durations of the puttable and straight bonds is illustrated in figure 6.2. As in figure 6.1 for the callable bond, this figure is only indicative as the price function for the puttable bond will be altered somewhat if different assumptions are made about variables such as the put price, put date and interest rate volatility. Again following the discussion in section 5.3, the duration can be heuristically interpreted as the slope of the bond price function. For a puttable bond that is approaching the put date, at puttable bond price levels well above (yield levels well below) the put exercise price, the puttable and straight bond price functions are identical ($P_{PB} = P_B$). In the formula for the modified duration, this implies that $\Delta_P = 0$ in this range of y . This follows because the puttable option is deep out-of-the-money in this region implying a zero or near zero price for the embedded option.

Figure 6.2 indicates that as the price level falls (yields rise) to a level approaching the put exercise price, the puttable and straight bond price functions start to diverge. This is an implication of the puttable option getting closer to the money and, as a result, the delta of the put option starts to take on a significant *negative* value, i.e., as straight bond prices fall the value of a put option increases ($\Delta_P < 0$). As the bond price falls below the put exercise price, the puttable option goes into the money and the delta of the option decreases below the at-the-money level of approximately minus one-half. As the option goes deep in the money, the delta goes to minus one and the duration of the puttable bond goes to zero. This implies that the puttable bond will have a price that is bounded below. This illustrates an important feature of puttable bonds: the downward price decrease of a puttable bond associated with rising yields is bounded below. With a

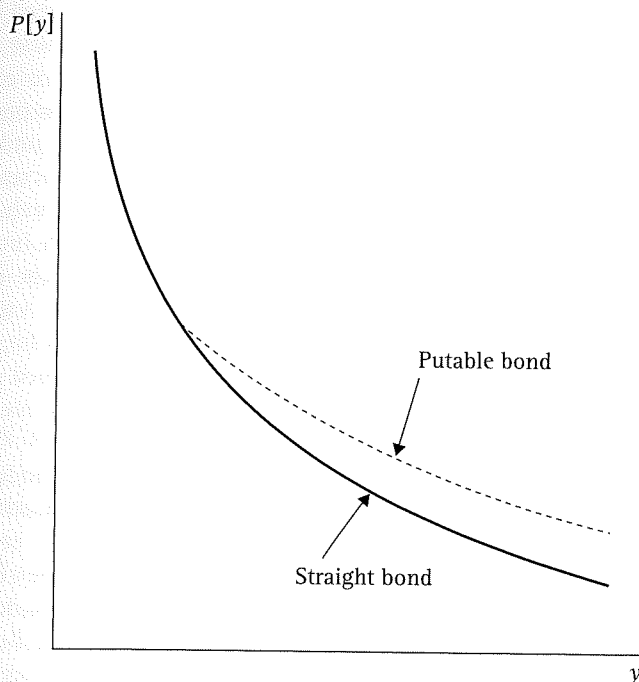


Figure 6.2 Price-yield function for a putable and a straight bond

zero notice period, this lower bound will be tighter for shorter put dates and higher levels of yield volatility.

As demonstrated by Chance (1990), Nawalka (1996) and Sarkar (1999), it is possible to extend the analysis to bonds subject to default risk and conversion features. Ignoring the topic of default risk for the moment, let the price of a convertible bond (P_{BW} , price of a bond with warrants) be equal to the straight bond price plus the value of the conversion feature that is set equal to W , the price of a warrant to purchase one share of stock, times N , the number of shares that can be obtained by fully converting the bond:

$$P_{BW} = P_B + NW \rightarrow D_{BW} = \frac{P_B}{P_{BW}} D_B - N \frac{1}{P_{BW}} \frac{dW}{dy}$$

This duration can be reexpressed as:

$$D_{BW} = D_B - \frac{NW}{P_{BW}} \left(D_B + \frac{1}{W} \frac{dW}{dy} \right)$$

When the rho of a warrant is, more or less, equal to the rho of an exchange traded call option, the Black-Scholes formula can be differentiated to show that $(dW/dr) > 0$, this implies that the duration of a convertible bond will be smaller than the duration of the straight bond component.

The formula for the duration of a convertible bond illustrates the need for a valuation model to make an adequate interpretation. For example, just because the rho of an exchange traded option calculated from the Black-Scholes formula is always positive, it does not necessarily follow that a warrant will have the same property in practice. The market value of a warrant depends on the market value of the underlying equity. Ignoring the potential dilution associated with warrant exercise, increases in interest rates can impact the value of the warrant by changing the market value of the assets and, hence, the value of the equity claim. Even though increases in interest rates will reduce the market value of the debt, it is theoretically possible for the equity claim to fall in value enough to cause $(dW/dr) < 0$, if the fall in the market value of assets is sufficiently greater than the reduction in the debt value. This could happen in various ways. One scenario is where a large fraction of the firm's debt is funded with short term rollover. If the leverage ratio for the firm is very high, an increase in interest rates could force the firm into default.

Sarkar (1999) illustrates the implications of using a valuation model to determine the duration of a convertible bond. Following Merton (1974), the model assumes that the market value of the firm (V) follows a lognormal diffusion (see section 6.4) with drift of μV and volatility σV . The convertible bond is assumed to be zero coupon and non-callable with par value of M . Firm value is the only source of randomness, i.e., the riskless interest rate is not modeled as being stochastic. The "conversion fraction" x is determined as $x = \#/(N + \#)$ where $\#$ is the number of shares outstanding prior to conversion. The only debt in the capital structure is this zero coupon convertible. Given this, the price of a zero coupon convertible bond can be derived as:

$$P_{BW} = VN[d_1] + xVN[d_2] + Me^{-rt^*}(N[d_4] - N[d_3])$$

where:

$$d_1 = \frac{\ln\left[\frac{M}{V}\right] - (r + .5\sigma^2)t^*}{\sigma\sqrt{t^*}} \quad d_2 = \frac{\ln\left[\frac{xV}{M}\right] + (r + .5\sigma^2)t^*}{\sigma\sqrt{t^*}}$$

$$d_3 = d_1 + \sigma\sqrt{t^*} \quad d_4 = -d_2 + \sigma\sqrt{t^*}$$

Using this valuation formula, the duration for this special case of a convertible bond can be determined by evaluating the derivative with respect to the riskless interest rate r .

Given this, the duration can be derived as (Sarkar 1999, p. 180):

$$D_{BW} = t^* \frac{Me^{-rt^*}(N[d_4] - N[d_3])}{P_{BW}} = t^* \frac{Me^{-rt^*}(N[d_4] - N[d_3])}{VN[d_1] + xVN[d_2] + Me^{-rt^*}(N[d_4] - N[d_3])}$$

Setting $x = 0$ will give the duration of a corporate zero coupon bond without the conversion provision. Chance (1990) provides a solution, without the conversion provision, that incorporates stochastic interest rates as well as randomness in firm value.

Observing that, with no default and no conversion, the expected value of the convertible bond price is $Me^{-rt^*}(N[d_4] - N[d_3]) > 0$, it follows that the duration for the convertible bond in this valuation model will be smaller than the duration for the straight bond component. Sankar (1999, pp. 182–3) provides a range of results from comparative static analysis for the convertible bond duration, e.g., convertible bond duration is decreasing in the volatility of firm value because this increases the probability of both default and conversion causing duration to fall. Inspection of the comparative static results reveals that the assumption of non-stochastic interest rates has substantive implications for the interpretation of convertible bond valuation and duration models (see section 6.4).

6.3 OPTION ADJUSTED SPREAD ANALYSIS*

*Static spread and cash flow assumptions**

Option adjusted spread (OAS) analysis is an extension of the static spread that incorporates randomness in the cash flows from the security. In turn, the *static spread* is an extension of traditional yield spread analysis that accounts for the timing of the cash flows of the two securities being compared, where accounting for the timing of the cash flows is affected by the term structure of interest rates. Fabozzi (1989, p. 340) illustrates traditional yield spread analysis by comparing two straight bonds, a Treasury and a corporate bond both with an 8.8% coupon and 25 years to maturity. The Treasury sells at \$96.613 to yield 9.15% and the corporate sells for \$87.08 to yield 10.24%. *Traditional yield spread* analysis would compare the value of these two bonds by observing the difference in the yields is 109 basis points. This yield spread would then be compared with the Treasury-corporate yield spread on other bonds of similar coupon and yield to maturity that are being assessed. After due consideration of the credit risk of the various bonds a decision would be made about the appropriate bond to purchase.

Traditional yield spread analysis has the attractive feature of simplicity. However, because promised yields to maturity are being compared, this approach fails to account for the shape of the term structure of interest rates. Consider valuing the corporate and Treasury bonds using implied zero interest rates. Because the implied zeros are derived from the Treasury yield curve, the price of the Treasury bond (P_B) would be evaluated as:

$$P_B = \sum_{t=1}^{2T} \frac{C/2}{\left(1 + \frac{z_t}{2}\right)^2} + \frac{M}{\left(1 + \frac{z_T}{2}\right)^{2T}} = \sum_{t=1}^{2T} \frac{C/2}{\left(1 + \frac{y}{2}\right)^2} + \frac{M}{\left(1 + \frac{y}{2}\right)^{2T}}$$

where coupons are paid semi-annually, T is the term to maturity in years and z_t and y are annualized. Comparing the yield to maturity calculation with the implied zero calculation it is apparent that if, say, the implied zero curve slopes up then the yield will “over discount” near term cash flows and “under discount” later cash flows.

In the same fashion that the yield to maturity "averages" over the yield curve by over and under discounting cash flows at specific points in time, traditional yield spread analysis will also be subject to "over and under discounting" specific cash flows. The static spread aims to address this problem by adding a fixed number of basis points to each of the implied zero interest rates to determine the price of the (semi-annual coupon) corporate bond (P_C):

$$P_C = \sum_{t=1}^{2T} \frac{C/2}{\left(1 + \frac{z_t}{2} + \frac{ss}{2}\right)^2} + \frac{M}{\left(1 + \frac{z_T}{2} + \frac{ss}{2}\right)^{2T}} = \sum_{t=1}^{2T} \frac{C/2}{\left(1 + \frac{y_C}{2}\right)^2} + \frac{M}{\left(1 + \frac{y_C}{2}\right)^{2T}}$$

where ss is the (annualized) static spread and y_C is the yield to maturity on the corporate bond. Using a spot rate curve that starts at 7% for the six month maturity and increases to 10.8% at 25 years, Fabozzi (1989, p. 345) demonstrates that evaluation of ss produces a value of 120 basis points, instead of the 109 basis point spread calculated using traditional yield curve analysis.

The static spread is a deterministic value that is used to construct the option adjusted spread (OAS) which is determined by doing a contingent claims valuation. Just like the static spread, the OAS gives a basis point value that captures the spread above Treasuries. However, due to the presence of randomness, there will not be one OAS but, rather, a range of OAS estimates that vary with the assumed volatility for the random variable(s). This is because the presence of an embedded option means that the future cash flows on the fixed income security are contingent on the future realization of some random variable. For example, the cash flows from a callable bond are contingent on the future realization of interest rates. If interest rates on the call date have fallen below the level at which it is cost effective to call the bond issue then there will be a cash flow at that point equal to the call price (plus accrued interest) and future cash flows from that point will be zero. However, if interest rates on the call date are not below the level at which it is cost effective to call the bond issue then there will continue to be a cash flow equal to the scheduled coupon payment.

In addition to the randomness in the future cash flows, the future behavior of interest rates also has to be modeled. Following the discussion in section 5.2, there are a number of different procedures that can be employed to generate future interest rates. The binomial process used in section 5.2 proceeded by generating paths for one period implied zeros. By working backwards through the binomial tree it was possible to determine the associated zero coupon bond prices for different maturities that could be used to solve for the implied forward rates. Monte Carlo methods provide an alternative procedure that is well adapted to solving for the OAS. In the Monte Carlo method, a large number of individual paths for the one period zeros are generated. Along each path, the one period zeros can be treated as implied forward rates that can be used to discount the random cash flows. The different values obtained can then be summed across each of the equally likely paths to obtain an estimate of the price of the contingent cash flows.

Generating the future spot rates and solving OAS*

Though it is possible to employ other methods, e.g., binomial trees as in Windas (1993), the Monte Carlo procedure for calculating the OAS is, perhaps, the easiest to illustrate. The sequence of steps involved in the Monte Carlo calculation of OAS is described by Fabozzi (1989, pp. 351-2):

- 1 Calculate the implied zero coupon interest rates from the observed Treasury yield curve. Use the implied zeros to calculate a full set of one period ahead implied forward rates ($f_{t,t+1}$). These values are used to calibrate the parameters used in the Monte Carlo process.
- 2 Identify the stochastic interest rate model that will be used in the Monte Carlo procedure to generate the one period Treasury spot rates. A range possible processes can be selected such as the self-calibrating one factor models of Black, Derman and Toy (1990) or Ho-Lee. The specification of the interest rate model will involve the selection of the volatility parameter.
- 3 Use the selected volatility and interest rate model to generate a sufficiently large number of interest rate paths. Each of the paths will be composed of a sequence of one period zeros that apply to future time periods. Treating the one period zeros as forward rates, construct the spot rates along each path.
- 4 Specify the decision rules used to determine the random cash flows. For example, a callable bond requires a rule for determining at what interest rate level the bond will be called.
- 5 Path by path, examine the relevant interest rate and determine when or whether the bond will be called along that path. Generate the contingent cash flow along that path.
- 6 Using the spot rates for each path, determine the present value of the cash flow along each path for a given level of the static spread.
- 7 For each static spread value, sum the present values across all the generated paths and divide by the number of paths. Repeat the process for a range of static spread values.
- 8 Compare the present values that were determined by averaging across the paths with the observed market price of the bond with the embedded option. When the observed bond price is equal to the value calculated from the Monte Carlo process, the associated value of the "averaged-across-paths" static spread is the option adjusted spread (OAS).
- 9 By repeating steps 3-8, recalculate the OAS for a different volatility assumption.

This 9 step process can be clarified by dissecting the simple example given in Fabozzi (1989, pp. 346-9).¹¹

For simplicity, assume that the Monte Carlo simulation only requires ten interest rate paths to be generated. In practice, 1,000 or more paths would be used but this number is impractical for pedagogical purposes. Fabozzi is concerned with valuing an 8.8% semi-annual coupon, 10 year corporate bond that is currently callable with a call price of \$103. The observed price of the corporate bond in the market is, say, \$103.30. Observe that in step (2), the generated interest rate paths are for Treasury zeros. As there will be a credit spread between the Treasury curve and the corporate curve for bonds

Table 6.6 Monte Carlo simulation of ten binomial paths for 6 month one period Treasury zeros; initial zero rate of 7% with 10% binomial rate changes^a

Semi-annual period	Short-term one period zero paths									
	1	2	3	4	5	6	7	8	9	10
1	7.0%	7.0%	7.0%	7.0%	7.0%	7.0%	7.0%	7.0%	7.0%	7.0%
2	7.7	7.7	7.7	6.3	6.3	6.3	7.7	6.3	7.7	7.7
3	8.5	6.9	6.9	6.9	5.7	5.7	8.5	5.7	8.5	8.5
4	7.6	6.2	6.2	7.6	5.1	6.2	9.3	6.2	9.3	9.3
5	6.9	6.9	5.6	8.4	4.6	6.9	8.4	6.9	8.4	8.4
6	7.5	7.5	5.1	9.2	4.1	6.2	7.5	7.5	7.5	7.5
7	8.3	8.3	4.5	10.1	4.5	5.6	6.8	8.3	6.8	6.8
8	9.1	9.1	5.0	11.2	5.0	5.0	6.1	9.1	6.1	6.1
9	10.0	10.0	5.5	12.3	5.5	4.5	5.5	8.2	6.7	6.7
10	9.0	9.0	6.1	13.5	6.1	4.1	5.0	7.4	6.1	7.4
11	9.9	8.1	6.7	12.2	6.7	4.5	4.5	8.1	5.4	6.7
12	8.9	8.9	7.3	10.9	7.3	4.0	4.0	8.9	4.9	6.0
13	8.1	9.8	8.1	9.8	8.1	4.4	3.6	9.8	4.4	5.4
14	8.9	8.9	7.2	8.9	7.2	4.0	3.2	10.8	4.0	4.9
15	9.7	8.0	6.5	8.0	6.5	4.4	2.9	9.7	4.4	5.3
16	8.8	7.2	7.2	8.8	7.2	3.9	3.2	10.7	4.8	4.8
17	7.9	6.5	7.9	9.6	7.9	3.5	3.5	9.6	5.3	4.3
18	7.1	7.1	8.7	10.6	8.7	3.2	3.9	10.6	5.8	4.8
19	7.8	7.8	7.8	11.7	9.6	2.9	4.3	11.7	6.4	5.2
20	8.6	7.0	8.6	10.5	8.6	3.2	4.7	12.8	7.0	4.7

^a Each interest rate in a given path is the (annualized) 6 month zero coupon interest rate simulated for that path. Each value is calculated by randomly multiplying the interest rate for the previous period by either $(1.1 = 1 + 0.1)$ or $(0.9 = 1 - 0.1)$. Each path provides a different random result for the 6 month interest rate process.

with a given credit rating, this has to be taken into account in the valuation. Though it is possible to incorporate credit spreads into the binomial interest rate process, e.g., Finnerty (1999), the OAS will typically include both the credit risk and callable option values. The forward rate paths given in table 6.6 are determined in an elementary fashion by using a binomial process where up and down moves are equally likely and the size of the move is plus or minus 10% of the previous period yield. Given the initial starting value of 7%, path one proceeds with an up move to 7% $(1.1) = 7.7\%$ in period two followed by another up move to $(7.7\%) (1.1) = 8.47\%$ (which is rounded to 8.5%) and a down move to $8.5\% (.9) = 7.65\%$ (rounded down to 7.6%).

The interest rate process used in table 6.6 suffers from a number of limitations. In particular, the process does not recombine, the step increments are relatively arbitrary and the process is not constructed to satisfy absence of arbitrage. In addition, the generating process does not capture the empirical property of mean reversion or allow the credit spread to change with interest rate level. As depicted in step (1), all this is assumed not to matter substantively if the generated paths are able to recreate the

Table 6.7 Calculated Treasury spot rates for the ten paths in table 6.6^a

Semi-annual period	Paths 1	2	3	4	5	6	7	8	9	10
1	7.00%	7.00%	7.00%	7.00%	7.00%	7.00%	7.00%	7.00%	7.00%	7.00%
2	7.35	7.35	7.35	6.65	6.65	6.65	7.35	6.65	7.35	7.35
3	7.72	7.21	7.21	6.74	6.32	6.32	7.72	6.32	7.72	7.72
4	7.70	6.97	6.97	6.96	6.02	6.30	8.12	6.30	8.12	8.12
5	7.53	6.95	6.69	7.25	5.73	6.41	8.17	6.41	8.17	8.17
6	7.53	7.05	6.42	7.57	5.46	6.37	8.07	6.60	8.07	8.07
7	7.64	7.22	6.15	7.94	5.33	6.26	7.89	6.84	7.89	7.89
8	7.83	7.46	6.01	8.34	5.29	6.10	7.66	7.13	7.66	7.66
9	8.07	7.75	5.95	8.77	5.31	5.92	7.42	7.25	7.56	7.56
10	8.17	7.88	5.96	9.24	5.39	5.73	7.17	7.26	7.41	7.54
11	8.33	7.90	6.02	9.51	5.50	5.62	6.93	7.34	7.23	7.46
12	8.38	7.99	6.13	9.62	5.65	5.48	6.68	7.48	7.03	7.34
13	8.36	8.13	6.28	9.64	5.84	5.40	6.44	7.66	6.83	7.19
14	8.39	8.18	6.35	9.59	5.94	5.30	6.21	7.88	6.63	7.02
15	8.48	8.17	6.36	9.48	5.98	5.24	5.99	8.01	6.47	6.91
16	8.50	8.11	6.41	9.43	6.05	5.15	5.82	8.17	6.37	6.78
17	8.46	8.01	6.50	9.45	6.16	5.06	5.68	8.26	6.31	6.63
18	8.39	7.96	6.62	9.51	6.30	4.95	5.58	8.39	6.28	6.53
19	8.36	7.95	6.68	9.62	6.47	4.84	5.51	8.56	6.28	6.46
20	8.37	7.90	6.78	9.67	6.58	4.76	5.47	8.77	6.32	6.37

^a Annualized semi-annual zero coupon interest rates calculated from the one period zeros in table 6.6 by assuming the local expectations hypothesis and observing that the hypothesis implies: $z_{t,t+1} = f_{t,t+1}$.

observed Treasury zero curve (even though such simple processes typically will not be able to accomplish such recreation). In the simple example in table 6.6, "calibration" to the observed Treasury zero curve could be accomplished by changing the up/down probabilities by deleting certain paths and adding others. For example, in table 6.6 only four of the ten paths start with a down move. This is followed by five up and five down moves and then another four down/six up. Unequal numbers of up moves relative to down moves result in a generating process with different up/down probabilities. This simplistic calibration procedure captures an element of arbitrage free processes that work by fitting the process by estimating the arbitrage free probabilities.

Under the (local) unbiased expectations hypothesis the one period ahead zero rates in table 6.6 can be treated as implied forward rates. This result permits the associated Treasury spot rate curve along each path to be calculated. Table 6.7 presents the calculated values for the Treasury spot rate curves. For example, along path one the period two value spot rate is determined as:

$$\left(1 + \frac{z_2}{2}\right)^2 = \left(1 + \frac{z_1}{2}\right)\left(1 + \frac{f_{1,2}}{2}\right) = \left(1 + \frac{.07}{2}\right)\left(1 + \frac{.077}{2}\right) = 1.03675 \rightarrow z_2 = 7.35\%$$

Averaging across the ten equally likely paths gives a two period spot rate (z_2) of 7.07%. Similar calculations give a three period spot rate (z_3) of 7.1% (see end of chapter questions). Given these spot rates, the OAS can be determined by adding a fixed static spread to the spot rates along each path to determine a value for the bond along that path, as in step (6). However, this requires the random cash flows along each path to be determined, as in steps (4) and (5).

An important limitation of Monte Carlo methods is the difficulty these methods have in valuing American style options. This happens because the decision rule about when to exercise along a given path is complicated to incorporate into the path-by-path simulation process. Though methods of getting around such difficulties have been available at least since Tilley (1993), the rule for exercising the callable bond proposed in the example of Fabozzi (1989, p. 346) is relatively naive:

The refinancing opportunity for the corporation will be based not on the short-term forward rate but on a longer-term rate that reflects how much the issuer would have to pay to refund a bond issue. Let's make the simple assumption that the refinancing rate for the corporation is 100 basis points higher than the short-term forward rate . . . Suppose that the following rule is established for calling of the bond. If the refinancing rate is below 5.8% with at least three years remaining to maturity, then the bond will be called.

To arrive at table 6.8a, Fabozzi adds 100 basis points to the values in table 6.6. Each path is then examined to determine if a value below 5.8% is obtained. If so, then the bond is called. This procedure is used to generate the cash flows in table 6.8b. For example, there are no values in paths one and two that are below 5.8%. On path 3, a value of 5.5% is obtained in period 7. Given a call price of \$103, plus the 4.4% semi-annual coupon, the payout in period 7 on path 3 is \$107.40. As the bond is then retired, all future cash flows on that path are zero.

While useful as a general illustration of the Monte Carlo method, there are some apparent difficulties with this decision rule. For example, the refinancing rate selected is determined by adding 100 basis points to the one period Treasury spot rates. This assumes a constant credit spread at the short end of the yield curve. More importantly, the decision rule assumes that the firm will look at one period interest rates instead of the potential financing rate on any new debt that may be issued to refund the issue that is being called. Even if a refunding is not used, e.g., the issue is redeemed out of cash on hand, the operative refinancing decision would probably not be done by comparing the short-term rate with a target short-term rate. Other problems that arise include the use of a 300 basis point spread over the current coupon to determine redemption instead of comparing the call price with the market price of the bond. By comparing prices, this would permit different call prices to be used, consistent with the market practice of using call prices that decline over time. In addition, a 300 basis point buffer seems to be a relatively large value to use.

The final step in the OAS calculation involves the calculation of a fixed static spread along each path to determine a bond value, as in table 6.9. The bond values along each path are then averaged across all the paths and an average price determined associated with the assumed level of the static spread. This average value is then compared with

Tables 6.8a and 6.8b Refinancing rate paths and callable bond cash flow for each refinancing path^a

6.8a										
Semi-annual period	Refinancing rate paths									
	1	2	3	4	5	6	7	8	9	10
1	8.0%	8.0%	8.0%	8.0%	8.0%	8.0%	8.0%	8.0%	8.0%	8.0%
2	8.7	8.7	8.7	7.3	7.3	7.3	8.7	7.3	8.7	8.7
3	9.5	7.9	7.9	7.9	6.7	6.7	9.5	6.7	9.5	9.5
4	8.6	7.2	7.2	8.6	6.1	7.2	10.3	7.2	10.3	10.3
5	7.9	7.9	6.6	9.4	5.6	7.9	9.4	7.9	9.4	9.4
6	8.5	8.5	6.1	10.2	5.1	7.2	8.5	8.5	8.5	8.5
7	9.3	9.3	5.5	11.1	5.5	6.6	7.8	9.3	7.8	7.8
8	10.1	10.1	6.0	12.2	6.0	6.0	7.1	10.1	7.1	7.1
9	11.0	11.0	6.5	13.3	6.5	5.5	6.5	9.2	7.7	7.7
10	10.0	10.0	7.1	14.5	7.1	5.1	6.0	8.4	7.1	8.4
11	10.9	9.1	7.7	13.2	7.7	5.5	5.5	9.1	6.4	7.7
12	9.9	9.9	8.3	11.9	8.3	5.0	5.0	9.9	5.9	7.0
13	9.1	10.8	9.1	10.8	9.1	5.4	4.6	10.8	5.4	6.4
14	9.9	9.9	8.2	9.9	8.2	5.0	4.2	11.8	5.0	5.9
15	10.7	9.0	7.5	9.0	7.5	5.4	3.9	10.7	5.4	6.3
16	9.8	8.2	8.2	9.8	8.2	4.9	4.2	11.7	5.8	5.8
17	8.9	7.5	8.9	10.6	8.9	4.5	4.5	10.6	6.3	5.3
18	8.1	8.1	9.7	11.6	9.7	4.2	4.9	11.6	6.8	5.8
19	8.8	8.8	8.8	12.7	10.6	3.9	5.3	12.7	7.4	6.2
20	9.6	8.0	9.6	11.5	9.6	4.2	5.7	13.8	8.0	5.7

6.8b										
Cash flow paths										
Period	1	2	3	4	5	6	7	8	9	10
1	4.4	4.4	4.4	4.4	4.4	4.4	4.4	4.4	4.4	4.4
2	4.4	4.4	4.4	4.4	4.4	4.4	4.4	4.4	4.4	4.4
3	4.4	4.4	4.4	4.4	4.4	4.4	4.4	4.4	4.4	4.4
4	4.4	4.4	4.4	4.4	4.4	4.4	4.4	4.4	4.4	4.4
5	4.4	4.4	4.4	4.4	107.4	4.4	4.4	4.4	4.4	4.4
6	4.4	4.4	4.4	4.4	0.0	4.4	4.4	4.4	4.4	4.4
7	4.4	4.4	107.4	4.4	0.0	4.4	4.4	4.4	4.4	4.4
8	4.4	4.4	0.0	4.4	0.0	4.4	4.4	4.4	4.4	4.4
9	4.4	4.4	0.0	4.4	0.0	107.4	4.4	4.4	4.4	4.4
10	4.4	4.4	0.0	4.4	0.0	0.0	4.4	4.4	4.4	4.4
11	4.4	4.4	0.0	4.4	0.0	0.0	107.4	4.4	4.4	4.4
12	4.4	4.4	0.0	4.4	0.0	0.0	0.0	4.4	107.4	4.4
13	4.4	4.4	0.0	4.4	0.0	0.0	0.0	4.4	0.0	4.4
14	4.4	4.4	0.0	4.4	0.0	0.0	0.0	4.4	0.0	4.4
15	4.4	4.4	0.0	4.4	0.0	0.0	0.0	4.4	0.0	4.4
16	4.4	4.4	0.0	4.4	0.0	0.0	0.0	4.4	0.0	4.4
17	4.4	4.4	0.0	4.4	0.0	0.0	0.0	4.4	0.0	4.4
18	4.4	4.4	0.0	4.4	0.0	0.0	0.0	4.4	0.0	4.4
19	4.4	4.4	0.0	4.4	0.0	0.0	0.0	4.4	0.0	4.4
20	104.4	104.4	0.0	104.4	0.0	0.0	0.0	104.4	0.0	104.4

^a Refinancing rate paths are determined by adding 100 basis points to the one period Treasury zero paths in table 6.6. Cash flow for each refinancing path is for an 8.8% semi-annual coupon 10 year corporate bond callable at 103. The call rule used is that if the refinancing rate falls to 5.8% or less and the bond has at least three years to maturity then the bond is called on the coupon payment date at 103 to produce a payment of $103 + 4.4 = 107.4$.

Table 6.9 Present value of each path and average present value using various spreads

Trial OAS	Present value of path										Average PV
	1	2	3	4	5	6	7	8	9	10	
40 bp	100.6	103.5	107.5	93.73	108.5	111.5	107.9	100.4	107.7	112.4	105.4
50 bp	100.0	102.9	107.3	93.14	108.3	111.1	107.4	99.77	107.2	111.6	104.9
60 bp	99.35	102.2	107.0	92.55	108.0	110.7	106.9	99.13	106.7	110.8	104.3
70 bp	98.70	101.5	106.7	91.97	107.8	110.3	106.5	98.49	106.2	110.1	103.8
80 bp	98.06	100.8	106.4	91.40	107.5	109.9	106.0	97.86	105.7	109.3	103.3
90 bp	97.43	100.2	106.1	90.83	107.3	109.5	105.5	97.24	105.3	108.6	102.8
100 bp	96.80	99.58	105.9	90.26	107.1	109.1	105.1	96.62	104.8	107.9	102.3
110 bp	96.18	98.93	105.6	89.70	106.8	108.7	104.6	96.00	104.3	107.1	101.8
120 bp	95.56	98.29	105.3	89.14	106.6	108.3	104.2	95.39	103.8	106.4	101.3

the observed market price of the bond to identify the static spread value that was able to produce the market price. This averaged-across-paths static spread value is the option adjusted spread (OAS). Recalling that the market price was assumed to be \$103.30, it follows that the OAS for this bond is 80 basis points. In effect, the OAS is the static spread that, when averaged across the interest rate paths, reproduces the observed market price. As such, the OAS will depend on the volatility assumption that was used to generate the one period spot interest rates. Because a different level of volatility will generate a different OAS, it is conventional to report a range of OAS for different volatility assumptions, which is indicated in step (9). Given this discussion, it is not surprising that considerable effort has been dedicated to identifying the pitfalls in the OAS technique.

*The OAS: examples and pitfalls**

Table 6.10 presents a number of different examples of OAS calculations for bonds with different features. In assessing the information in table 6.10, it is important to remember that an OAS calculation is not like a yield to maturity calculation. Different OAS values can be calculated for the same bond, depending on the procedure employed to estimate the OAS. For example, as illustrated by Kupiec and Kah (1999), significantly different OAS estimates are calculated for the same security by MBS dealers on a regular basis, e.g., on a 7.5% MBS participation certificate the spread between the Lehman Bros and Bear Stearns OAS estimates was over 60 basis points with one estimate being almost double the other. Table 6.10 illustrates the impact of changing volatility on OAS values: as the predicted volatility of interest increases, the OAS will fall. This happens because the value of the option is subtracted from the straight bond price to determine the callable bond price or MBS price. Increasing predicted volatility raises the cost of the embedded option reducing the spread above Treasuries. Put differently, increasing predicted volatility will increase the probability the option will be exercised, lowering the "option adjusted spread" above Treasuries.

Table 6.10 Option adjusted spreads for various types of bonds with embedded options^a

<i>Callable Bonds (9/88)</i>									
Issuer	S&P rating	Maturity	Next call date	Next call price	Current price	Coupon (%)	Yield	Treasury yield spread	OAS at volatility of:
									10% 15% 20%
ITT Fin.	A	07/01/92	07/01/89	100.00	101.58	10.800	10.27	168	57 30 -4
Marriot	A-	02/01/96	02/01/93	100.00	99.59	9.625	9.70	85	65 50 37
GMAC	AA-	07/15/07	now	104.00	84.19	8.000	9.86	82	71 54 41
<i>Mortgage backed pass through (PT) vs. fixed coupon agency (6/88)</i>									
Security	Price	Term to maturity	Average life	Yield	Treasury spread	OAS at volatility of:			
						10% 15% 20%			
FNMA 9% PT	95.3125	28.05	9.80	9.98	115	104 84 64			
FNMA 9.35%	101.1875	7.7	7.70	9.13	38	38 38 38			
<i>GNMA 9 and collateralized mortgage obligations (5/88)</i>									
Class	Par amount	Price ^a	Coupon	Average life	Yield	Treasury spread	OAS at volatility of:		
							10% 15% 20%		
GNMA 9	100	92.656	9.0	11	10.48	123	104 88 71		
A	35.1	98.378	8.0	2	9.10	100	67 48 21		
B	19.0	95.508	8.5	5	9.85	115	83 59 25		
C	18.0	94.748	9.0	7	10.27	125	103 80 52		
Z	27.9	80.999	9.0	17	10.87	150	102 79 52		

^a Callable bond values are based on closing prices and Treasury rates on September 20, 1988. The MBS values are based closing prices on June 29, 1988. All prices shown in decimal form. The GNMA 9 is assumed to have a remaining term to maturity of 28 years; the CMO is priced at 100% PSA. This table is adapted from Hayre and Lauterbach (1991).

For the simplifying case of an annual cash coupon bond, the OAS calculation can be expressed compactly as, e.g., Kupiec and Kah (1999), Babbel and Zenios (1992):

$$P_{OB} = \frac{1}{S} \sum_{s=1}^S \sum_{t=1}^T \frac{CF_t^s}{\prod_{i=1}^t (1 + z_{i-1,i}^s + OAS)}$$

where P_{OB} is the market price of the bond with the embedded option, S is the number of possible equally likely future states, T is the term to maturity, OAS is the option adjusted spread, CF_t^s is the cash flow at time t ($= 1$ to T) in state s ($= 1$ to S), and $z_{i-1,i}^s$ is the state s one period spot interest rate starting at $i - 1$ and ending at i for discounting cash flows occurring at $t = i$ with $z_{0,1}^s = z_1$ the one period zero rate observed on the pricing date $t = 0$. Under the local expectations hypothesis, the one period zeros will be equal to the one period ahead forward rates. The product operator in the numerator is used for analytical purposes, e.g., because the cash flows at time t in different states will not necessarily be the same; as with the static spread, the OAS is added to each of the one period zeros.¹²

The widespread use of OAS in assessing value for bonds with embedded options has led to a number of contributions aimed at identifying pitfalls in the implementation and interpretation of OAS , e.g., Babbel and Zenios (1992), Kopprasch (1994), Kupiec and Kah (1999). Some of these pitfalls have already been illustrated in detailing the Monte Carlo example of Fabozzi (1989). In particular, it was observed that the OAS calculation for a callable bond depended on the rule that was adopted to model the issuer's call decision. A similar observation applies to other types of bonds with embedded options, e.g., MBS where a prepayment model has to be specified. Kupiec and Kah (1999, p. 91) take an extreme position on this pitfall: "That there is an OAS in mortgage-backed securities at all is a consequence of misspecification of the prepayment function . . . In general it is not possible to modify the prepayment functional form or parameter values to fully compensate for the omission of factors that influence prepayment behavior." Another pitfall occurs with the use of one number, the OAS , to account for more than one option, e.g., default risk and the callability option for a callable corporate bond.

In addition to these concerns, another pitfall is that the OAS depends on the stochastic process selected to model the interest rate. Since the emergence of OAS , considerable effort has been dedicated to addressing this pitfall, such as the development of the self-calibrating processes proposed by Ho-Lee and Black, Derman and Toy. Unfortunately, though such processes do address certain fundamental problems, other problems such as the selection of a volatility assumption still remain. In addition to the size of volatility, the behavior of volatility over time needs to be addressed, e.g., is it appropriate to hold volatility constant over time? Security rankings based on OAS can change depending on assumptions made about the temporal behavior of volatility. This problem is exacerbated by another pitfall: the assumption of a constant OAS across time. As an option approaches its expiration date, the impact on the option value is non-linear. Yet the OAS calculation assumes that the OAS in the last period is the same size as in the first period.

The list of potential pitfalls in OAS modeling and interpretation goes on. Yet, despite the potential pitfalls, OAS is still a commonly used technique that permits practitioners to extend yield spread valuation techniques to bonds with embedded options. Practitioners have to make comparative valuations on a regular basis and OAS does have attractive features for this purpose. The ability to handle the impact of interest rate volatility and other potential sources of randomness on cash flow patterns requires some valuation methodology that can handle these contingencies and OAS is, in many cases, the best alternative available. Though there are pitfalls and limitations in OAS, such problems are not avoided by using other contingent claim valuation methodologies, e.g., Bookstaber (1991). For example, using much simpler valuation techniques, Longstaff (1992) values the embedded option in callable US Treasury securities and find estimates of negative prices for the embedded option. Such results raise more questions about the valuation methodology than the apparent conclusion that a basic property of option valuation (non-negative prices) has been violated.

In the end, OAS is a model dependent calculation. Alterations in the various modeling assumptions can, and likely will, impact the OAS that is estimated. Despite the analytical complexity, the problem of *ex post* estimates being used to make *ex ante* decisions cannot be avoided, e.g., in determining the appropriate level of predicted interest rate volatility to use. As illustrated in the Fabozzi (1989) example, much can be said about the value inherent in the process of arriving at the estimate, rather than the actual estimate that is obtained. The process requires various assumptions to be made about: the decision rules associated with exercising the option; the interest rate process used to generate the one period zeros; the presence of other options; the impact of other random variables; and, the impact of randomness on the future cash flows. Going through this process aids considerably in developing the intuition needed to arrive at a practical assessment of the relative value. The process is more relevant to the security analysis than the actual OAS number that is determined.

6.4 MODELING AND ANALYZING DEFAULT RISK[#]

Early approaches

In addition to developing the notion of duration, Macaulay (1938, pp. 60–2) also provides insight into the modeling of default risk. The discussion begins by observing the failures of a possible method of handling default probabilities:

The simplest of all probability hypotheses is that the probability of each and every promised future payment is always the same, 9/10, for example. With this assumption, the price of the bond would, on any specified date, be 90 per cent of the price on the same date of a bond containing an identical set of promises but rated by the market as “absolutely secure.” However, unless the bonds were perpetuities, the yield of the lower grade bond would fluctuate less than the yield of the “absolutely secure” bond.

Macaulay recognized that using bond price as a method of assessing credit risk is inadequate because taking the price of a lower grade bond as some percentage of the

price of a higher grade bond reduces to assuming that the probabilities of all future payments are equal, a severe assumption that assumes the independence of the payment probabilities.

Macaulay proceeds to develop the implications of using the "much more appealing" approach of comparing yields to measure default risk. Explicitly recognizing that using yields can also lead to "strange conclusions." Macaulay (1938, p. 61) considers the problem of determining the default probabilities:

The simplest assumption having any appreciable air of reality is that no payment will be met unless all preceding payments have first been met, but that, as soon as one payment has been met, the probability that the next payment will be met is the same as previously had been the probability that the preceding payment would be met. Under this compound assumption, if the probability that the first payment will be met is designated as p , the probability (as of the same date) that the second payment will be met will be p^2 , and the probability that the n th payment will be met will be p^n .

Letting P_L be the price of a low grade bond, y the yield on an "absolutely secure" bond and, using Macaulay's notation, p the compounding payment probability, Macaulay proceeds to demonstrate that this leads to the result:

$$P_L = \sum_{t=1}^T \frac{p^t C}{(1+y)^t} = \sum_{t=1}^T \frac{C}{\left(\frac{1+y}{p}\right)^t}$$

Hence, in this case yield on the low grade bond (y_L) can be determined as $(1+y_L) = (1+y)/p$. This leads to the assessment of the default probability as $p = (1+y)/(1+y_L)$.¹³

While recognizing the importance of analyzing default risk in terms of yields and not prices, Macaulay is able to identify a number of problems with this compound probability approach. Being primarily concerned with preparing an indepth empirical analysis of interest rate and bond yield movements, Macaulay was concerned that the compound probability model did not accurately describe the empirical behavior of bond yields, i.e., the model predicts that yields for low grade bonds fluctuate less in percentage terms than high grade yields which is contrary to what Macaulay observed empirically. To see this let $y = 2\%$ and $p = .99$, it follows that $y_L = 3.03\%$. If the absolutely secure yield doubles to 4% then the low grade yield will increase to 5.05% . Macaulay observes that, even though it is possible to specify probabilities that would produce a larger percentage change in the low grade yield, "any such hypothesis would necessarily be a mere mathematical curiosity" (p. 62).

Macaulay demonstrates a subtle understanding of the empirical relationship between low grade and high grade bonds:

startling as it may sound, we do not *know* that lower grade bonds actually do fluctuate in yield appreciably more than do higher grade bonds. Indeed, there are strong reasons for suspecting that the excessive fluctuations in the yields of so-called lower grade bonds is primarily an indication of fluctuation in the *grades* assigned by the market to such bonds

rather than an indication that violent fluctuations in yield normally accompany low, but unchanging, market grading.

Even though Macaulay was writing at a time when ratings agencies were considerably less sophisticated and information about bonds was substantially less, Macaulay (p. 65) makes a telling observation about bond grades: "The average and the whole distribution of grades undoubtedly drift in great secular swings, immense waves, and even up and down with the movements of the business cycle." Changes in the yield spreads between different bond grades reflect these swings, waves and cycles.

Macaulay identifies numerous factors that influence the yield on bonds: security, duration, taxation, marketability, embedded options such as convertibility and callability, legal restrictions (e.g., eligible for "trustee" investment), and currency of denomination. Macaulay also makes the following interesting observation:

If the earnings of a corporation cover the interest charges on a bond thirty times, the market takes little or no notice of a change in conditions such that the charges are covered only twenty times. On the other hand, a change from one and a half times to once only will probably be considered extremely serious. The yield may rise violently. The lower the grade of a bond the more it tends to act like a common stock. It comes to be significantly affected not only by interest rate factors but also by potential earnings.

Yet, despite insightful comments, Macaulay (1938) is ultimately a study about the economics of the empirical behavior of bond yields and not about the methods to be used in picking individual bonds. It is a study about averages and tendencies.

In contrast, Graham, Dodd and Cottle (1962) (GDC) is directly concerned with picking individual bonds. Where Macaulay is concerned with average yields between bond grades, GDC is concerned with the determinants of specific bond yields. The focus is on the properties of the individual security and not the average. GDC (p. 310) structures the security analysis for corporate bonds around "Four Principles for the Selection of Issues of the Fixed-income Type":

- I. Safety is measured not by specific lien or other contractual rights but by the ability of the issuer to meet all its obligations.
- II. This ability should be measured under conditions of recession and depression rather than of prosperity.
- III. Deficient safety cannot be compensated for by an abnormally high coupon rate alone.
- IV. The selection of all senior securities for investment should be subject to rules of exclusion and to specific quantitative tests.

GDC make it explicit that analysis of a corporate bond needs to focus on the characteristics and strengths of the business issuing the debt and not on the property that is securing the issue. The basic intuition of this approach is illustrated in section 4.4 where empirical evidence was presented that the recovery rate on secured debt is significantly less than 100%.

Based on this general framework GDC make a number of recommendations about corporate bond selection. For example (p. 313): "The theoretically correct procedure for

bond investment, therefore, is first to select a company meeting every test of strength and soundness, and then to purchase its highest-yielding obligation, which would usually mean its junior rather than its first-lien bonds." This recommendation is tempered with the observation that the junior issue must provide a sufficiently higher yield, which is usually the case. In general, selecting first-mortgage issues reflects a lack of confidence in the analysis of the business which begs the question: is it advisable to buy any bonds of a company that requires a first-lien to protect against adverse consequences associated with the business? GDC quotes Walter Bagehot with admiration: "If there is any difficulty or doubt the security should be declined." Unlike equity selection where the costs of rejecting a specific security can be considerable, bond selection is primarily a "*negative art*" where the avoidance of capital loss is of central concern.

Perhaps the most telling statement in GDC (p. 320) about bond selection is the following:

In the traditional theory of bond investment a mathematical relationship is supposed to exist between the interest rate and the degree of risk incurred. The interest return is divided into two components, the first constituting "pure interest" – i.e., the rate obtainable with *no* risk of loss – and the second representing the premium obtained to compensate for the risk assumed. If, for example, the "pure interest rate" is assumed to be 3 percent, then a 4 percent investment is supposed to involve one chance in a hundred of losing the entire principal in one year.

GDC strongly question this view of bond interest rates. In contrast to the numerous theories that have been discussed in part II, GDC (p. 320) maintain there is no mathematically identifiable relationship between yield and risk:

This view . . . seems to us to bear little relation to the realities of bond investment. Relative bond yields usually do reflect the rating of the market as to the relative degrees of risk involved in the investments, but these yields have not been computed by means of mathematical equations involving chance of loss on the one hand and required income on the other.

The actuarial computation of investment risks is "not a practicable undertaking." Despite the substantial amount of information about the history of bond performance, the nature of the "risks" contains enough "uncertainty" that mathematical formulations are impractical.

*The continuous time framework**

The scepticism of Macaulay and, particularly, GDC, about the modeling of default risk stands in stark contrast to the approach of modern Finance to this subject. The techniques of contingent claims pricing have overtaken the subject. The initial connection between default risk and option pricing theory can be found in Merton (1974). Since this time, the continuous time approach to modeling default risk has been developed and extended in a number of ways, e.g., Jones et al. (1984), Ritchken (1987, ch. 14),

Kim et al. (1993) and Culp (2001). Though simplified, the Merton (1974) approach provides a foundation that requires description. The basic idea follows by extending the conventional embedded option relationship to the case where the default decision is viewed as an option for the firm to "put" the assets of the firm back to the bondholders. In other words: market value of a risky loan = market value of a default and option free loan - put value associated with default.

In this formulation the exercise price on the put option is the promised value of the outstanding fixed income securities. More precisely, assume that the firm has a relatively simple capital structure composed of common stock and continuously compounded zero coupon bonds maturing to a par value of M at time T . Assuming perfect capital markets the conventional market valuation equation for the firm applies: $V(t) = S(t) + B(t)$, where S is the market value of equity (number of shares times price per share), B is the market value of the debt and V is the market value of the firm. Observing that the zero coupon bonds permit the risk of default to be shifted to T , then at $t = T$ the holders of debt will receive: $\min[M, V(T)] = M - \max[0, M - V(T)] = B(T)$, where $\min$ ($\max$) refers to the functions that select the minimum (maximum) of the values within the brackets. Observe that the value of a put option on the firm with an exercise price equal to M is $PM(T) = \max[0, M - V(T)]$. Discounting the value back to time t at the riskless rate r and taking (risk neutral) expectations gives: $B(t) = M \exp[-rt^*] - PM(t)$. Hence, the market value of the corporate debt equals the difference between the price of a riskless default free bond and a short position in the put option.

It is possible to approach this specification from another direction. More precisely, it is also possible to have: $\min[M, V(T)] = V(T) - \max[0, V(T) - M] = B(T)$. Again, discounting back at the riskless rate and taking expectations gives: $B(t) = V(t) - C_V(t)$ where C_V is the value of call option on the firm with exercise price equal to M . The interpretation of this relationship is that in issuing debt the common stockholders have, effectively, sold the firm to the bondholders in exchange for a call option that permits the stockholders to repurchase the firm at maturity in exchange for payment of the face value of the debt, i.e., $V(t) - B(t) = C_V(t) = S(t)$. In the absence of any coupon payments, all the default risk occurs at maturity of the debt and the option can be modeled as a European option. Assuming that the market value of the firm follows a geometric Brownian motion with volatility parameter σ_V , a Black Scholes equation for the call can be derived as:

$$C_V(t) = V(t)N[d_1] - e^{-rt^*}MN[d_2]$$

where:

$$d_1 = \frac{\ln \left[\frac{V(t)}{M} \right] + (r + .5\sigma_V^2)t^*}{\sigma_V t^*} \quad d_2 = d_1 - \sigma_V t^*$$

It is difficult not to see this interpretation of the market value of the firm's equity as a remarkable insight.

Developing the alternative put option approach involves viewing the shareholders as having purchased the market value of the firm, plus a put option, offsetting the associated costs by borrowing funds with time T maturity value of M , i.e., $B(t) = V(t) - S(t) = M \exp[-rt^*] - PM(t)$ or $S(t) = V(t) + PM(t) - M \exp[-rt^*]$. Observing that this is a put on the market value of the firm, using the same assumptions as for the call the value of the put can be specified:

$$PM(t) = e^{-rt^*} MN[-d_2] - V(t)N[-d_1]$$

This can now be used for a value of $B(t)$ as:

$$B(t) = Me^{-rt^*} - PM(t) = Me^{-rt^*} \left[1 - N[-d_2] + \frac{V(t)N[d_1]}{Me^{-rt^*}} \right] \equiv Me^{-rt^*} K$$

This result is sufficient to derive the perfect markets, continuous time representation of the default risk spread.

To derive the default risk spread, let the risky interest rate be R and observe that the market value of corporate debt at t is $B(t) = M \exp[-Rt^*]$. It follows that:

$$Me^{-Rt^*} = Me^{-rt^*} K \rightarrow -Rt^* = -rt^* + \ln[K] \rightarrow R - r = \frac{\ln[K]}{t^*}$$

This provides a direct relationship between the credit (default risk) spread and K . Closer inspection of K reveals a number of important variables including the leverage ratio, the volatility of the market value of the firm and the term to maturity of the debt. While the presence of these variables is hardly surprising, what the Merton (1974) approach provides is a precise specification of how these particular variables interact. Following Culp (2001, p. 388), empirical implementation of this model is complicated by the difficulty of obtaining two important inputs, σ_V and the debt to equity ratio: "the volatility of equity tends to be a very noisy proxy for asset volatility and can be extremely sensitive to asset price changes . . . the capital structure of a firm is rarely as simple as equity plus zero-coupon debt with a single maturity date."¹⁴

*Recent developments in valuing defaultable bonds**

Since Merton (1974), considerable effort has been given to the extension and application of contingent claims pricing models to the valuation of defaultable fixed income securities. Empirical evidence on the basic Merton model has demonstrated an inability to explain the large size of observed credit spreads, e.g., Kim et al. (1993). Explanations for the failure of the basic model focus on: the assumption that all default risk occurs at maturity; limitations of the assumed absolute priority rule; and, the assumption of constant riskless interest rates. Following Cathcart and El-Jahel (1998), relaxation of each of these assumptions has proceeded by extending the "structural" continuous time

default model of Merton.¹⁵ Extensions to the structural model have incorporated default prior to maturity by introducing first passage time to a boundary that may be either random or deterministic. Randomness in the short term riskless interest rate is incorporated into the structural model in a fashion that is all too familiar, for example, from the discussion in section 5.2. In addition, some contributions to the structural model also allow for exogenous violations of the absolute priority rule.

Longstaff and Schwartz (1995) is a useful example of the structural model extensions where the firm defaults on debt obligations when the value of assets falls below an exogenously determined threshold. Stochastic interest rates are modeled using a mean-reverting OU process. Deviations from the absolute priority rule are permitted by exogenously setting the payment to bondholders when default occurs. Cathcart and El-Jahel (1998) address some limitations in Longstaff and Schwartz by using a mean reverting square root process for the riskless short term interest rate and introducing a "signaling process" that determines default. As discussed in section 5.1, the square root process avoids the possibility of negative interest rates. The use of a signaling process avoids the need to use the value of assets to specify the occurrence of default. The signal is treated as a random variable following a diffusion. Default occurs when the signal variable hits a lower threshold. At this point, the debt is "reorganized" and the bondholder receives $(1 - \delta)$ in new default free discount bonds. The value of δ ($0 \leq \delta \leq 1$) is exogenous with $\delta = 1$ representing a complete loss of bond value. The assumed process for the signal follows a geometric Brownian motion. No rationalization is given for the choice of this particular process and, though stylistically different, the conceptual gains compared to using the value of assets, as in Longstaff and Schwartz, does not appear substantive.

The linchpin of continuous time valuation models is the "fundamental partial differential equation" (PDE). For example, in section 5.3 a fundamental partial differential equation was derived for all interest rate dependent securities. Given the usefulness of closed form solutions, the trick is to structure the specifics of the valuation problem in such a way that the PDE can be solved in closed form. When there are two stochastic processes that have to be addressed, as in Cathcart and El-Jahel with the interest rate and signal processes, the PDE is more complicated. In order to reduce the complexity, it is analytically convenient to assume that the two random variables are not correlated. By making this assumption, Cathcart and El-Jahel can specify the PDE for the defaultable bond value (price) function $H[x, r, t^*]$ as:

$$rH = H_r \kappa(\mu - r) + H_x \alpha x + .5[H_{rr} r \sigma_r^2 + H_{xx} x^2 \sigma_x^2] - H_t$$

where x is the random signal that follows a lognormal diffusion, r is the random riskless interest rate that follows a mean reverting square root process and $t^* = T - t$ (measured in years) is the time to maturity of the defaultable bond.

By making reference to the discussion of Ito's lemma in section 5.2, the definitions of the diffusions, and the associated parameters, are apparent from this specification of the PDE. The assumption that the short term riskless interest rate and the signal are not correlated allows the cross partial derivatives terms to be set to zero. Given this PDE, for the problem to be "well posed" appropriate initial and boundary conditions also have to

be specified. Cathcart and El-Jahel provide five conditions: $H[x, r, 0] = 1$, if the bond reaches maturity then a par value of \$1 is paid; $H[\infty, r, t^*] = P[r, t^*]$, as the signal goes to infinity, the price of the defaultable bond goes to the value of the riskless bond, $P[r, t^*] = P(t, T)$ (the notation used in chapter 5); $H[x = x_L, r, t^*] = (1 - \delta)P[r, t^*]$ when the signal reaches the lower threshold default value (x_L) then the "reorganization" of the debt will occur; and, two technical conditions associated with r , $0 < H[x, 0, t^*] < \infty$ and $H[x, \infty, t^*] = 0$. Because the riskless interest rate follows a square root process, closed form solutions for $P[r, t^*]$ are available (see end of chapter 5 questions).

With this mathematical setup, Cathcart and El-Jahel are able to solve for the price function for the defaultable bond as:

$$H[x, r, t^*] = P[r, t^*] - \delta P[r, t^*] \Lambda[x, t^*] = P[r, t^*] (1 - \delta \Lambda[x, t^*])$$

where $\Lambda[x, t^*]$ is the inverse Laplace transform defined as:

$$\Lambda[x, t^*] = \frac{1}{2\pi i} \int_{c-100}^{c+100} \frac{e^{qt^*}}{q} \left(\frac{x}{x_L} \right)^{\lambda[q]} dq$$

with:

$$\lambda[q] = \frac{-\alpha + .5\sigma_x^2 - \sqrt{(\alpha - .5\sigma_x^2)^2 + 2\sigma_x^2 q}}{\sigma_x^2}$$

In this specification, q is the Laplace parameter and, heuristically, c is chosen so that the inverse Laplace transform integration makes sense. Leaving aside the obvious questions about how to actually do this calculation, observe that the credit spread in this model ($R - r$ in the Merton model) can be solved as: $-\ln[1 - \delta \Lambda[x, t^*]]/t^*$.

To the uninitiated, all this analytical structure might seem somewhat contrived. As discussed in sections 1.3 and 2.4, the propensity for practitioners of modern Finance to generate more heat than light is difficult to ignore. What relevance are the "structural models" of default risk to security analysts trying to determine the value of a defaultable bond? The simple answer lies in the functional form used to specify the default probability. The security analyst is interested in finding differences between the assessment of default (and the recovery rate) as reflected in the market price of a defaultable bond and the "true" *ex ante* assessment whether there will be a default and, if so, what the payout to bondholders will be. In the Cathcart and El-Jahel framework the probability of default is $\Lambda[x, t^*]$ and the recovery rate $(1 - \delta)$ is set exogenously. The function $\Lambda[x, t^*]$ can be associated with the probability of the first passage time from a given signal value x to the lower threshold x_L . As indicated, this function depends on the drift and volatility parameters of the assumed signal diffusion process. Though *ex post* estimates of the parameters could be obtained, and some rationalization given for x_L , the comments concerning the value of "process" made about the OAS procedure also seem to apply concerning models of default assessment.

Questions



1 (CFA Level II) The table below shows prices as a function of yields for four tranches of a collateralized mortgage obligation (CMO). Using the prices and yields in the Table, determine the effective duration of tranche T2. Assume that the relevant current interest rate is 6%. Identify the tranche with negative convexity. Justify your choice.

Table: price vs. yield for four CMO tranches

Yield	T-1	T-2	T-3	T-4
5.0	113	109	103.5	107
5.5	106	104	101.5	104
6.0	100	100	100	100
6.5	95	97	99.5	95
7.0	91	95	99.25	89

2 (CFA Level II) The table below shows the CMO option adjusted spreads declining as price volatility increases. Explain this pattern of option adjusted spreads.

Table: CMO option adjusted spreads for different pricing volatilities

Pricing Volatility	8%	12%	16%
Option Adjusted Spread (bp)	93	74	51

3 "For a callable bond, it is inappropriate to use modified duration (and convexity) because the expected cash flow changes as the yield changes... A change in interest rates will affect the price volatility of the noncallable bond component depending on the duration of the noncallable bond. It will also affect the price of the embedded call option."

Explain how the option adjusted duration measure is derived and how this measure can be used in the analysis of callable bonds and mortgage backed securities.

4 An important drawback of "traditional yield spread analysis" is the "failure to take into account future interest rate volatility that would affect the expected cash flow" of a fixed income security. What is option adjusted

spread analysis and how does this technique correct for this limitation of traditional yield spread analysis in the analysis of bonds with embedded option features. What are some important pitfalls of option adjusted spread analysis?

5(a) In the Fabozzi example of section 5.3, finish calculating the empirical spot rate curve, for the remaining 17 cash flow dates, by averaging the spot rates across the paths. Compare these values to the Treasury spot rate curve given in Fabozzi (1989, pp. 342–3). What adjustments could be made to the process to make these two curves approximately equal?

(b) Reconstruct the ten paths given in table 6.6 using an interest rate process that is mean-reverting. Repeat the exercise using the Black, Derman and Toy model and calibrate the model to the empirical spot rate curve in (a). (Hint: see Sundaresan 2002, pp. 605–8 and 612–16).

6(a) Using the Merton credit spread model of section 6.4, consider a firm with a market value of $V = 30$ and a zero coupon debt that matures in one year with a par value $M = 20$ and a term to maturity of $T = 1$. If the riskless interest rate $r = 10\%$ and the volatility of the market value $\sigma_V = .3$. Show that the market value of the bond is 17.96. Similarly, solve for the credit spread to show that $R - r = .7\%$ (70 basis points). (Hint: see Ritchken 1987, pp. 334–6.)

(b) Extend the Merton default risk model to the case where the firm issues both senior debt B_1 , with par value M_1 , and subordinated debt B_2 with par value M_2 . In this case, $S(T) = \text{Max}[0, V(T) - (M_1 + M_2)]$, $B_1(T) = \text{Min}[V(T), M_1]$ and $B_2(T) = \text{Max}[\text{Min}[V(T) - M_1, M_2], 0]$.

7(a) Given the duration for a putable bond derived in section 6.2, derive the formula for the convexity of a putable bond. What will happen to the value of convexity as interest rates change? Is it possible for a putable bond to have negative convexity?

(b) Given the duration of a convertible bond provided by Sarkar (1999) examined in section 6.2, derive a formula for the convexity of the convertible. What are the implications of introducing subordinated debt into the firm's capital structure? (Hint: see Sarkar, pp. 189–93.)

NOTES

- 1 What is a closed form? Though this terminology appears in numerous articles in modern Finance, a precise definition is difficult to identify. The basic intuition is that a closed form is an "exact formula." For example, numerical methods are used to solve option pricing problems when

closed forms (exact formulas) are not available. Given this, when is a formula an exact formula? The Black-Scholes option pricing formula is clearly an exact formula as are the discounted cash flow formulas for pricing fixed income securities such as default free fixed

term annuities, perpetuities and option and default free bonds. However, as valuation formulas become more complicated terms may appear in formulas that, for example, are sums that are used to approximate an unknown function, e.g., Buttler and Waldvogel (1996, p. 62). For example, a truncated Taylor series could be used as an approximation to some unknown function. This sum is then truncated at some arbitrary point creating an "exact formula." Such formulas are little more than disguised numerical methods dressed up as formulas. In some sense, almost all numerical methods can be reduced to "exact formulas" describing the solution procedure. To avoid the associated semantic confusions, a closed form is defined here as an "exact formula" in the sense that a numerical method is not required to implement the formula. Heuristically, the formula can be solved by "plugging in" values. One important advantage of a closed form is that it can be differentiated to obtain closed form solutions for the Greeks.

- 2 The relevant legal citation is *Morgan Stanley Co. vs. Archer Daniels Midland Co.*, 570 F. Supp. 1529, 1532-33, (S.D.N.Y. 1983). Blanc and Gordon (1999) examine the relevant legal issues of this case and other situations where firms seek to undertake surprise calls of issues that are either non-callable or have restrictive call provisions.
- 3 More detail on MBS can be found in Fabozzi (2001, chs 24-8).
- 4 Both before and after the introduction of the system for issuing mortgage backed securities, the supply of long term fixed rate mortgage funds in the US was facilitated by the savings and loan network of financial institutions. This network was given a variety of favorable institutional and legal preferences, e.g., Regulation Q restricting the payment of interest on demand deposits, that provided a supply of relatively low cost savings deposit funds that, in turn, were used to fund mortgages. The collapse of these preferences during the 1970s led, ultimately, to the end of this method of funding mortgages.
- 5 Technically, GNMA does not formally issue pass-throughs. Rather, GNMA adds a guarantee

to privately-issued pass-throughs that are backed by FHA and VA mortgages. FNMA and FHLMC both issue and guarantee pass-throughs.

- 6 In addition to CMOs, pass-throughs are also used to collateralize stripped MBS. These securities unbundle the pass-through cash flow into interest only (IO) and principal only (PO) classes. The first stripped MBS were introduced in 1986. Some presentations, e.g., Carron (1992), treat stripped MBS as a type of CMO.
- 7 Even though "vega" is not a Greek letter, this is the name given to this "Greek."
- 8 For those unfamiliar with futures contracts on Eurodollars, Treasury bills and the like, the method of quoting futures prices is $EU(t) = 100 - eu(t)$, where $eu(t)$ is expressed in % terms. For example, $EU(t) = 96.45$ means that the add-on interest rate for the three month Eurodollar deposit to be delivered at maturity of the futures contract will be 3.55%. This method of quoting is consistent with the market practice of quoting money market securities in terms of yields while retaining the futures market convention that long (short) positions make money when "prices" rise (fall). The method of quoting futures contracts for longer term debt securities, e.g., Treasury notes and bonds, differs from the money market securities by using the "price" of a theoretical deliverable.
- 9 The reference to callable option is not meant to imply that the option is callable. Rather, the reference is a short form for the expression "the embedded option that permits the issuer to call the bond issue prior to the stated maturity date." This terminology is used in preference to referring to the embedded option as a call option which confuses the terminology with exchange traded and other call options that are buyer's options.
- 10 Unlike the callable bond, the reference to a puttable option is meant to imply that the option is to put the bond back to the issuer, i.e., the option is a buyer's option. More precisely, the reference to puttable is a short form for the expression "the embedded option that permits the bondholder to put the bond issue back to the issuer at a stated put price on the stated put date(s)."

- 11 Later editions of Fabozzi (1989) replaced the Monte Carlo example with a binomial solution to valuing the embedded option in a callable bond based on Kalotay et al. (1993). Recognizing the advantages of Monte Carlo methods for dealing with path dependent options and options subject to a number of random variables, Fabozzi (2000) retains the OAS method to value MBS.
- 12 As stated, the compact formula for the OAS differs slightly from the Monte Carlo calculation described in Fabozzi (1989). This is due to the OAS being added to the one period zero rate in the compact formula and to the z_t spot rate in the Fabozzi presentation. Even though the product of t (one plus the) one period ahead zeros will equal the product of (one plus the) z_t spot rate (raised to the t), it does not follow that adding the OAS to each of the one period zeros will give the same result as adding the OAS to the z_t spot rate. However, the difference will be typically small enough to ignore for all but the most precise calculations.
- 13 Based on the relationship between the yields of low grade and high grade bonds, Macaulay is able to conclude that, for the same term to maturity, the low grade bond will have a shorter duration than for a high grade bond.
- 14 Despite the potential complications, there are a number of commercial applications of credit risk measurement that have been inspired by the Merton credit risk model. The most well known of these applications is the KMV Corporation Credit Monitor® model. This model calculates the expected default frequency (EDF) for companies with publicly traded common stock, e.g., Culp (2001, pp. 388–9), Sundaresan (2002, pp. 645–6). The EDF calculation is used to predict future changes in a firm's credit rating. The EDF measure is an extension of the Merton approach that uses actual default probabilities to substitute for the cumulative normal distribution function value $(K - 1)$ which is the default probability in the Merton model.
- 15 In conjunction with the extension of the structural models, there has also been the development of "reduced form" models, e.g., Jarrow and Turnbull (1995). According to Cathcart and El-Jahel (1998, p. 66): "The reduced-form models are models in which default time is a stopping time of some given hazard function." A simpler description would be to describe these models as extensions of the self-calibrating binomial models where the objective is to estimate the martingale default probabilities. This definition would include, say, Finnerty (1999) in the reduced form category.