Econ 302: Microeconomics II - Strategic Behavior

Problem Set #8 – July 5, 2016

1. True/False/Uncertain? Repeated interaction allows players to condition their actions on past behaviour, which facilitates reward-like or punishment-like strategies.

True. Whether or not strategies that include rewards/punishments are being successfully played in equilibrium, however, is another matter. Take a prisoners' dilemma that is repeated for 2 periods for example: One possible strategy of the players could be "Tit-for-Tat", i.e., a player always cooperates in period 1, and continues to cooperate in period 2 if the other player also cooperated in period 1, if the other player defected in period 1, however, the strategy calls for the player to defect in period 2. However, these strategies are not subgame perfect as long as the repetition is finite: cooperation cannot be sustained. Using backward induction, the unique SPE is (Defect,Defect) in both periods. If the PD is repeated an infinite (or uncertain) number of times, however, Tit-for-Tat can be part of a SPE, inducing cooperation, if players are sufficiently patient.

2. The following is an interpretation of the rivalry between the United States and the Soviet Union for geopolitical influence in the 1970's and 1980's. Each side can either be Aggressive or Restrained. The Soviet Union wants to achieve world domination, so being Aggressive is its dominant strategy. The United States wants to prevent the Soviet Union from achieving world domination; it will match Soviet aggressiveness with aggressiveness, and restraint with restraint.

		S.U. Restrained	S.U. Aggressive
U.S.	Restrained	$4, 3$	$1, 4$
	Aggressive	$3, 1$	$2, 2$

a) Suppose the two countries move simultaneously. Find the Nash equilibrium. *The game is dominance solvable. Aggressive is the Soviet Union's dominant strategy, and then Aggressive is the U.S.'s best response. The unique Nash equilibrium is (Aggressive, Aggressive).*

b) Now, consider three different alternatives ways in which the game could be played: (i) the U.S. moves first, (ii) the S.U. moves first, and (iii) the S.U. moves first and the U.S. moves second, but then the S.U. has a further move in which it can change its first move. For each case, draw the game tree and find the subgame perfect equilibrium. *I omit the game trees, which are easy to draw. The*

SPE's are (i) (Aggressive, Always Aggressive) (ii) (Restrained, Restrained if Restrained/Aggressive if Aggressive), (iii) the SPE strategies are: U.S. always aggressive at its only move in the game, S.U.: Aggressive or Restrained at its first move, and Always Aggressive at last move, regardless of history of game.

c) What are the key strategic matters (commitment, credibility, etc.) for the two countries? *Both the US and the SU are better off in the situation in which the US moves last. This is the situation where the US retains its flexibility and respond to the actions of the Soviets (second mover advantage). With this order of timing, the SU's awareness of the US's ability to respond to its action discourages it from acting aggressively. The Soviets' now that if they did so, the US would match their aggression, and they would suffer as a result. In turn, the SU knows that restraint will be met by restraint.*

3. (Cournot– Stackelberg Competition). Carl and Simon are two rival pumpkin growers who sell their pumpkins at the local Farmer's market. The demand function is given by $p = 2-(q_C+q_S)/1600$ where q_C (respectively, q_S) is the number of pumpkins grown by Carl (respectively, Simon). The marginal cost of producing a pumpkin for both farmers is \$ 0.5.

a) Assume first that every spring, both farmers simultaneously decide how many pumpkins to grow. Each pumpkin grown will be sold in the fall. Calculate the best response functions and the Cournot Nash equilibrium (quantities, price, profit). *To find each farmer's response function, we have to maximize profit, taking rival output as given. The first-order conditions equate marginal revenue with marginal cost:*

$$\textit{Carl}: \qquad 2-(2q_C-q_S)/1600 = \frac{1}{2} \quad \Rightarrow \quad q_C = 1200 - \frac{1}{2}q_S.$$

$$\textit{Simon}: \qquad 2-(2q_S-q_C)/1600 = \frac{1}{2} \quad \Rightarrow \quad q_S = 1200 - \frac{1}{2}q_C.$$

Substituting Simon's best response into Carl's best response (or vice versa) gives the NE quantities $q_1^* = q_2^* = 800$. *The price is* $p^* = 1$ *and each farmer makes a profit of* $\pi_C^* = \pi_S^* = 400$.

b) Now assume the snow thaws off Carl's pumpkin field a week before it thaws off Simon's pumpkin field. Therefore, Carl can plant his pumpkins a week earlier than Simon can. Since Simon lives just down the road, he can tell by looking at Carl's field how many pumpkins Carl will produce (and sell) in the fall before making his own decision on how many pumpkins to grow.

i) If Simon sees that Carl has planted q_C pumpkins, how many should he plant? (Hint: remember the reactions functions from part a)) *If Carl planted q_C pumpkins, Simon maximizes profit by planting $q_S = 1200 - \frac{1}{2}q_C$ pumpkins. Of course, this is nothing but his best response function from part a).*

ii)* When Carl plants his pumpkins, he understands how Simon will make his decision. Therefore, Carl realizes that the amount that Simon produces will be determined by what Carl himself produces. If Carl plants q_C pumpkins, what will be the **total** output, taking into account how many pumpkins Simon produces as a function of q_C? Write down a function of Carl's revenue as a function of his own output, taking Simon's best response into account. *If Carl plants q_C, Simon will plant $q_S = 1200 - \frac{1}{2}q_C$. The total amount of pumpkins planted is thus $q_C + q_S = 1200 + \frac{1}{2}q_C$. Therefore, Carl knows that if his own output is q_C, the price on the market will be $p = \frac{5}{4} - \frac{1}{3200}q_C$ and his revenue is $(\frac{5}{4} - \frac{1}{3200}q_C)q_C$.*

iii)* Find the profit maximizing output for Carl, the equilibrium output of Simon, and the equilibrium price of pumpkins. How much profit does either producer make? *Given the calculation in ii), Carl's profit as a function of q_C (taking into account how Simon reacts) is given by*

$$\pi_C = (\frac{5}{4} - \frac{1}{3200}q_C)q_C - \frac{1}{2}q_C$$

The first order conditions give $q_C^* = 1200$. Simon would then produce $q_S^* = 1200 - \frac{1}{2}q_C = 600$. The price is $p^* = \frac{7}{8}$. Carl's profit is $\pi_C = 450$ and Simon's profit is $\pi_S = 225$.

iv)* If he wanted to, Carl could delay his planting until the same time that Simon planted, so that neither of them would know each other's plans at a time when he planted. Would it be in Carl's interest to do this? Explain your finding intuitively! *No, Carl's profit in the equilibrium where he moves first is higher than in the equilibrium where they move simultaneously (Carl has a first-mover advantage). Why? Because Carl realizes that Simon's response function is downward sloping (the more he produces, the less will Simon produce), he can by producing **more** than his Cournot quantity cause Simon to cut back production, which is beneficial for him (recall that output is a negative externality).*

This type of sequential competition is called Stackelberg competition. The leader moves first, the follower moves second. The follower's output depends on what he expects the leader to produce. His reaction function is constructed the same way as for Cournot competition. The leader knows the reaction function of the *follower, and gets to choose his output first. The leader thus knows that total output is the sum of his output plus the reaction function of the follower. The price and the revenue and profit function can be expressed as functions of the leader's output only, and are derived by substituting in for the reaction function of the follower. Using the first order conditions for the leader's profit max, we can solve for the leader's equilibrium output. Plug this in the response function of the follower to get the follower's output.*

4. Enforcing a cartel. There are two firms in a market with demand function $p(Q) = 24 - Q$, where $Q = q_1 + q_2$. The marginal cost of both firms are zero.

a) Solve for the Cournot-Nash equilibrium quantities, price, and profits. *Maximizing profits taking the other firms output as given yields reaction functions* $q_i = 12 - \frac{1}{2}q_j$. *Solving for the NE gives* $q_1^* = q_2^* = 8$, $p^* = 8$ *and* $\pi_1^* = \pi_2^* = 64$.

b) Solve for the cartel (=joint profit maximizing) quantities, price, and profits, assuming that the firms agree to share the market equally. *The joint profit maximizing aggregate quantity is the monopoly quantity* $Q_m = 12$. *Since they share the market equally by assumption, each firm produces* $q_i^* = 6$. *The price is the monopoly price* $p_m = 12$ *and profits are* $\pi_1^m = \pi_2^m = 72 > 64$.

c) Suppose one firm produces its cartel quantity, but the other firm deviates and produces the profit maximizing output, given the cartel output of the other firm. Solve for quantities, price, and profits. *Using the reaction function derived in a), the profit-maximizing quantity for firm* i *if firm* j *produces* $q_j = 6$ *is* $q_i = 9$. *The price is* $p = 9$ *and the profits are* $\pi_i = 81 > 72$ *for the cheating firm and* $\pi_j = 54 < 64$ *for the firm that honors the agreement.*

d) The firms interact repeatedly, but they know that their interaction will end after a certain finite number of rounds. What is the SPE of this finitely repeated game? *The unique NE of the stage game is the Cournot outcome. If the game is repeated finitely many times, we can use backward induction: in the last period* T, *both firms will set their Cournot quantities, regardless of the history of the game. But then, in the period before that,* $T-1$, *what they do does not influence their behavior in* T. *So the best thing they can do is to set their Cournot quantities in* $T-1$, *regardless of what happened in prior periods. But then their strategy in* $T-2$ *does not influence what happens in* $T-1$ *and* T, *i.e., this argument works for every period. We can conclude that they play Cournot in each period.*

e) Suppose now that the firms do not know how long they will be in the market. Let δ be the probability that they interact again tomorrow (firms do not discount the future otherwise). In order to sustain collusion, the firms adopt 'grim' strategies of the following kind: each firm produces the cartel quantity if the opponent has produced the cartel quantity last period; otherwise (i.e., if there was cheating in the cartel), it produces the Nash equilibrium quantity of the stage game from then on. Under what condition on δ can the cartel be sustained as an SPE? Give an intuition for your finding. *Given the strategies of the opponent, the present value of firm i's profit from honoring the cartel is (which will be rewarded by the opponent honoring the cartel as well)*

$$V_i(\textit{set cartel quantity}) = 72(1 + \delta + \delta^2 + \ldots) = \frac{1}{1-\delta}72.$$

If it instead cheats, it will make a one-period profit of 81*, which however is followed by reverting to the NE from then on. The present value from cheating is thus:*

$$V_i(\textit{cheat}) = 81 + 64(\delta + \delta^2 + \ldots) = 81 + \frac{\delta}{1-\delta}64.$$

Sticking to the agreement is better than cheating if $V_i(\textit{set cartel quantity}) > V_i(\textit{cheat})$ *or*

$$\frac{1}{1-\delta}72 > 81 + \frac{\delta}{1-\delta}64 \quad \Leftrightarrow \quad \delta \geq \frac{9}{17}.$$

f)* Now assumed a firm only observes that its opponent cheated with a one period lag, i.e, if a firm cheated in period t, cheating will be discovered in period $t + 1$. How does your analysis in f) change? Explain. *If cheating is observed with a one-period lag, the cheating firm can cheat twice before play reverts to the Cournot quantities. The present value from cheating becomes:*

$$V_i(\textit{cheat}) = 81 + \delta 81 + 64(\delta^2 + \delta^3 + \ldots) = 81(1 + \delta) + \frac{\delta^2}{1-\delta}64.$$

Sticking to the agreement is better than cheating if $V_i(\textit{set cartel quantity}) > V_i(\textit{cheat})$ *or*

$$\frac{1}{1-\delta}72 > 81(1+\delta) + \frac{\delta^2}{1-\delta}64 \quad \Leftrightarrow \quad \delta \geq \sqrt{\frac{9}{17}} > \frac{9}{17}.$$

The discount factor now has to be higher than in case f) for the cartel to be sustainable. The intuition is that cheating has become more profitable due to the lag in detection. So the future must matter more for the 'reward' of cartel profits and the 'punishment' of Cournot profits to work.

g)** Going back to the game in f), is the punishment to convert to the Nash equilibrium of the stage game indeed the worst punishment that they can inflict on each other for a deviation in a SPE of the infinitely repeated game? In other words, is there a (credible!) punishment that is more severe than the Cournot equilibrium? *Perhaps surprisingly, there is. First, it is easy to see that a firm can punish the opponent for cheating harder by producing* ***more*** *than the Cournot quantity. This will drive prices down even further and will hurt the opponent - its profits will drop below those of the Cournot equilibrium. More severe punishments are obviously desirable in the sense that the cartel will become more stable; formally, the cooperative outcome could be sustained even for discount factors $\delta < 9/17$. What is more difficult to see is that such punishments can be credible. The basic idea is as follows: we punish each other not only for deviating from the collusive outcome, but also for deviating from the punishment strategy.*

Further questions for review:

1. Reconsider the fundraising event game played between Harry and Sally in Problem Set 6, Question 4 but modify utility functions as follows. Sally values the prize at 72$ and Harry values the price at 144$, so $u_S = 72\frac{x_S}{x_S+x_H} - x_S$ and $u_H = 144\frac{x_H}{x_S+x_H} - x_H$. Calculate first the Nash equilibrium if Harry and Sally simultaneously choose how many tickets $x_i, i = H, S$ to purchase. Next, assume Sally moves first: Harry can observe Sally's choice of tickets x_S, before choosing how many tickets to buy himself, x_x. Derive the SPE of the sequential game. Compare the outcome with the simultaneous move game, and explain the difference(s)! Does Harry have a second-mover advantage?

Analogous to Question 4 in PS 6, we can solve for the NE in the simultaneous move game to obtain $x_S = 16$ and $x_H = 32$. Harry purchases twice as many tickets, and is twice as likely to win the prize, as does Sally. In the sequential game, Harry will continue to behave in accordance with his best response function $x_H(x_S) = 12\sqrt{x_S} - x_S$ for any given observed choice of x_S. Sally, however, will recognize that Harry's choice of x_H depends on her own choice of x_S via the best-response function $x_H(x_S)$, that is, instead of taking x_H as given, Sally will recognize that x_H varies with her own choice x_S according to $\partial x_H/\partial x_S = 6/\sqrt{x_S} - 1$. At her optimal choice in the simultaneous move game, namely $x_S = 16$, this derivative is positive. In other words, if Sally reduced her choice of x_S by 1 ticket, Harry would reduce his number of tickets as well. This means Sally's perceived cost of reducing the number of tickets she purchases is lower than it was previously – she realized that Harry will respond by also reducing his tickets, thereby not reducing the odds of her winning by as much as if he kept the number of ticket constant. We would thus expect her to further lower her ticket purchases in the sequential game. This is indeed what happens. Plugging Harry's best response into u_S, taking derivatives with respect to x_S and simplifying gives $x_S = 9 < 16$. Accordingly, $x_H = 27 < 32$. The odds of winning have now shifted in Harry's favour (he is 3 times *more likely to win) but if you calculate equilibrium utilities, you will notice that Sally gains more from the sequential nature of the game than does Harry (Why?)*

2.* A ship of 15 pirates has just claimed a treasure of 100 indivisible gold coins. Their traditional procedure for dividing the treasure is as follows. The pirates are ordered by age and then the oldest pirate has the honor of making the first proposal. After any proposal, all the pirates vote "Yes" or "No". The proposer is allowed to vote as well. If at least 50% of the votes are in favor, the proposal is adopted and the game ends. If the proposal is voted down, then the pirate who made the proposal is thrown overboard to the sharks and then the next oldest pirate is asked to make a proposal. Ties are broken in favor of the proposal. Coins cannot be divided in fractional units. Assume throughout that a pirate will never vote for a proposal that gives him 0 coins.

c) No. This is not a repeated game, as the game that is played in $t+1$ differs depending on the choice of the players in t. For example, if the proposal is adopted in t, the game ends so there is no game in $t+1$.

b) Suppose we reach the point where only two pirates remain and thirteen have been thrown overboard. At that point, only one pirate needs to vote for the next proposal in order for it to be adopted. What will be the outcome in this contingency? *Throughout the problem, I denote the pirates by descending age. So P1 is the oldest, and P15 is the youngest. For simplicity, let's assume the pirate making the offer votes in favor of his own proposal (this is weakly dominant, and does not change the analysis in any meaningful way). A pirate's strategy includes whether he votes "yes" or "no" for each possible offer by older pirates, plus the offer he makes if it gets to be his turn. The backward induction solution is: P14 offers (100,0) and keeps all the money. Pirate 15 votes "no" but ties are decided in favor of the proposal so the outcome is (100, 0) [Note: P15 can vote "yes" or "no" (or any mixture of the two) since he is indifferent but remember we assume a pirate never votes yes for a proposal that gives him nothing.]*

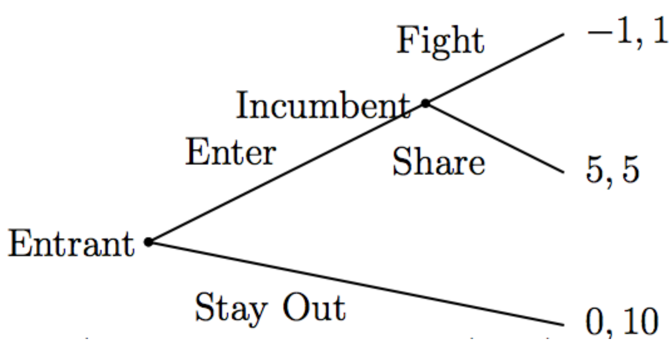
c) Now suppose only three pirates remain. Use the answer to (a) to identify the backward induction outcome in this case. *If P13 is thrown overboard, P14 ends up with 100, and P15 gets 0 from the analysis in part a. P14 weakly prefers voting"no" regardless of the offer P13 makes (this is only a weak preference, because if both P13 and P15 vote yes, P14's vote makes no difference; from here on out I will assume pirates vote according to their weak preference for simplicity's sake. Meanwhile, P15 will vote yes to any offer which gives him at least 1 coin, since if P13 is thrown overboard, he will get 0. Based on this logic, the backward induction solution is for P13 to offer (99, 0, 1), and P13 and P15 to vote in favor of that division.*

d) What will be the backward induction outcome in the game with all 15 pirates on board? *Iterating the logic from above, the backward induction outcome in a four-pirate game will be (99, 0, 1, 0), with P12 and P14 voting in favor, and P13 and P15 voting against (all of these are again weak preferences). With 5 pirates: (98, 0, 1, 0, 1). This continues such that the oldest remaining pirate gives 0 to the second-oldest, 1 to the third-oldest, 0 to the fourth-oldest, 1 to the fifth-oldest, etc., taking the rest for himself. All pirates who get a positive number of coins vote in favor, and those getting 0 vote against. This leads to the following outcome with 15 pirates: (93,0,1,0,1,0,1,0,1,0,1,0,1,0,1).*

e) How would the answers to parts (a) through (c) be different if the voting rule is adjusted slightly so that the proposal requires a majority of the votes (more than 50%) to be adopted? *Now the two-player game produces the opposite result: P15 will vote "no" to any proposal, since then he gets to throw P14 overboard and keep all the coins. Three-player game: P13 now needs P14's support, and can get it by offering (99, 1, 0). Four-player game: P12 now needs support of two other pirates to get more than 50% of the vote. So P12 needs to offer (97,0,2,1). 15 players: (92,0,1,2,0,1,0,1,0,1,0,1,0,0,1) Finally, note that if we instead had started with with 7 players, there is an arbitrary choice of which pirate to give 2 coins to, resulting in slightly different possible answers. As for part d, we would still get a range of Nash equilibria, but the division would be* $(x, 100-x, 0)$*. The logic is the same in part d above, except we switch P14 and P15.*

3.* The following problem shows that repeated interaction cannot only solve the problem of cooperation in prisoner's dilemma type of situation, but also opens up the possibility to build a reputation for being 'tough'.

Reconsider the entry game from class,



and recall that if this game is played only once, the unique SPE (by backward induction) is (Enter, Share if Enter).

a) What is the SPE of the repeated game with finitely many periods? Explain!

Solving the game backwards, the Incumbent will always "Share" in the last period $t = T$ and the Entrant will therefore "Enter". By the same argument as in the lecture, the same is then true for $t = T - 1$, since play tomorrow is independent of today, and so forth - the unique SPE is (always enter, always share) so entry occurs in period $t = 1$.

b)** Now assume that the game is repeated an infinite number of times. Propose strategies for both players that could result in Entry being deterred, and show that — if adopted – these strategies form a SPE in which the Entrant indeed never enters if both players' common discount factor satisfies $\delta \geq \frac{4}{9}$ (Note: in order to do this you will have to show that these strategies are optimal for **each** history of the game). Give an intuition for this result.

The easiest way to think of this game is a situation in which the Incumbent is facing an infinite number of potential entrants, one each period. Now, we already know that if the Incumbent can ***credibly*** *threaten to fight if the Entrant enters, the Entrant will stay out. But how can fighting, albeit being costly, still be credible? The answer must lie in the fact that not fighting (sharing the market) is "punished" by having the Entrant enter in all subsequent periods. In constructing the corresponding strategies, though, we have to take good care of the strategies being sequentially rational at each node of the tree, i.e., for each history of the game. Let's see how this works. Consider the following pair of strategies:*

Incumbent: Fight entry in the first period and in all subsequent periods, unless there was market entry in the past and you shared the market. In all other circumstances, fight.

Entrants: Stay out in the first period and in all subsequent periods unless you have seen an entry occurring and the Incumbent sharing the market in the past; in particular, if an entry ever occurred in the past and the incumbent fought, stay out.

Let's check when these strategies are optimal for each history of the game. In period t, there are three broad possibilities for what happened previously. First, entry never occurred. In this case, if the Incumbent fights entry in t (assuming entry happens), this will keep entrants out in all subsequent periods given their strategy. We have:

$$V_i(\text{fight}) = 1 + (\delta + \delta^2 + \ldots)10 = 1 + \frac{\delta}{1-\delta}10.$$

If the Incumbent instead shares the market, entry will occur in all periods $t + 1$

onwards. The present value from sharing is thus:

$$V_i(\text{share}) = 5(1 + \delta + \delta^2 + \ldots) = \frac{1}{1-\delta}5.$$

Fighting is thus better than sharing in t if $V_i(\text{fight}) > V_i(\text{share})$ or

$$\frac{1}{1-\delta}5 < 1 + \frac{\delta}{1-\delta}10 \quad \Leftrightarrow \quad \delta \geq \frac{4}{9}.$$

We still have to check whether the strategies are sequentially optimal for the other possible histories, though! The second possibility is that entry has occurred in the past and the monopolist fought. In this case, the strategies call for the Incumbent to fight and for the entrant to stay out. The Incumbent will fight if $\delta \geq 4/9$ by the same argument as above, and thus it is rational for the entrants to keep staying out. Third, entry occurred and the monopolist shared the market. In this case, the entrants will enter according to their strategy, and the monopolist will share. This is clearly sequentially rational, if only because it is a SPE of the one-shot game (and thus continues to be a SPE of the repeated game). The intuition for the result that entry is now deterred is that there is no last period in this model. Each period, the Incumbent faces an infinite number of times entry (today and into the future), which makes fighting optimal given that subsequently, the entrant won't enter if the Incumbent fought entry in the past.

c) A different of the game with the same incumbent but a series of entrants in distinct geographic markets is called *Chain Store Game*: a monopolist has n branches in n towns, and plays the Entry Game in each town, with entrants observing the incumbent's behaviour in other towns. Assume n is large but finite – would your answer in a) change? In experiments designed to study actual behavior, people rarely play according to the theoretical predictions: incumbents tend to fight early on and entry is deterred. Can you think of ways to modify the original game so as to generate equilibria with this property? *We can use backward induction (rollback). Suppose we reach the last town (final round of the game); the Incumbent will do better to "Share" than to "Fight". Correctly anticipating the Incumbent's best response, the nth Entrant will always "Enter". Note that this will happen irrespective of the history of the game, i.e.,* ***irrespective of whether the Incumbent has fought entry before or not****. Now consider the second to last town. The Incumbent knows that even if he fights the $n-1$th entrant, he will not be able to deter entry by the nth entrant (as we just argued, this entrant will always enter) Thus, fighting in the $n-1$th city cannot deter entry in the nth city – the outcome in the nth city is independent of what the incumbent did previously. As a result, the best response to entry by the $n-1$th* *entrant is to share, not to fight. Anticipating this best response, the entrant in the $n-1$th city will "Enter". Again this is* ***independent of the history up to that point****. Applying this argument to all previous cities (rounds) we arrive at the unique SPE which has the entrant "Enter" and the Incumbent "Share" in all n cities. This result is called the Chain Store Paradox, since it is seemingly counter intuitive: the monopolist should fight in a market (even if that is costly) given that early aggression could deter entry later. But the problem with the chain store paradox in the finitely repeated game is that there is a last period in which the incumbent firm will share the market because its reputation plays no longer a role. But this essentially makes the second-to-last period the last period, and so the game unravels. If there is no last period, however, as in the infinite game, reputation is always important: If the incumbent faces a never-ending stream of potential entrants, fighting an entrant can be part of the SPE. Another – and entirely different – possibility is imperfect information on the part of the entrant. Suppose that from the entrants' point of view, there is a (possibly very small) probability that the Incumbent is of a crazy type. Crazy types are not rational and always fight, even if that's not a best response. In such a situation, it could pay a rational Incumbent to fight the first entrant to make all subsequent entrants believe he is the crazy type. This would give rise to an equilibrium in which entry is fought with positive probability even if the Incumbent is perfectly rational.*