Econ 302: Microeconomics II - Strategic Behavior

Problem Set #7 – June 28, 2016

1. Ned and Ruth love to play "Hide and Seek." It is a simple game, but it continues to amuse. It goes like this: Ruth hides upstairs or downstairs. Ned can either look upstairs or downstairs but not in both places. If he finds Ruth, Ned gets one scoop of ice cream, and Ruth gets none. If he does not find Ruth, Ruth gets one scoop of ice cream and Ned gets none.

a) Describe the payoff matrix for this game.

		Ruth: Upstairs	Ruth: Downstairs
Ned	Upstairs	$1, 0$	$0, 1$
	Downstairs	$0, 1$	$1, 0$

b) Is this a zero-sum game? What are the Nash equilibria in pure strategies? *This is a constant sum game which is equivalent to a zero-sum game (because we could deduct -1/2 from all payoffs without changing the Nash equilibria). There are no Nash equlibria in pure strategies. We can verify this with cell-by-cell inspection: for instance, (Upstairs, Upstairs) cannot be NE because the best-response for Ruth to Ned playing (Upstairs) is (Downstairs). In this way, we can eliminate all 4 cells as potential NE.*

c) Find a Nash equilibrium in mixed strategies. *Using the line of argument we developed in class, the following is a unique Nash equilibrium in mixed strategies: Ruth hides downstairs and Ned searches downstairs with probability 1/2; Ruth hides upstairs and Ned searches upstairs with probability 1/2. Ned will find Ruth 1/2 of the time.*

d) After playing this game for a while, Ned and Ruth decide to give it a new twist. Now, if Ned finds Ruth upstairs, he gets two scoops of ice cream, but if he finds her downstairs, he gets one scoop of ice cream. If Ned finds Ruth, she gets no ice cream, but if he doesn't find her, she gets one scoop. Describe the new payoff matrix and find any (pure strategy or mixed) Nash equilibria.

There are still no NE in pure strategies. In the unique mixed strategy equilibrium, Ruth hides downstairs 2/3rds of the time, and Ned looks downstairs 1/2 of the time. Again, Ned will find Ruth 1/2 of the time. Note that now, Ruth hides

		Ruth: Upstairs	Ruth: Downstairs
Ned	Upstairs	$2, 0$	$0, 1$
	Downstairs	$0, 1$	$1, 0$

downstairs ***more often*** *in equilibrium as compared to the previous situation,* ***although her payoff hasn't changed: she still gets the same payoffs irrespective of where Ned finds/does not find her.*** *The reason that her previous strategy is no longer an equilibrium is that, if she did mix with equal probability, Ned would be looking upstairs most of the time (because finding her upstairs gives a higher payoff). But if Ned would concentrate his search on the upstairs, Ruth would no longer be indifferent between the two hiding places.*

2. *Paper, Scissors, Rock* is a game in which two players simultaneously choose Paper (hand held flat), Scissors (hand with two fingers protruding to look like scissors) or Rock (hand in a fist). Paper beats Rock, Rock beats Scissors, and Scissors beats Paper. This game has the structure:

		Column: Paper	Column: Scissors	Column: Rock
Row	Paper	$0, 0$	$-1, 1$	$1, -1$
	Scissors	$1, -1$	$0, 0$	$-1, 1$
	Rock	$-1, 1$	$1, -1$	$0, 0$

Paper, Scissors, Rock

a) Show that this game has no pure strategy Nash equilibria. *We can do this with cell-by-cell inspection: for instance, (Paper,Paper) cannot be NE because the best-response for each player to his opponent playing (Paper) is (Scissors). (Paper, Scissors) cannot be NE because the best response to one's opponent playing (Scissors) is (Rock). In like manner, you can check all other 7 possible strategy combinations and find that there are no pure strategy NE's.*

b) Show that playing all three actions with equal probability is a mixed strategy Nash equilibrium. *Suppose Row plays each three actions with equal probability*

1/3. *Then Column's expected payoffs from each of the actions are:*

$$\begin{aligned} \textit{Paper:} \quad & \frac{1}{3}0 + \frac{1}{3}(-1) + \frac{1}{3}1 = 0 \\ \textit{Scissors:} \quad & \frac{1}{3}1 + \frac{1}{3}0 + \frac{1}{3}(-1) = 0 \\ \textit{Paper:} \quad & \frac{1}{3}(-1) + \frac{1}{3}1 + \frac{1}{3}0 = 0 \end{aligned}$$

So Column is indifferent between any of the three actions, which implies that he/she is equally willing to mix with any probability and in particular, with equal probability. By symmetry, the same is true for Row, so playing each action with equal probability is a mixed strategy Nash equilibrium.

3. The Stag-Hare Game. Two hunters set out on a hunt together. The simultaneously can choose to hunt a stag (deer) or hunt a hare. If the hunter hunts a stag, he must have the cooperation of his partner in order to succeed. A hunter can get a hare by himself, but a hare is worth less than an equal share of the stag. The payoff-matrix is:

		Column: Stag	Column: Hare
Row	Stag	5, 5	0, 2
	Hare	2, 0	2, 2

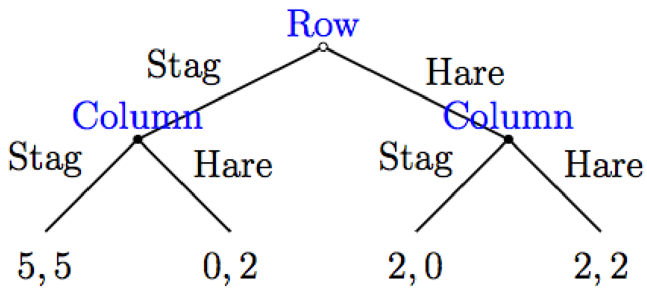
The Stag Hunt Game

a) Find *all* Nash equilibria of this game (pure and mixed strategies)

There are two pure strategy NE's and one mixed strategy NE. Given player Column chooses Stag, Stag is better than Hare for Row (5 as opposed to 2), and vice versa. So (Stag, Stag) is an NE. Also, given Column chooses Hare, choosing Hare likewise is better than choosing Stag for Row (2 as opposed to 0), and vice versa. So (Hare, Hare) is also a NE. To determine the mixed strategy NE, let p *be the probability with which Column plays Stag and* $1-p$ *the probability with which Column plays Hare. Row's expected payoff is then* $p5 + (1-p)0 = 5p$ *if he plays Stag and* $p2 + (1-p)2 = 2$ *if he plays Hare. In a mixed strategy equilibrium, Row must be indifferent between playing Stag and playing Hare; we thus must have* $5p = 2$ *or* $p = 2/5$. *By symmetry, the same is true for Column. The mixed strategy NE is thus: both players play Stag with probability* $2/5$ *and Hare with probability* $3/5$.

b) Now suppose that the hunters move sequentially: Row chooses first, and then Column (observing what Row did) makes his choice.

i) Draw the game tree, write down the possible strategies for each player, and determine the unique Subgame Perfect Equilibrium of the game.



The strategies are:
– Row has two strategies: (Stag), (Hare)
– Column has four strategies (Stag regardless), (Hare regardless), (Stag if Stag, Hare if Hare), (Stag if Hare, Hare if Stag).

To determine the SPE, we can use backward induction, we see that Column will hunt (Stag) if Row hunts (Stag) and (Hare) if Row hunts (Hare). Compare the payoffs for Row shows that Row is better off hunting (Stag). The SPE is thus: Row's strategy is (Stag), Column's strategy is (Stag if Stag, Hare if Hare). The outcome is that they both hunt Stag and payoff's are 5, 5*. Intuitively, Row anticipates that Column will be hunting whatever he hunts, so he has a first mover advantage and can pick the prey. Since Stag is better than Hare, he picks Stag.*

ii) Consider the outcome where both hunters hunters hunt a hare. Can this outcome be supported as *Nash* equilibrium of the sequential game? Explain!

Yes, it can - the outcome where both hunt Hare is still a Nash equilibrium of the sequential *game. Suppose Column's strategy is to play Hare regardless, i.e., (Hare if Hare, Hare if Stag). A best response to this strategy by Row is to play (Hare) as well. But given that Row plays (Hare), it is optimal for Column to play (Hare regardless).This Nash equilibrium is not an SPE because the part of Column's strategy that prescribes (Hare if Stag) is not sequentially rational: if Row played Stag, Column should play (Stag) as well - not (Hare). So (Hare regardless) is not a sequentially optimal choice (it is not optimal for a node in the game tree that isn't reached – namely if Row plays Stag), and thus does not constitute a subgame perfect strategy. It is still Nash, however, because since Row never plays (Stag), only (Hare), (Hare regardless) is a best response.*

Further questions for review:

1. Consider the game:

		Column L	R
Row	T	$0, 3$	$3, 0$
	B	$2, 1$	$1, 2$

a) Does the game have any pure strategy Nash equilibria? *No. The best response to "Left" is "Bottom" to which "Right" is best response. And the best response to "Right" is Top, to which "Left" is best response.*

b) Determine the Nash equilibrium in mixed strategies! *Suppose Row plays "Top" with probability p. Then Column's expected payoffs from each of the actions are:*

$$\textit{Left:} \quad 3p + 1(1-p) = 1 + 2p$$
$$\textit{Right:} \quad 0p + 2(1-p) = 2 - 2p$$

So Column is indifferent if $1 + 2p = 2 - 2p$*, or* $p = 1/4$*. Suppose Column plays "Left" with probability q. Then Row's expected payoffs from each of the actions are:*

$$\textit{Top:} \quad 0q + 3(1-q) = 3 - 3q$$
$$\textit{Bottom:} \quad 2q + 1(1-q) = 1 + q$$

So Row is indifferent if $3 - 3q = 1 + q$*, or* $q = 1/2$*. The mixed strategy NE is Row plays "Top" 25 percent of the time, and Column mixes equally between "Left" and "Right".*

2. Helping the victim of a crime. A group of n people observe an old lady being attacked by a stranger. Each person would like the lady to receive help, but prefers that someone else interferes. Specifically, payoffs are as follows: if someone else helps the lady, a person gets v. If the person himself/herself goes after the attacker, he/she gets $v - c > 0$. If no one helps the lady, each person gets 0. Note that this game is a variant of the game from exercise 5 of Problem Set 5 with $k = 1$.

a) Show that this game has n asymmetric pure strategy Nash equilibria, in each of which exactly one person helps. Argue that this is a coordination problem, and that social norms may help to select among equilibria. *First, we show that any equilibrium in which exactly one person helps is a Nash equilibrium. Suppose person i helps and all others don't. Then this is optimal for all non-helpers because $v > v - c$. It is also optimal for person i because $v - c > 0$, i.e., helping is better than not helping, given that no other person helps. Since this argument holds for any arbitrary person i, there are in total n Nash equilibria of this kind. There are no other NE's in pure strategies: if nobody helps, one person has an incentive to deviate and help. If more than one person helps, each helper has an incentive to deviate and stay put.*

*The people face a coordination problem because the payoff that any one player receives in **each** of the **multiple** Nash equilibria is **higher than in any other potential outcome** (it is strictly higher for everybody than if nobody helps; compared to outcomes where more than one person helps, it is strictly higher for the helpers, and just as good for the non-helpers). In other words, it is better to coordinate on any one of the NE than to fail to play a Nash equilibrium. Social norms help here, because they would dictate that mid-age male observers go after the attacker (rather than the elderly or women).*

Note the similarity to the problem of getting the right amount of people into the life-boats of a sinking ship, which is also a coordination problem (given that the boat will sink if too many try to squeeze in). Here, the norm is that children and women go first.

b) * If people cannot coordinate on one pure strategy Nash equilibrium, another likely outcome is a symmetric, mixed strategy Nash equilibrium, in which each person helps with probability p. Calculate the equilibrium value of p as a function of v, c and n. What is the probability that no one helps in equilibrium? And how does it change with the size of the group?

In any such mixed strategy equilibrium, the payoff of each person must be equal if they help and if the do not. If they do not help, then either no one helps the lady or somebody else will. Hence, the equilibrium condition is:

$$v - c = 0 \times \Pr(\text{no one else helps}) + v \times \Pr(\text{at least one other person helps}),$$

which is equivalent to

$$\begin{aligned} & v - c = v(1 - \Pr(\text{no one else helps})) \qquad (*) \\ \Leftrightarrow \quad & \frac{c}{v} = \Pr(\text{no one else helps}). \end{aligned}$$

Denote by p the probability with which each person helps. The probability that no one else helps is the probability that every one of the $n - 1$ other people does not *help, namely,* $(1-p)^{n-1}$. *Thus, the equilibrium condition becomes* $c/v = (1-p)^{n-1}$ *or*

$$p^* = 1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}}.$$

This number is between zero and one, so we can conclude that the game has a unique symmetric mixed strategy equilibrium in which each person helps with probability $p^*1 - (\frac{c}{v})^{\frac{1}{n-1}}$. *How does this equilibrium change as the size of the group increases? As* n *increases, the probability that any one of the* n *people helps decreases (note that as* n *increases,* $1/n-1$ *decreases and so* $v/c^{\frac{1}{n-1}}$ *increases). But the size of the group gets larger, so what about the probability that* ***at least one person helps****? There are two ways to calculate this. We can either calculate* $1-p^*$ *directly, or we can note that*

$$\Pr(\text{no one helps})) = \Pr(\text{person } i \text{ does not help})) \times \Pr(\text{no one else helps})).$$

Now, the probability that any arbitrary person i *helps in this equilibrium is decreasing in* n. *From* $(*)$, *the joint probability that none other than* i *helps is equal to* v/c *and is* ***independent*** *of* n. *That is, the larger the group , the* ***less likely it is that someone will help the lady.*** *While the finding that larger group sizes make any individual person less likely to help is intuitive, the result that the absolute probability of the lady receiving help drops is more surprising. It is an implication of the equilibrium: any one person in the larger group is no less concerned about the lady getting help and the group is larger – still* ***in equilibrium****, she has less chances of receiving help.*

3.*(Ultimatum Game) Consider the following 2–stage game, where two players have to split a pile of 100 gold coins. Player 1 moves first. His action is a proposal $1, 2, \ldots, x, \ldots 100$, where action x means that player 1 proposes to keep x of the gold coins for himself and give $100-x$ to player 2. Player 2 learns the choice of player one, and then takes one of two actions in response: A (accept) or R (reject). If Player 2 accepts (plays A) , the pile is split according to the proposal of Player 1. If Player 2 rejects (plays R), both get zero.

a) Describe the extensive form version of the game using a game tree. *omitted*

b) Describe the normal form of the game. It suffices to specify the strategies and the payoffs of both players (Hint: Player 2 has 2^{101} pure strategies.) *Solution: Player 1's strategy is simply the proposed split* $x \in \{0, 1, ..., 100\}$. *Player 2's strategy is to accept or reject depending on the proposal of Player 1. We can write it as follows:* $d \in \{0, 1\}^{100}$ *where the interpretation is that* $d(x) = 1$ *if Player 2* *accepts when Player 1 plays* x $d(x) = 0$ *if Player 2 rejects when Player 1 plays* x*, where* $0 \leq x \leq 100$ *and the payoff functions are given by* $u_1(x, d) = xd(x)$ *and* $u_2(x, d) = (100 - x)d(x)$*.*

c) Identify a Nash equilibrium of the normal form game where the players split the pile in half, i.e., the payoffs are 50 for Player 1 and 50 for Player 2. *The following is a NE. Player 1 proposes* $x = 50$ *and Player 2 plays* $d(x) = 1$ *if* $x \leq 50$ *and* $d(x) = 0$ *if* $x > 50$*. That is, Player 2 accepts if Player 1 plays* $x \leq 50$ *and rejects otherwise.*

d) Identify the subgame perfect equilibria of the extensive form game. (Hint: There are two of them.) *Consider the subgame after Player 1 has made a proposal. If* $x \leq 99$*, Player 2 is strictly better off to accept (payoff is* $100 - x$*) than to reject,* **so in a subgame perfect equilibrium, Player 2 must accept whenever** $x \leq 99$*. So one SPE is that Player 1 propses* $x = 99$ *and Player 2 accepts. Can Player 1 get even more? When* $x = 100$*, Player 2 is indifferent between accepting or rejecting but* **one possible strategy** *consistent with utility maximization is* $d(100) = 1$*. So there is another SPE where Player 1 plays* $x = 100$ *and Player 2 accepts everything, i.e., plays* $d(x) = 1$ *for any* $x \in \{0, \ldots, 100\}$*.*