Optimal Crime Networks – Theory and Lessons for Policy

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Abstract

We construct a social network model of criminal activity. Agents' payoffs depend on the number and the structure of their connections with each other and are determined in a Nash equilibrium of a crime activity supply game. Unlike much of the literature which takes the network structure as given, we study *optimal networks*, defined as the networks that maximize the sum of agents' payoffs. We characterize the Nash equilibrium in crime activity and use our theoretical results to identify the optimal network for given cost and benefit parameters using an algorithm that searches over all possible non-isomorphic graphs of given size. We also analyze, via simulations, the effects of different anti-crime policies (both expected and unexpected) on the optimal crime network structure and the overall crime level – removing agents, removing links, and/or varying the probability of apprehension.

PRELIMINARY AND INCOMPLETE

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1 Introduction

Recently there has been considerable theoretical work as well as emerging empirical studies integrating social networks, defined as graphs with nodes being economic agents and edges being various links connecting them, into economic models (Jackson, 2003, 2004, 2006, 2007, 2008). Various economic applications have benefited from this approach, including studies of delinquency, crime, prisons, job search, social norms, human capital investment and social mobility, among many others. However, much of the research on networks in economic contexts continues the tradition from sociology and psychology and takes networks as exogenously given.[1] Clearly some social networks are and should be treated as exogenous to the actor's decision process. For example, one does not choose one's relatives or kin relations and such fixed social relationships have been important applications of network theory. In principle, however, there is no reason why the network structure itself should not be part of the decisions made by economic agents. Our approach here is to model the network structure as endogenously emerging from solving an optimization problem.

Specifically, we develop a formal social network model of criminal activity. Social networks are more relevant in situations in which markets fail, for example as a consequence of high transaction costs caused by asymmetric information, limited enforcement, externalities and the like. Crime is a prime example of an economic activity conducted in such environment – illegal organizations cannot rely on formal means of enforcing contracts or sharing information, thus costly alternative mechanisms for performing their activities must be used. Informal institutions based on inter-personal interaction such as social networks naturally provide such a mechanism.

In our model, the agents' payoffs (net incomes) depend on the number and structure of the links connecting them. These links can be viewed as information channels, indicators of the agents' ability to meet or work together, and the like. Importantly, the agents' payoffs also depend on an individually chosen action – the level of "crime effort" supplied by each criminal network member. We assume that these crime efforts are determined in a Nash equilibrium of a simultaneous move game. We then analyze the optimal network structure – that is, the patterns of links among agents that maximize the aggregate payoff of the networked agents, as well as the associated total crime level.

On the theory side, our analysis sheds light on the following questions. What conditions ensure the existence or uniqueness of Nash equilibrium in the crime effort choice game? Is the equilibrium interior or features 'corners', that is, idle agents supplying zero effort? For given cost and benefit parameter values for which a Nash equilibrium exists, what network structures maximize total payoff and what is the associated criminal activity level?

We first derive theoretical results that guarantee the existence and uniqueness of Nash equilibria for interior and corner solutions in effort choice. We obtain these results using a *potential function* associated with individual payoffs.[2] We show that, as long as the parameter that controls the strength of congestion in the network is greater or equal to the parameter that determines the positive effect of being connected to other agents, there exists a Nash

[1] There is a small literature that considers network choice, e.g., Hojman and Szeidl (2006) look at dyadic decisions that lead to an equilibrium network.

[2] We borrow and adapt to our setting techniques from Bramoullé, Kranton and D'Amours (2011).

equilibrium. We also show that equilibrium uniqueness depends on the relative strength of the costs and benefits of being in the network and the minimum eigenvalue of a matrix derived from the network adjacency matrix.

We then characterize the equilibrium total crime effort and total payoff. We prove that in our setting the individual equilibrium payoff is quadratic in the optimal effort level and that agents who exert zero effort in equilibrium are linked to agents who supply less effort in comparison to the neighbors of an agent with positive effort in equilibrium. We also compare the individual (Nash) effort choice problem with the corresponding planner's problem and prove that the full network (completely connected) maximizes both the total crime level and the overall payoff if the planner maximizes the sum of payoffs. This is not the case in Nash equilibrium where the total payoff may be maximized for a different network structure.

Analytically, we model an N-member network. Each of the N agents has a payoff function, U_i representing the net benefit ("income" net of costs) that the agent obtains from interacting with others. The network structure, G is a crucial determinant of an agent's benefits and costs. In our application to criminal networks we assume that an agent's payoff depends on: (i) the level of criminal activity ("effort") an agent performs, e_i and (ii) the number of connections (links) he has with other agents. The number and pattern of links affects on the one hand the income (benefit) of an agent (e.g., through cooperation) and, on the other hand, his costs (e.g., by raising the probability of apprehension). The total amount of crime activity is determined in Nash equilibrium whereby each agent $i = 1, ..., N$ simultaneously chooses his effort level, e_i, taking the efforts of all other network members, e_j, $j \neq i$ as given.

Unlike much of the literature which takes the network structure as given, to find the optimal (joint-payoff maximizing) network we use a computational algorithm developed in applied mathematics (see Appendix 1) to search over all possible *non-isomorphic* graphs of given size, that is, all networks that cannot be obtained from each other by relabeling the nodes. It becomes computationally infeasible very quickly to search over all possible networks of given size without focusing on non-isomorphic networks. Finding all such networks (also known as 'simple graphs') is a complex combinatorial problem for which no algorithm to solve it for any N exists.[3] The problem has been solved, however, for small network sizes ($N = 2$ to $N = 11$ players).[4] We use the computed solution data – that is, the list of all non-isomorphic networks of a given size, as an input to our search algorithm. For each non-isomorphic network and set of model parameters for which a Nash equilibrium in crime efforts exists, we compute the individual and aggregate crime and payoff levels. We then use this information to find the optimal network, defined as the aggregate payoff maximizing network among all non-isomorphic networks of size N) for a large representative set of model parameters. We analyze the structure of the optimal network and associated crime levels as the cost and benefit structural parameters vary as well as the frequency with which various structures emerge as optimal.

Finally, we use the numerical simulations data described above to assess the relative effectiveness of various possible crime-reducing policies: removing players, removing links, and/or

[3]In fact, this problem belongs to its own class in complexity theory called 'graph isomorphism complete' and is thought to be non-verifiable in non-deterministic polynomial time (NP-complete) – see Skiena (1981).

[4]See McKay (1981) for an early algorithm description as well as B. McKay's webpage: http://cs.anu.edu.au/people/bdm/

varying the probability of apprehension. We look at the effect of these policies on the optimal network and answer the following questions. For a particular network structure, what constitutes the optimal crime deterrence policy? How does the optimal network structure respond to the crime prevention technique used? Conversely, given a particular deterrence policy, what is the optimal network structure that arises to minimize the damage to the criminal network inflicted by the policy? What structures and crime levels emerge as the joint outcome of trying to find the most effective policy and the criminal network most robust to damage given that policy? The results indicate that in many cases the optimal structures are special networks identified in the theoretical literature: the "line", the "wheel", the "star" and the "complete" network (Bala and Goyal, 2000). We also find evidence of the optimality of "cell" type structures. The policy analysis is able to take into account the optimal network structure emerging as a result of an announced policy, both in the short run holding the network as fixed, and in the long run when the network can be re-optimized.

Related literature

The role of networks in organized crime has begun to be studied by criminologists – e.g., see Sarnecki (2001), Bruinsma and Bernasco (2004). Kenny (2007) reviews the recent literature. Other areas of crime have also benefited from more explicit use of network theory, including human trafficking, crime groups formed by youth, and drug distribution (Hughes, 2000; Coles, 2001; Frank, 2001 and Hoffer, 2002). The events of September 11 and the discovery of the "cell"-type network structure of Al Qaeda have spawned policy work devoted to combating terrorist networks (for example, Carley et al., 2001; Krebs, 2001; Raab and Milward, 2003). Kenny (2007) reviews much of the criminology literature that uses some form of network analysis. Easton and Karaivanov (2009) provide a non-technical version of the model studied here and give some simple examples of policy applications.

In economics, Ballester, Calvo-Armengol and Zenou (2004) (hereafter, BCZ) provide one of the first economic treatments of crime networks. They concentrate on identifying the 'key player' in a given network – that is, the agent whose removal leads to the greatest decline in criminal activity. While our approach borrows from BCZ (2004), we adopt different assumptions about how the agents' costs and benefits depend on the network structure and total effort level. Additionally, in contrast to BCZ we do not take the network structure as exogenous but optimize over all possible networks of a given size, as well as across different sizes. Furthermore, we analyze and compare the effects of various alternative crime prevention policies in addition to the "removal of key player" policy.

The paper is organized as follows. Section 2 describes the basic theoretical model. In Section 3, we introduce the potential function and derive necessary and sufficient conditions for existence and uniqueness of Nash equilibria in crime effort. Section 4 discusses various properties of the equilibrium and the associated crime networks, including results that we derive by comparing the Nash equilibrium effort choice problem to a planner's problem. In Section 5 we motivate and discuss how our theoretical setting can be used to study optimal criminal networks and the crime level under alternative crime-deterring policy environments. Section 6 concludes.

2 The Model

2.1 Basics

There are N agents whose interaction we model as a social network (graph): a set of nodes representing individual agents (players) and the links between them. We assume that all links are bidirectional, so that if player i is connected to player j, then the reverse is also true. The network structure can be fully summarized by its 'adjacency matrix', G – an N-by-N matrix with zeros on the main diagonal (by convention) and elements, g_{ij} equal to 1 if players i and j are connected and 0 otherwise. The assumed bidirectional links implies that the matrix G is symmetric.

For a given network structure G, each agent, $i = 1, ..., N$ decides on the level of criminal activity (hereafter, "effort"), $e_i \geq 0$ to supply in order to maximize his payoff (benefits minus costs). The crime efforts are chosen by all agents simultaneously and non-cooperatively, in a Nash equilibrium (see Section ?? in which we study the cooperative game). We assume a quadratic form for the benefit and cost functions, similarly to BCZ (2004). This results in a linear system of equations to be solved for the equilibrium crime levels. An important difference with BCZ is that we assume that benefits from criminal activity *increase* in the number of connections an agent has to others, while the total amount of crime, that is the sum of crime efforts, creates a congestion effect which raises one's costs of being in the network(for example, because of greater likelihood of being detected or harder to find opportunities for crime).[5] In addition, we also allow for costs of maintaining a link between agents and for a 'standalone' cost of effort, independent of other's actions.

Specifically, let agent i's payoff, U_i (which can be thought of as utility or net income) be given by:

$$U_i(G, \mathbf{e}) = y_i(G, \mathbf{e}) - c_i(G, \mathbf{e})$$

where:

$$y_i(G, \mathbf{e}) = e_i(1 + \gamma \sum_{j=1}^{N} g_{ij} e_j)$$

and

$$c_i(G, \mathbf{e}) = e_i(\pi + \lambda \sum_{j=1}^{N} e_j + \delta \sum_{j=1}^{N} g_{ij})$$

and where $\mathbf{e}$ denotes the vector of the agents' efforts, $\mathbf{e} \equiv (e_1, ..., e_N)$.

The parameter $\gamma \geq 0$ determines the strength of the benefit from having links ($g_{ij} = 1$) with other agents, while the parameter $\lambda \geq 0$ determines the strength of the 'congestion' effect from aggregate crime activity. We allow for the possibility of costly link maintenance through the parameter $\delta \geq 0$. The parameter $\pi \in [0, 1)$ is the standalone cost of unit of effort.

The optimal effort choices e_i^* for all $i = 1, ...N$ are determined in a Nash equilibrium whereby each agent maximizes his payoff U_i taking the other agents' effort levels as given and subject

[5] In contrast, BCZ (2004) assume that individual benefits increase in the total amount of crime while individual costs decrease in the number of connections to other criminals.

to the non-negativity constraints $e_i \geq 0$. The resulting first order conditions (best response equations), in matrix form, are:[6]

$$[\beta_1 \mathbf{I} - \phi_2 G]\mathbf{1} - [(\mathbf{J} + \mathbf{I})\phi_1 - G]\mathbf{e} \leq \mathbf{0} \text{ with equality if } e_i > 0 \tag{1}$$

where $\phi_1 \equiv \frac{\lambda}{\gamma}$, $\phi_2 \equiv \frac{\delta}{\gamma}$, $\beta_1 \equiv \frac{1-\pi}{\gamma}$ and where $\mathbf{1}$ is a N-by-1 vector of ones, $\mathbf{I}$ is the N-by-N identity matrix and $\mathbf{J}$ is an N-by-N matrix of ones. Since U_i are strictly concave in e_i the first-order conditions (1) are necessary and sufficient for optimum.

Definition 1.

A Nash equilibrium (NE) is a vector of crime efforts, $\mathbf{e} = (e_1, ..., e_N)$ *such that (1) hold for all* $i = 1, ..., N$. *We call a NE 'interior' if* $e_i > 0$ *for all* i. *A NE is called 'corner' if* $e_i = 0$ *for at least one* $i = 1, ..., N$.

In the interior NE case, equations (1) form a linear system the solution of which (when it exists) is the equilibrium vector of crime levels.[7] Clearly, as long as $\det((\mathbf{J} + \mathbf{I})\phi_1 - G) \neq 0$ which happens on a set of Lebesgue measure zero, $\mathcal{Z}$, the system

$$[\beta_1 \mathbf{I} - \phi_2 G]\mathbf{1} - [(\mathbf{J} + \mathbf{I})\phi_1 - G]\mathbf{e} = \mathbf{0} \tag{2}$$

has a unique solution. Note that $\beta_1 \mathbf{I} - \phi_2 G \neq \mathbf{0}$ since $\beta_1 > 0$ and since the diagonal elements of G are zero. We assume $\phi_1 \notin \mathcal{Z}$. Using arguments similar to those in Proposition 1 in BCZ (2004), it is easy to show that if $\phi_1 \notin \mathcal{Z}$ is large enough and if ϕ_2 is small enough, the system (2) has a solution consisting of strictly positive e_i for all i. BCZ (2004) restrict attention only to parametrizations which satisfy these conditions.

In general, however, there is no guarantee that a NE defined by (1) is interior. We thus analyze a more general class of equilibria (or parameter configurations), only requiring that agents' efforts be non-negative instead of strictly positive (that is, we allow corner solutions of the best response equations). In words, given the crime efforts chosen by everyone else, in a NE agent i should have no incentive to vary his effort, unless he is constrained, in which case he might like to reduce e_i but this is infeasible. Because of the non-negativity constraints one cannot find the equilibrium choices e_i^* by simply solving the linear system (2). We therefore adopt a different approach, based on quadratic programming, which allows us to obtain all Nash equilibria for a wider range of parameter values for which a non-negative solution to (1) exists.[8] The full details are presented in Appendix 2.

2.2 Existence of equilibrium

To characterize the Nash equilibria (NE) in crime efforts, we initially focus on the case $\delta = 0$, that is, when there are no link maintenance costs. We relax this assumption for some results

[6] The inequality sign is interpreted element by element.

[7] Appendix 2 shows an example for $N = 4$.

[8] In the numerical simulations we first solve the system (2) ignoring the non-negativity constraints, and if we have negative e_i we switch to the (computationally slower) quadratic programming approach from Appendix 2.

later on, as indicated. In the following analysis of existence and uniqueness of NE we draw upon some results by Bramoullé, Kranton and D'Amours (2011) (hereafter, BKD).

Given $\delta = 0$ we can write the individual payoff function as,

$$U_i(G, \mathbf{e}) = e_i(1 - \pi) - \lambda \sum_{j=1}^{N} e_i e_j + \gamma \sum_{j=1}^{N} g_{ij} e_i e_j$$

From the first order conditions (1), the optimal effort level of agent i given the efforts of others, e_j satisfies:

$$e_i = \max\{0, \beta - \phi \sum_{j=1}^{N} a_{ij} e_j\}.$$

where $\beta \equiv \frac{1-\pi}{2\lambda}$, $\phi \equiv \frac{\gamma}{2\lambda}$ and where $\{a_{ij}\}_{i,j=1}^{N}$ are the elements of the N-by-N matrix

$$\mathbf{A} = \big[\frac{\lambda}{\gamma}(\mathbf{J} - \mathbf{I}) - \mathbf{G}\big].$$

Normalizing all efforts $\mathbf{e}$ by $\beta \neq 0$ by calling $\hat{\mathbf{e}} = \frac{\mathbf{e}}{\beta}$, the following must hold in a NE:

$$\hat{e}_i = f_i(\mathbf{e}, \mathbf{A}) \equiv \max\{0, 1 - \phi \sum_{j=1}^{N} a_{ij} \hat{e}_j\} \text{ for all } i = 1, ..., N \tag{3}$$

Therefore, when $\delta = 0$, a Nash equilibrium is a vector of efforts, $\hat{\mathbf{e}}$ which satisfies $\hat{e}_i = f_i(\hat{\mathbf{e}}, \mathbf{A})$ for all $i = 1, ..., N$.

To interpret the best response functions (3), look closer at the elements of the matrix $\mathbf{A}$,

$$\begin{cases} a_{ij} = \frac{\lambda}{\gamma} - 1 & \text{if } g_{ij} = 1 \\ a_{ij} = \frac{\lambda}{\gamma} & \text{if } g_{ij} = 0 \\ a_{ij} = 0 & \text{if } i = j \end{cases}$$

Observe that $\mathbf{A}$ is obtained from the adjacency matrix G and can be thought of as representing a network in which all agents are connected but the "weight" of the links between them differs depending on whether or not the pair of agents are connected in the original network G. Depending on the values of the cost and benefit parameters λ and γ, the elements a_{ij} for $i, j = 1, ..., N$ can be positive, zero or negative. If $a_{ij} \geq 0$, we can think of the effort supply game as a game of strategic substitutes while when $a_{ij} \in [-1, 0]$ we have a game of strategic complements.

The following Lemma states the condition for existence of Nash equilibria.

Lemma 1. ***Existence of equilibrium***. *The vector-valued best response function* $\mathbf{f}(\mathbf{e}, \mathbf{A})$ *has a fixed point and hence at least one Nash equilibrium (NE) exists for all networks* G *of given size* N, *if and only if* $\phi \leq 1/2$ *(that is,* $\gamma \leq \lambda$*).*

Proof: By Brouwer's Theorem, the best response function $\mathbf{f}(\mathbf{e}, \mathbf{A})$ with elements $f_i(\mathbf{e}, \mathbf{A}) =$

$\max\{0, 1 - \phi \sum_{j=1}^{N} a_{ij}\hat{e}_j\}$ has a fixed point if and only if it is a continuous function mapping the convex and compact set $[0,1]^N$ into itself. Note that f_i are continuous and non-negative by construction, so we only need to verify that $f_i \leq 1$ for all i. Suppose $\phi \leq 1/2$ (or, $\lambda \geq \gamma$) holds. This implies $a_{ij} \geq 0$ for all i, j, which, since $\phi > 0$ and $e_j \geq 0$, is a sufficient condition for $f_i \leq 1$ for all i. To show that $\phi \leq 1/2$ is necessary, note that, by the definition of f_i, if $1 - \phi \sum_{j=1}^{N} a_{ij}\hat{e}_j \leq 0$, then $\hat{e}_i = 0 = f_i < 1$. We also have $\hat{e}_i = 1 - \phi \sum_{j=1}^{N} a_{ij}\hat{e}_j$ if $\hat{e}_i > 0$. In this case, requiring $f_i = 1 - \phi \sum_{j=1}^{N} a_{ij}\hat{e}_j \leq 1$ implies $\phi \sum_{j=1}^{N} a_{ij}\hat{e}_j \geq 0$, or $\frac{\gamma}{2\lambda} \sum_{j \neq i} (\frac{\lambda}{\gamma} - g_{ij})\hat{e}_j \geq 0$, that is, $\lambda \sum_{j \neq i} \hat{e}_j \geq \gamma \sum_{j \neq i} g_{ij}\hat{e}_i$. Note that by the definition of G the inequality $\sum_{j \neq i} \hat{e}_j \geq \sum_{j \neq i} g_{ij}\hat{e}_i$ is true for all G and holds with equality for the full network ($G = \mathbf{J} - \mathbf{I}$, for which $\hat{e}_i > 0$). Therefore, to have $f_i \leq 1$ for *all* G, it is necessary that $\lambda \geq \gamma$ (equivalently, $\phi \leq 1/2$). ■

In the next sub-section, we derive necessary and sufficient conditions for uniqueness of Nash equilibria and discuss the conditions for interior and corner equilibria.

2.3 The potential function and uniqueness of equilibrium

We next study the question of uniqueness of Nash equilibria. To proceed, it helps to express the agent's maximization problem in terms of a *potential function* associated to the payoff function U_i. This sub-section draws on some techniques from BKD (2012) and applies them to our different setting.

Definition 2. Potential function

The function $\Phi(e_i, e_{-i})$ is a potential function of the game $\{U_i, E_i\}_{i=1}^{N}$ with strategy sets E_i which are real intervals and continuously differentiable payoffs U_i, if and only if Φ is continuously differentiable and, for all $i = 1, ..., N$,[9]

$$\frac{\partial U_i(e_i, \mathbf{e}_{-i})}{\partial e_i} = \frac{\partial \Phi(e_i, \mathbf{e}_{-i})}{\partial e_i} \quad \forall e_i \in E_i,\ \forall \mathbf{e}_{-i} \in \mathbf{E}_{-i}$$

Proposition 1. *The effort levels, $\mathbf{e}$ is a NE of the effort supply game with best response functions $f_i = \max\{0, 1 - \phi \sum_{j=1}^{N} a_{ij}e_j\}$ if and only if $\mathbf{e}$ satisfies the Kuhn-Tucker first-order conditions of the problem*

$$\max_{e_i} \ \Phi(\mathbf{e}, G) \quad s.t. \quad e_i \geq 0 \text{ for all } i \tag{4}$$

where Φ is the potential function of the effort supply game, defined as:

$$\Phi(\mathbf{e}, G) = \sum_{i=1}^{N} [e_i(1 - \pi) - \lambda e_i^2] - \frac{\gamma}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} e_i e_j$$

All NE correspond to the set of maxima and saddle points of problem (4).

Proof: We first show that Φ defined above is a potential function of our effort supply game. Clearly Φ is continuously differentiable. By definition, Φ is a potential function for our game

[9] See Monderer and Shapley (1996) for details.

with payoffs U_i if and only if $\frac{\partial U_i}{\partial e_i} = \frac{\partial \Phi}{\partial e_i}$ for all $i = 1, ..., N$. This can be verified directly by taking the partial derivatives of Φ with respect to e_i:

$$\frac{\partial \Phi}{\partial e_i} = (1-\pi) - 2\lambda e_i - \gamma \sum_{j=1}^{N} a_{ij} e_j = \frac{\partial U_i}{\partial e_i}$$

We also have,

$$\frac{\partial \Phi}{\partial e_i} = 0 \quad \Rightarrow \quad e_i^* = \frac{1-\pi}{2\lambda} - \frac{\gamma}{2\lambda} \sum_{j=1}^{N} a_{ij} e_j^* = \beta - \phi \sum_{j=1}^{N} a_{ij} e_j^*$$

Therefore the N individual maximization problems of finding the optimal crime efforts, e_i can be re-written as the single constrained optimization problem:

$$\max_{e_i} \quad \Phi(\mathbf{e}, G) \quad \text{s.t.} \;\; e_i \geq 0 \text{ for all } i$$

Among the Kuhn-Tucker conditions associated with this problem are:

$$\frac{\partial \Phi}{\partial e_i} = 0 \;\; \text{if} \;\; e_i > 0$$
$$\frac{\partial \Phi}{\partial e_i} \leq 0 \;\; \text{if} \;\; e_i = 0$$

It is easy to see that these conditions correspond to the individual best response functions f_i exhibited earlier. Hence, the set of NE for any given network G coincides with the set of extrema and saddle points of the potential function $\Phi(\mathbf{e}, G)$ on $\mathbb{R}_+^n$.■

Re-write the potential function Φ using matrix notation:

$$\begin{aligned}
\Phi(\mathbf{e}, G) &= \sum_{i=1}^{N} [e_i(1-\pi) - \lambda e_i^2] - \frac{\gamma}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} e_i e_j \\
&= \lambda \Big\{ \sum_{i=1}^{N} [(\frac{1-\pi}{\lambda}) e_i - e_i^2] - \frac{\gamma}{2\lambda} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} e_i e_j \Big\} \\
&= \lambda \Big\{ (\frac{1-\pi}{\lambda}) \mathbf{e}^{\mathbf{T}} \mathbf{1} - \mathbf{e}^{\mathbf{T}} (\mathbf{I} + \phi \mathbf{A}) \mathbf{e} \Big\}
\end{aligned}$$

This implies that the Hessian matrix of Φ is,

$$\nabla^2 \Phi = -\lambda(\mathbf{I} + \phi \mathbf{A})$$

The next result provides conditions for the uniqueness of Nash equilibrium in our setting.

Proposition 2. *Suppose $\phi \leq 1/2$. A NE is unique if and only if $\phi < -\frac{1}{\alpha_{\min}(\mathbf{A})}$ where $\alpha_{\min}(\mathbf{A})$ is the smallest eigenvalue of the matrix $\mathbf{A}$.*

Proof: The quadratic potential function Φ has a unique maximum if and only if the matrix $\mathbf{I} + \phi\mathbf{A}$ (the negative of $\Phi's$ Hessian) is positive definite. BKD (2011) show that the matrix $\mathbf{I}+\phi\mathbf{A}$ is positive definite if and only if $\phi < -\frac{1}{\alpha_{\min}(\mathbf{A})}$, where $\alpha_{\min}(\mathbf{A})$ is the minimum eigenvalue of the matrix $\mathbf{A} = [\frac{\lambda}{\gamma}(\mathbf{J} - \mathbf{I}) - \mathbf{G}]$.■

2.4 Types of equilibria

In Proposition 3 we showed that if $\phi < -\frac{1}{\alpha_{\min}(\mathbf{A})}$ the potential function is strictly concave and there exists a unique NE in the effort supply game. This equilibrium can be either interior (all $e_i > 0$) or corner (there is some $e_i = 0$). When $\phi > -\frac{1}{\alpha_{\min}(\mathbf{A})}$, the function Φ is not strictly concave and there may exist multiple equilibria.[10] It is easy to see that for such parameters, there always exists a corner equilibrium – because Φ is a non-concave function, there is a direction along which it increases without bound, therefore there exists at least one maximum that is not interior. In addition, we know from Proposition 3 that any vector $\mathbf{e}(\phi, \mathbf{A})$ which globally maximizes Φ is a Nash equilibrium. Therefore, in this range of parameters there is always a Nash equilibrium which features a corner solution.

Lemma 2. *If $\phi > -\frac{1}{\alpha_{\min}(\mathbf{A})}$ and $\phi \leq 1/2$, then there exists a corner NE (some i for which $\hat{e}_i = 0$).*

However, we can show that it is never optimal for all agents to choose zero effort level in a Nash equilibrium.

Proposition 3. *There does not exist a NE in which $e_i = 0$, $\forall i = 1, ..., N$.*

Proof: By the first order conditions of the individual maximization problems,

$$e_i^* = \frac{1-\pi}{2\lambda} - \frac{\gamma}{2\lambda}\sum_{j\neq i}(\frac{\lambda}{\gamma} - g_{ij})e_j \quad \text{if} \quad e_i > 0$$
$$e_i^* = 0 \quad \text{if} \quad \frac{1-\pi}{2\lambda} < \frac{\gamma}{2\lambda}\sum_{j\neq i}(\frac{\lambda}{\gamma} - g_{ij})e_j \tag{5}$$

If $e_j = 0$, $\forall j \neq i$, then agent i's optimal effort level is $\frac{1-\pi}{2\lambda} > 0$. Conversly, the only way that agent i would choose zero effort is when there are some agent(s) whose effort level is larger than zero so that the right hand side of inequality (5) is larger than the left hand side. Therefore it is never optimal for all agents to choose zero effort in equilibrium. ■

3 Crime networks – properties

In the previous section we showed conditions for existence and uniqueness of Nash equilibria in agents' efforts. We now characterize the properties of the Nash equilibria. Re-arrange agent

[10] If $\phi = -\frac{1}{\alpha_{\min}(A)}$ multiple equilibria are possible and all equilibria form a convex set yielding the same aggregate effort (see BKD).

i's best response equation at an interior solution as

$$e_i = \frac{1-\pi}{2\lambda} - \frac{\delta}{2\lambda}\sum_{j=1}^{N} g_{ij} + \frac{\gamma}{2\lambda}\sum_{j=1}^{N} g_{ij}e_j - \frac{1}{2}\sum_{j\neq i}^{N} e_j$$

The first term, $\frac{1-\pi}{2\lambda}$, can be thought of as a "standalone" or autarky level of effort. The sum in the second term $\frac{\delta}{2\lambda}\sum_{j=1}^{N} g_{ij}$ can be broken into two parts: agents who are directly linked to i ($g_{ij}=1$) and agents who are not directly linked to i ($g_{ij}=0$). Consequently, it equals $\frac{\delta}{2\lambda}l_i$ where l_i is the number of direct links that player i has with other agents. Denoting with $L(i)$ the set of agents with whom agent i has direct links, we can express agent i's equilibrium effort at an interior solution as:

$$e_i = e^{autarky} + \frac{\gamma}{2\lambda}\sum_{L(i)} g_{ij}e_j - \frac{\delta l_i}{2\lambda} - \frac{1}{2}\sum_{j\neq i}^{N} e_j \tag{6}$$

In the above expression the autarkic effort level, $e^{autarky} \equiv \frac{1-\pi}{2\lambda}$ is augmented by the benefit associated with links to others $\frac{\gamma}{2\lambda}\sum_{L(i)} g_{ij}e_j$ and reduced by the costs of links with others $\frac{\delta l_i}{2\lambda}$ and the congestion term (e.g., due to the increased likelihood of being caught), $\frac{1}{2}\sum_{j\neq i}^{N} e_j$.

Proposition 4. *Individual equilibrium payoffs are quadratic in own equilibrium effort, e_i^*, that is, $U_i^* = \lambda(e_i^*)^2$ for all i.*

Proof: From the earlier expression for the agent's payoff, U_i, we have:

$$\begin{aligned}
U_i^* &= e_i^*\left[(1-\pi) + \gamma\sum_{j=1}^{N} g_{ij}e_j^* - \lambda\sum_{j=1}^{N} e_j^* - \delta\sum_{j=1}^{N} g_{ij})\right] = \\
&= e_i^*\left[(1-\pi) + \sum_{j=1}^{N}(\gamma g_{ij}-\lambda)\, e_j^* - \delta l_i\right] = \\
&= e_i^*\left[(1-\pi) - \delta l_i + (\gamma-\lambda)\sum_{j\in L(i)} e_j^* - \lambda\sum_{j\notin L(i)} e_j^* - \lambda e_i^*\right]
\end{aligned}$$

On the other hand we know that at an interior solution (from the FOCs):

$$\begin{aligned}
e_i^* &= \frac{1-\pi}{2\lambda} - \frac{\delta l_i}{2\lambda} - \frac{1}{2}\sum_{j\neq i} e_j^* + \frac{\gamma}{2\lambda}\sum_{j\in L(i)} e_j^* \\
&= \frac{1}{2\lambda}\left[1-\pi-\delta l_i + (\gamma-\lambda)\sum_{j\in L(i)} e_j^* - \lambda\sum_{j\notin L(i)} e_j^*\right].
\end{aligned} \tag{7}$$

Combine the last expression for the payoff and the last expression for e_i^*, we have that:

$$U_i^* = \lambda(e_i^*)^2 \qquad \forall i = 1, ..N$$

Thus, for an interior e_i^* interior, the agent's payoff is always positive and proportional to the square of the agent's own effort in equilibrium. If the agent chooses a corner solution ($e_i^* = 0$) we trivially have $U_i^* = 0 = \lambda(e_i^*)^2$ as well.■

We go on to characterize some properties of the aggregate crime level in equilibrium.

Proposition 5. *Suppose $\delta = 0$. Then:*
(a) The network maximizing aggregate crime, $E \equiv \sum_{i=1}^N e_i^*$ is the "full network" – the network in which all agents are connected to all other agents.
(b) If, for given N, the network G' is obtained from G by adding more links, that is, $g_{ij} \leq g_{ij}^{'}$ $\forall i, j$, then total crime acticity is higher in network G' compared to in network G, that is, $\sum_{i=1}^N e_i^*(G) \leq \sum_{i=1}^N e_i^*(G')$.
(c) The maximum value of total crime in equilibrium is increasing in the size of the network, N.

Proof: (a) From Proposition 3 we know that as long as $\pi < 1$, there does not exist a NE with $e_i = 0, \forall i$. Therefore, there is always at least one agent who chooses positive effort level. When $\delta = 0$, this agent's effort is:

$$\begin{aligned}
& e_i = \frac{1-\pi}{2\lambda} + \frac{\gamma}{2\lambda}\sum_{j=1}^N g_{ij}e_j - \frac{1}{2}\sum_{j\neq i}^N e_j \\
& \Rightarrow \quad \sum_{i=1}^N e_i \leq \frac{N(1-\pi)}{2\lambda} + \frac{\gamma}{2\lambda}\sum_{i=1}^N\sum_{j=1}^N g_{ij}e_j - \frac{1}{2}\sum_{i=1}^N\sum_{j\neq i}^N e_j \\
& \text{Note} \quad \sum_{j=1}^N g_{ij}e_j \leq \sum_{j=1}^N e_j - e_i \quad \Rightarrow \quad \sum_{i=1}^N\sum_{j=1}^N g_{ij}e_j \leq (N-1)E \\
& \text{Also} \quad \sum_{j\neq i}^N e_j = \sum_{j=1}^N e_j - e_i \quad \Rightarrow \quad \sum_{i=1}^N\sum_{j\neq i}^N e_j = (N-1)E \\
& \Rightarrow \quad E \leq \frac{N(1-\pi)}{2\lambda} + \frac{\gamma(N-1)}{2\lambda}E - \frac{N-1}{2}E \\
& \Rightarrow \quad E \leq \frac{1-\pi}{\frac{\lambda+\gamma}{N} + \lambda - \gamma}
\end{aligned}$$

for all networks G, with equality only if G is the full network. Thus total crime, E is maximized when G is the full network since ϕ_1, β_1 and N do not depend on G.

(b) This result follows similarly to as in BCZ, Proposition 2.

(c) From (a) we know that for the full network, $E = \frac{1-\pi}{\frac{\lambda+\gamma}{N}+\lambda-\gamma}$ which is increasing in N since $\lambda, \gamma > 0$.■

The result in part (c) together with that in (a) imply that if $\delta = 0$ and N is bounded from above, then maximum crime is achieved for the full network of maximum possible size. Of course, this may be too costly and so the optimal (profit maximizing, rather than crime maximizing) network can be of smaller size or be different than the full network (see the next section). Also, the individual equilibrium effort for the full network with N players if $\delta = 0$ is $e^{full}(N) = \frac{\beta_1}{\phi_1+1+N(\phi_1-1)}$ which is weakly decreasing in N since $\phi_1 \geq 1$ (equivalent to $\lambda \geq \gamma$ – our condition for existence of equilibrium).

The next question we study is the relationship between the equilibrium effort levels and the position of agents in the network when corner NE are possible. A close look at the expessions for the equilibrium levels of effort in (6) yields:

Proposition 6. *In a Nash equilibrium,*
(a) if $\delta = 0$, the directly connected agents to any inactive agent (with $e_i^* = 0$) supply lower total effort than the directly connected agents to any active agent (with $e_j^* > 0$).
(b) if $\delta = 0$ and if the sets of direct connections of two active agents are nested, then the agent with larger number of direct connections supplies higher effort level.
(c) if $\delta \neq 0$ and two active agents share the same set of active agents to which they are directly connected, the agent with a larger number of inactive direct conections supplies lower effort in equilibrium.

Proof: a) Suppose $e_i^* = 0$ for some $i = 1, ..N$ and $e_j^* > 0$ for some other agent. Then the FOCs imply:

$$\frac{1-\pi}{\lambda} - \frac{\delta l_i}{2\lambda} - \sum_{k=1}^{n} e_k^* + \frac{\gamma}{\lambda}\sum_{k\in L(i)} e_k^* < 0 \qquad \text{for agent } i \text{ and,}$$

$$e_j^* = \frac{1-\pi}{\lambda} - \frac{\delta l_i}{2\lambda} - \sum_{k=1}^{n} e_k^* + \frac{\gamma}{\lambda}\sum_{k\in L(j)} e_k^* \qquad \text{for any agent } j \text{ with } e_j^* > 0.$$

Thus, at $\delta = 0$, it must be that $\sum_{k\in L(i)} e_k^* < \sum_{k\in L(j)} e_k^*$. This suggests that if there are no costs to maintain links, then for an individual who supplies no effort, his links are to agents who supply lower effort in total compared to someone who exerts positive effort. Of course if $\delta \neq 0$, it still may be the case that those linked with agents who provide no effort provide less effort so long as they have sufficiently many costly connections, or if the productive colleague has sufficiently few connections. These observations are summarized by the necessary condition:

$$\frac{\delta\left(l_j - l_i\right)}{\gamma} + \sum_{k\in L(i)} e_k^* < \sum_{k\in L(j)} e_k^*$$

b) Compare two active agents (with $e_i^* > 0$ and $e_k^* > 0$) whose sets of direct neighbors $L(i)$ and $L(k)$ are nested, $L(k) \subseteq L(i)$. The latter implies that $\sum_{j\in L(i)} e_j^* \geq \sum_{j\in L(k)} e_j^*$ with strict inequality if at least one agent in the set $L(i)\backslash L(k)$ is active. From the FOCs,

$$e_l^* = \frac{1-\pi}{\lambda} + \frac{\gamma}{\lambda}\sum_{j\in L(l)} e_j^* - \sum_{m=1}^{N} e_m^* \quad \text{for } l = i, k$$

which implies $e_i^* \geq e_k^*$.

c) Suppose there is a cost of link maintenance, $\delta \neq 0$. The FOC at an interior solution implies:

$$e_i^* = \frac{1-\pi}{\lambda} + \frac{\gamma}{\lambda}\sum_{j\in L(i)} e_j^* - \sum_{j=1}^{N} e_j^* - \frac{\delta}{\lambda} l_i$$

Suppose agents i, j have the exact same set of active direct neighbors, but agent i has more inactive direct connections. It is easy to see that $l_i > l_k$ implies $e_i^* < e_k^*$. This result suggests that when an active agent is connected to inactive agents he puts less effort into the criminal activity. Being linked with an active agent may have positive or negative effects. As long as $\phi e_j^* - \delta/\lambda$ is positive, befriending that active agent will increase agent's i's effort level holding all else constant. But a less active (inactive) agent (with $\phi e_j^* - \delta/\lambda < 0$) can decrease agent i's effort level. ■

3.1 The Planner's Problem

So far we have studied the strategic decisions making by each individual. In our setting individuals in the network impose both positive and negative externalities to each other. To characterize the latter more clearly, in this sub-section we contrast these results by the solutions to a planner's problem. Specifically, we now set up the effort choice problem from a planner's (criminal boss) perspective and characterize the optimal solution assuming this planner chooses efforts, $\mathbf{e}$ to maximize the overall payoff/profit generated by the criminal network. Unless stated otherwise, in this sub-section we focus on the interior solution case.

The total payoff in the criminal network G is:

$$\begin{aligned}
\tilde{U}(\mathbf{e}, G) = \sum_{i=1}^{N} U_i(\mathbf{e}, G) &= \\
&= \sum_{i=1}^{N} e_i(1-\pi) - \gamma \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{\lambda}{\gamma} - g_{ij}\right)e_i e_j \\
&= \sum_{i=1}^{N}[e_i(1-\pi) - \lambda e_i^2] - \gamma \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} e_i e_j
\end{aligned}$$

where a_{ij} are elements of matrix $A = [\frac{\lambda}{\gamma}(J-I) - G]$. Note that this equation is very similar to the potential function that we derived in the previous section, the only difference is that here the coefficient of the second term is γ instead of $\frac{\gamma}{2}$. As before, we could rewrite this equation using matrix notation:

$$\tilde{U}(\mathbf{e}, G) = \lambda\Big\{(\frac{1-\pi}{\lambda})\mathbf{e^T 1} - \mathbf{e^T}(\mathbf{I} + 2\phi\mathbf{A})\mathbf{e}\Big\}$$

where $\phi = \frac{\gamma}{2\lambda}$, for simplicity let us call $\eta = 2\phi = \frac{\gamma}{\lambda}$. Therefore the hessian of this matrix is:

$$\nabla^2\tilde{U} = -\lambda(\mathbf{I} + \eta\mathbf{A})$$

Using the same results as before, we know that there exists a unique solution if $\eta < -\frac{1}{\alpha_{min}(A)}$. Note that $\phi \leq \eta$, therefore, $\phi \leq \eta < -\frac{1}{\alpha_{min}(A)}$. In other words, if planner's problem has a unique solution, then individual optimization problem has a unique solution for sure. However, it might be the case that: $\phi < -\frac{1}{\alpha_{min}(A)} \leq \eta$. In which case, the planner's problem exhibits multiple equilibria (including corners), where as the individual solution is unique.

Proposition 7. *The "Planner's Problem":*
(a) The planner's problem has a unique solution if and only if $\eta = \frac{\gamma}{\lambda} < -\frac{1}{\alpha_{min}(A)}$.
(b) The total payoff in the planner's problem, $\tilde{U}(\mathbf{e}, G)$ *is smaller or equal to a fraction of the total equilibrium crime level,* $\tilde{E}(G)$ *– we have* $\tilde{U}(\mathbf{e}, G) \leq (\frac{1-\pi}{2})\tilde{E}(G)$.
(c) Suppose $\delta = 0$. *The full network maximizes both total effort and total payoff in the planner's problem. The optimal values of total effort and total payoff are both increasing in the number of agents in the network.*

Proof: (b) Note that the first order conditions of equation (8) is:

$$\tilde{e}_i = \begin{cases} \frac{1-\pi}{2\lambda} - \frac{\gamma}{\lambda}\sum_{j\neq i}(\frac{\lambda}{\gamma} - g_{ij})\tilde{e}_j & \text{if } \ \tilde{e}_i > 0 \\ 0 & \text{if } \ \frac{1-\pi}{2\lambda} \leq \frac{\gamma}{\lambda}\sum_{j\neq i}(\frac{\lambda}{\gamma} - g_{ij})\tilde{e}_j \end{cases}$$

Rewriting $\tilde{e}_i = \frac{1-\pi}{2\lambda} - \frac{\gamma}{\lambda}\sum_{j\neq i}(\frac{\lambda}{\gamma} - g_{ij})\tilde{e}_j$, we get $\gamma\sum_{j=1}^{N}(\frac{\lambda}{\gamma} - g_{ij})\tilde{e}_j = \frac{1-\pi}{2}$ for the interior solution. Also note that when $\tilde{e}_i = 0$, we have $\frac{1-\pi}{2\lambda} \leq \frac{\gamma}{\lambda}\sum_{j\neq i}(\frac{\lambda}{\gamma} - g_{ij})\tilde{e}_j$, which is equivalent to $\frac{1-\pi}{2} \leq \gamma\sum_{j=1}^{N}(\frac{\lambda}{\gamma} - g_{ij})\tilde{e}_j$. Therefore:

$$\gamma\sum_{j=1}^{N}(\frac{\lambda}{\gamma} - g_{ij})\tilde{e}_j \geq \frac{1-\pi}{2}$$

Substituting into the total payoff function, we obtain:

$$
\begin{aligned}
\tilde{U}(\mathbf{e}, G) &= \sum_{i=1}^{N} \tilde{e}_i(1-\pi) - \gamma \sum_{i=1}^{N}\sum_{j=1}^{N}(\frac{\lambda}{\gamma} - g_{ij})\tilde{e}_i\tilde{e}_j \\
&\leq \sum_{i=1}^{N} \tilde{e}_i(1-\pi) - \sum_{i=1}^{N}(\frac{1-\pi}{2})\tilde{e}_i \\
&\leq \sum_{i=1}^{N}(\frac{1-\pi}{2})\tilde{e}_i
\end{aligned}
$$

Let $\tilde{E}(G) = \sum_{i=1}^{N} \tilde{e}_i$, then we have $\tilde{U}(\mathbf{e}, G) \leq (\frac{1-\pi}{2})\tilde{E}(G)$, with equality for the interior solutions.

(c) Similar to proposition (5), it is easy to see that there is no solution to the planner's problem where all agents choose zero effort, i.e. $\tilde{e}_i = 0 \;\; \forall i$. Therefore, starting from an agent with positive effort:

$$
\begin{aligned}
& e_i = \frac{1-\pi}{2\lambda} + \frac{\gamma}{\lambda}\sum_{j=1}^{N} g_{ij}e_j - \sum_{j\neq i}^{N} e_j \\
& \Rightarrow \;\; \sum_{i=1}^{N} e_i \leq \frac{N(1-\pi)}{2\lambda} + \frac{\gamma}{\lambda}\sum_{i=1}^{N}\sum_{j=1}^{N} g_{ij}e_j - \sum_{i=1}^{N}\sum_{j\neq i}^{N} e_j \\
& \text{Note} \;\; \sum_{j=1}^{N} g_{ij}e_j \leq \sum_{j=1}^{N} e_j - e_i \;\; \Rightarrow \;\; \sum_{i=1}^{N}\sum_{j=1}^{N} g_{ij}e_j \leq (N-1)\tilde{E}(G) \\
& \text{Also} \;\; \sum_{j\neq i}^{N} e_j = \sum_{j=1}^{N} e_j - e_i \;\; \Rightarrow \;\; \sum_{i=1}^{N}\sum_{j\neq i}^{N} e_j = (N-1)\tilde{E}(G)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \;\; \tilde{E}(G) &\leq \frac{N(1-\pi)}{2\lambda} + \frac{\gamma(N-1)}{\lambda}\tilde{E}(G) - (N-1)\tilde{E}(G) \\
\Rightarrow \;\; \tilde{E}(G) &\leq \frac{N(1-\pi)}{2(\gamma + N(\lambda-\gamma))},
\end{aligned}
$$

with equality if G is the full network. Therefore, from part (b),

$$
\tilde{U}(\mathbf{e}, G) \leq \frac{N(1-\pi)^2}{4(\gamma + N(\lambda-\gamma))}
$$

with equality if G is the full network. It is clear from these results that both total effort $\tilde{E}$ and total payoff $\tilde{U}$ in the planner's problem are maximized when G is the full network. Also note that, for the full network, the values of $\tilde{E}$ and $\tilde{U}$ are increasing in the number of agents N.■

In Appendix 3 we further contrast the results from the Nash equilibrium and the planner's (cooperative) optimum by studying the total payoffs and crime levels in two extreme cases: the empty network and the full network.

4 Optimal networks and crime-deterrent policies (incomplete)

Given the equilibrium individual crime levels derived above we can compute the overall equilibrium crime level, $E = \sum_{i=1}^{N} e_i$, as well as the overall equilibrium profit (surplus) level, $\Pi = \sum_{i=1}^{N}(y_i - c_i)$ for each possible network structure G. These two aggregates play an important role in the subsequent analysis. On the one hand they are informative about the type of networks we would expect to observe if agents are optimizing according to our model. A standard competition or group selection argument suggests that the network that maximizes total surplus, Π, is likely to be the optimal network in the long run. The same result can be obtained if we assume that a "planner" (boss or Godfather) designs the network to extract maximum surplus. Figure(1) presents the optimal networks that arises for a 100 different combinations of the parameters of the model for networks with 4, 5, 6 agents.

On the other hand, the level of total crime that is likely to emerge optimally may be guiding the design and implementation of most effective crime-reducing policies that an outside authority (e.g. the police) would like to implement. We study the interaction between the optimal network structure in the following four different policy environments.

1. *removing a "key player" (RKP)* – as in BCZ (2004), the key player is defined as the (an) agent whose removal from the network (apprehension) results in the largest drop in the overall crime level

2. *removing an agent at random (RRP)* – each agent in G has equal probability, $1/N$ to be removed/apprehended

3. *removing an agent with probability proportional to his number of links (PRL)* – the probability of removing agent i equals $\frac{l_i}{\sum_{i=1}^{N} l_i}$ where l_i denotes the number of direct links of agent i

4. *removing a player with probability proportional to her effort level (RPE)* – the probability of removing agent i equals $\frac{e_i}{\sum_{i=1}^{N} e_i}$

The optimal network structure will in general reflect the policy environment in which the network operates. This is important and arises as an application of the standard Lucas critique (Lucas, 1976). Policy design must recognize that the actor's response may be a function of the policies that are employed. Moreover, the optimal network structure may change (potentially at a cost) as a given policy is implemented. Although it applies to any policy, consider the policy of "removing the key player" . BCZ take a given network G_0 and identify the "key player": the agent whose removal from the network results in the largest drop in the overall crime level.

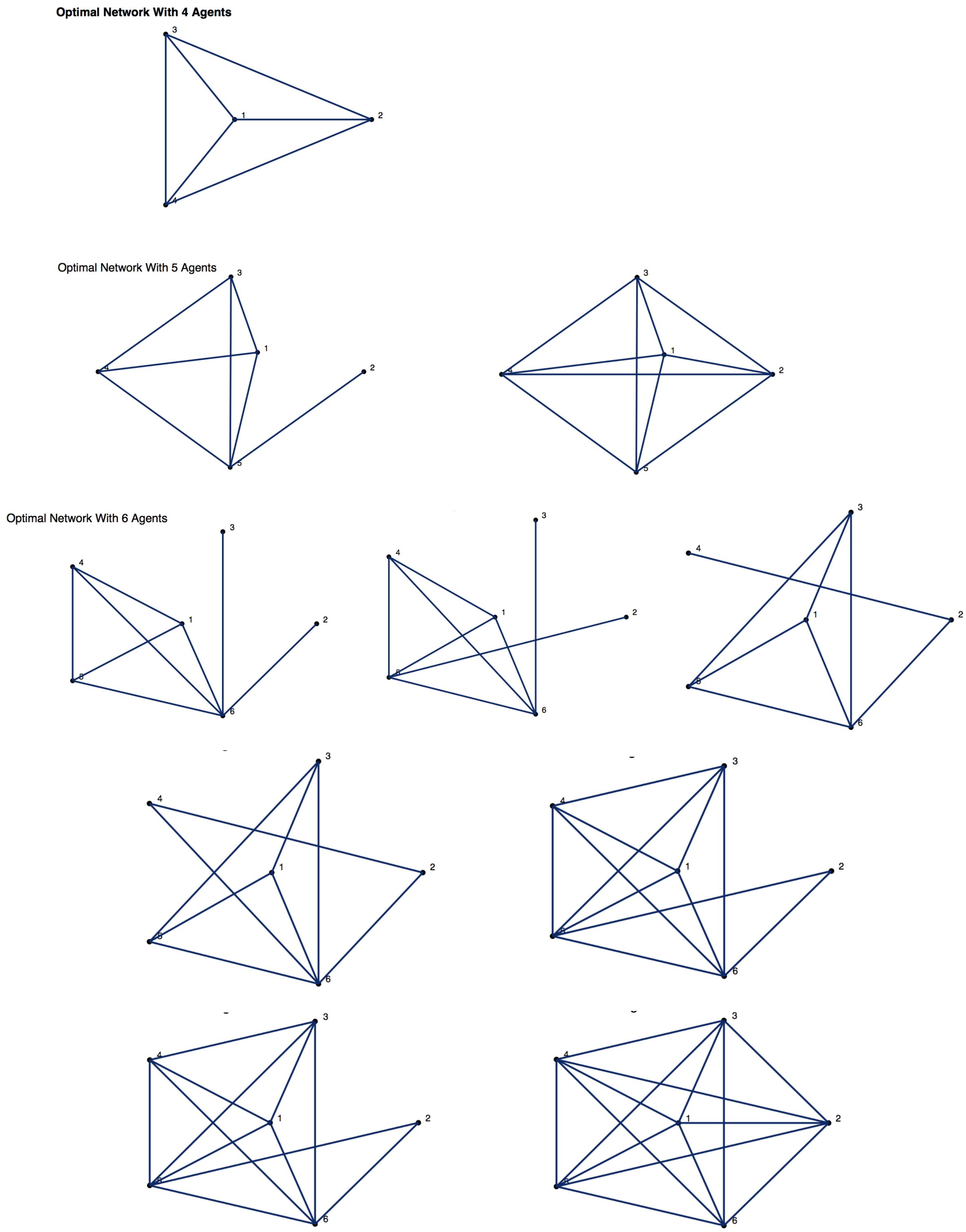

Figure 1: Optimal network with 4, 5, 6 agents for combination of parameters $\phi \in [0.01, 0.5], \pi \in [0.01, 0.99]$. When the cost of link formation is zero, the optimal networks are either complete networks or different kinds of kite networks.

There are two potential weaknesses of this approach as a guide to policy effectiveness. First, when determining who the "key player" is, we assume that after this player is removed, the remaining network does not re-optimize.[11] Second, and more importantly, the initial network is not necessarily the optimal network that would result were the criminals to recognize that a particular policy is in place.[12] An example of this case is presented in figure (2).

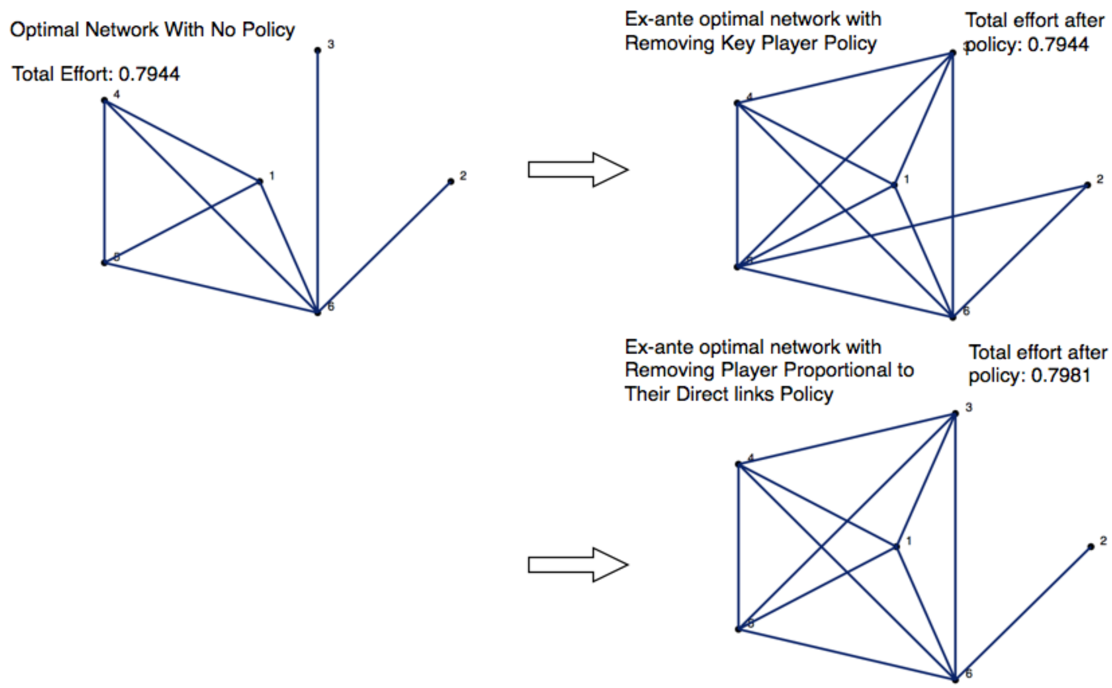

Figure 2: **Left:** optimal network with 6 agents for $\phi = 0.2767, \pi = 0.3367$ when there is no policy implemented. The total crime level is 0.7944. **Top Right:** The ex-ante optimal network if agents recognize that the policy is removing the key player. The total effort level after implementing the policy in this case is the same as before, i.e. there is no reduction in crime as a results of this policy. **Bottom Right:** The ex-ante optimal network if agents recognize that the policy is removing players with probability proportional to the number of their links. Total effort level after implementing the policy is 0.7981 which is even higher than the total effort in no-policy environment.

Practically, since analytical results cannot be derived easily, to find the optimal network for a given policy environment we need to search over all possible networks, compute the expected profits for each and choose the one with the highest value. In principle, for a given N and

[11]In principle, of course, links can be re-arranged to maximize profits so that the residual network is not simply the original network set of links less one player. The key player approach in BCZ assumes that no such re-arrangement of the network occurs. However, when it is possible to re-arrange links, the optimal network may be a different one.

[12]In BCZ (2004) of course the analysis is conducted with respect to a given network and is in no sense an optimal network.

our bidirectional link structure, all possible networks are $2^{N(N-1)/2}$ - a number that becomes huge even for very low N. However many of these networks are equivalent, or in graph-theory language, isomorphic. They can be obtained from each other by simply re-labeling the nodes. Obviously, in our setting all isomorphic networks will yield the same crime and profit levels so we only search over non-isomorphic networks when finding the optimal one. We achieve this by using computed data made publicly available by Brendan McKay from the Australian National University. This data set provides a list of the adjacency matrices (in a special compressed format, see Appendix 1) of all non-isomorphic graphs for $N = 2, 3, ..9$. The outcomes of the optimization for profit, effort and network structure depend on the parameter values for the set of λ, π, and γ. In the simulations we set $\delta = 0$. We use our theoretical results as a guideline to choose appropriate values for these values that guarantee the existence of solutions.

In practice, when we are examining several crime-combatting policies such as removing the key player, removing a random player, etc., we allow this policy to be successful with certain probability, parameterized by $p \in [0, 1]$ and look at (i) the case in which after the removal the network re-optimizes by re-organizes its structure optimally (long-run), and (ii) the case in which the network does not re-optimize (short-run). The case $p = 0$ can be thought of as lack of policing.

We also distinguish between surprise policies and expected policies. Fix some parameter vector $\theta = (\lambda, \gamma, \pi)$. Call $G^*(N|\theta)$ the optimal network of size N for these parameters. and $U(G^*|\theta)$ be the corresponding total payoff. Let $G(N)$ be a network of size N and $U(G|\theta)$ be the total equilibrium payoff for network G at parameters θ. Let also $\xi(G)$ denote the network obtained after the policy is applied to network G (e.g., the network G with its key player removed, etc). If the policy is expected, the crime organization chooses the optimal structure G that maximizes ex-ante expected total payoff. If the policy is unexpected, the total payoff is fixed. The baseline case is the case of no policy, in which case the crime organization receives total payoff

$$U(G^*(N|\theta)|\theta)$$

In the presence of policy with effectiveness (probability of success) p we study the following four scenarios:

1. **Surprise policy, no reoptimization (SNR)** – the crime organization receives payoff:

$$(1-p)U(G^*(N|\theta)|\theta) + pU(\xi(G^*(N|\theta))|\theta)$$

2. **Expected policy, no reoptimization (ENR)** – the crime organization solves:

$$\max_{G(N)} \ (1-p)U(G(N)) + pU(\xi(G(N))|\theta)$$

3. **Surprise policy, with reoptimization (SR)** – the crime organization receives payoff:

$$(1-p)U(G^*(N|\theta)|\theta) + pU(G^*(N-1|\theta)|\theta)$$

Note that the optimal network with $N-1$ members will be chosen if the policy is effective (the second term).

4. **Expected policy, with reoptimization (ER)** – the crime organization solves:

$$\max_{G(N)} \ (1-p)U(G(N)|\theta) + pU(G^*(N-1|\theta)|\theta)$$

As in case 3, the optimal network with $N-1$ members will be chosen if the policy is effective (the network will re-optimize). This implies that the value of the second term is fixed and hence the solution to the above problem is network $G^*(N|\theta)$ – the optimal network with N members. Hence the expected total payoff is exactly the same as in case 3 (SR) above. The ability to re-optimize ex-post makes the distincttion between surprise and expected policy irrelevant.

Note that cases 1–3 above could yield different predictions for the optimal crime network structure before and after the policy. Correspondingly, they can yield different predictions for the total crime level, both in the case in which the anti-crime policy is unsuccessful and in the case the policy is successful. Knowing the optimal network structures that arise for a given policy means that we can study the relative effectiveness of various policies that are defined as the reduction n the aggregate crime effort.

5 Discussion and conclusions

We develop a model that optimizes overall profits in a criminal network by varying both individual crime effort levels and the network configuration. We characterized conditions for existence and uniqueness of corner and interior Nash equilibria in efforts and gave a characterization of some properties of the solution, including individual and aggregate crime effort and total profits.

Finally, there are a variety of challenges to this methodology and a number of interesting questions to be posed. Many networks are larger than ten or eleven players. Can we deal with larger numbers in a systematic way that still preserve the spirit of optimization? If we know the observed structure of a crime network, how is that information to be integrated? Can it be used to reduce the number of networks over which we need to search? Can it help us identify the relevant parameters and their magnitudes? There are reasonable questions about what procedure is relevant when a network is stressed. Should the removal of a player simply mean that the network continues with one fewer members but leave links intact? Or should new links be forged without adding an additional player? Does the network learn? Among the questions that we can address are those related to knowledge about the network based on incomplete information about observed nodes. Can we say something about the size or structure of the network by observing one node? How much can we learn about the network knowing the links of one player to another? Is it the case that certain structures are favored in real world environments? And many more.

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Appendix 1 – Enumerating all non-isomorphic networks of given size

One of the main contributions of this paper is that we study "optimal networks", i.e. networks that maximize some economically relevant criterion among *all possible networks* of a given size. In principle, there are $2^{N(N-1)/2}$ networks of size N. Even at $N = 7$ these are already 2^{21} i.e. more than 2 million networks, thus if we were to compute an equilibrium for each of them the computational time required will grow exponentially with network size. On the other hand, it is clear that many of the possible networks are effectively the same modulo some permutation of the numbering of vertices. For example, for $N = 7$ there are only 1044 unique networks. Thus, for our purposes and to avoid costly duplication of time and effort we only need to compute the equilibria for the different, or as they are known in graph theory, *non-isomorphic* networks. As mentioned in the introduction, it turns out that generating all non-isomorphic graphs of a given size is a hard problem in graph theory and computer science, one that has not been solved for any N.

Fortunately for us, Brendan McKay from the Department of Computer Science at the Australian National University, has developed an algorithm to compute all non-isomorphic networks for up to $N = 12$ and has made the data, that is, the adjacency matrices of those networks publicly available on his website. These adjacency matrices data come in a special format designed by Prof. McKay (the "g6 format") which minimizes the amount of storage necessary. The algorithm stores the upper diagonal part of the adjacency matrices as a string of ASCII symbols. Below we explain how one can convert a .g6 file into the set of its corresponding adjacency matrices. A Matlab file performing the conversion is available upon request by the authors.

Matab algorithm for converting the McKay data into an adjacency matrix

1. Open a .g6 file provided by B. McKay at http://cs.anu.edu.au/people/bdm/data/graphs.html. A string of ASCII codes is returned, separated by "linefeed", ASCII=10 (use `fopen` to open the file and `fread` to read its contents).

2. Find the number of lines (number of linefeed symbols) – this corresponds to the number of networks in the file, M.

3. Eliminate the linefeed symbols and reshape the remaining ASCII symbols from Step 1 into a matrix with M rows each corresponding to a non-isomorphic network.

4. Subtract 63 from the ASCII codes of the elements of the matrix in Step 3.

5. For each row in the resulting matrix from Step 4 (that is, for each non-isomorphic network) perform the following:

(a) find the number of vertices, element 1 of each row

(b) convert the rest of the row elements from a decimal number into groups of 6-digit binary numbers (the g6 format uses only 6 bits).

(c) the result from (b) which is a sequence of zeros and ones forms the upper diagonal part of the adjacency matrix of the current network, going column by column, i.e. starting at element (1,2), then (1,3), ..(1,N), (2,3), etc.

Appendix 2 – Solving for Nash equilibria

Because of the non-negativity constraints, $e_i \geq 0$ we cannot find the Nash equilibrium efforts e_i by simply solving the linear system of FOCs taken as equalities, (2). Instead, we adopt a general approach which allows us to obtain all Nash equilibria, both interior and corner.

Proposition 8. *For any given network G, the set of Nash equilibria coincides with the set of solutions to the quadratic programming problem*[13]

$$\min_{\mathbf{e}} \mathbf{e}^T\{[(\mathbf{J}+\mathbf{I})\phi_1 - G]\mathbf{e} - (\beta_1\mathbf{I} - \phi_2 G)\mathbf{1}\} \tag{8}$$
$$s.t. \; [\phi_1(\mathbf{I}+\mathbf{J}) - G]\mathbf{e} - (\beta_1\mathbf{I} - \phi_2 G)\mathbf{1} \geq \mathbf{0} \; and \; \mathbf{e} \geq \mathbf{0}$$

Proof: Suppose $\mathbf{e}$ solves (QP). The objective function is equivalent to minimizing $\sum_{i=1}^N e_i\{[(\mathbf{J}+\mathbf{I})\phi_1 - G]\mathbf{e} - (\beta_1\mathbf{I} - \phi_2 G)\mathbf{1}\}_i$ where $\{.\}_i$ denotes the i-th vector element. Notice first that if $e_i > 0$ at the solution to (QP), then having $\{[(\mathbf{J}+\mathbf{I})\phi_1 - G]\mathbf{e} - (\beta_1\mathbf{I} - \phi_2 G)\mathbf{1}\}_i > 0$ cannot minimize the objective function since we can reduce e_i and reduce its value. That is $\{[(\mathbf{J}+\mathbf{I})\phi_1 - G]\mathbf{e} - (\beta_1\mathbf{I} - \phi_2 G)\mathbf{1}\}_i$ must equal zero at the optimum – the FOC of agent i is satisfied with equality and so e_i is his best response. On the other hand, if $e_i = 0$ at the solution (agent i is constrained), then we have, by the inequality constraint (which is the negative of the agent i's FOC) that $\{(\beta_1\mathbf{I} - \phi_2 G)\mathbf{1} - [(\mathbf{J}+\mathbf{I})\phi_1 - G]\mathbf{e}\}_i \leq 0$, i.e., agent i does not find it optimal to increase her effort from 0. Thus, any solution to (QP) is a NE.

Now suppose $\mathbf{e}$ is a NE. Because of the constraits, it is clear that the minimum value the objective of (QP) can take is zero and thus any vector e which achieves its value and satisfies the constraints is a solution to (QP). By the definition of NE, we have $e_i\{[(\mathbf{J}+\mathbf{I})\phi_1 - G]\mathbf{e} - (\beta_1\mathbf{I} - \phi_2 G)\mathbf{1}\}_i = 0$ for all i thus e achieves the minimum value of the objective (zero). Also, by (1), the constraints in (QP) are satisfied at a NE. Q.E.D.■

[13] The inequality signs apply to each vector element separately. The set of NE and solutions to (QP) could be empty.

Appendix 3 – Comparing the NE and planner's solutions for the empty and full networks

Suppose $\delta = 0$. Both the empty network and full network are symmetric and it is easy to verify that as long as a solution exists it is interior (in both the Nash and the planner's problems). From above, the payoff and optimal effort choice of an individual in a criminal network G are:

$$U_i(\mathbf{e}, G) = e_i(1-\pi) - \gamma \sum_{j=1}^{N} (\frac{\lambda}{\gamma} - g_{ij}) e_i e_j$$
$$\text{and } e_i^* = \frac{1-\pi}{2\lambda} - \frac{\gamma}{2\lambda} \sum_{j \neq i} (\frac{\lambda}{\gamma} - g_{ij}) e_j^*$$

On the other hand, the total payoff and the optimal effort choices in the planner's problem are:

$$\tilde{U}(\mathbf{e}, G) = \sum_{i=1}^{N} e_i(1-\pi) - \gamma \sum_{i=1}^{N} \sum_{j=1}^{N} (\frac{\lambda}{\gamma} - g_{ij}) e_i e_j$$
$$\text{and } \tilde{e}_i = \frac{1-\pi}{2\lambda} - \frac{\gamma}{\lambda} \sum_{j \neq i} (\frac{\lambda}{\gamma} - g_{ij}) \tilde{e}_j$$

- **The empty network** ($g_{ij} = 0$ for all $i, j = 1, ..., N$)

  1. Nash solution:

$$e^*_{i,\text{empty}} = \frac{1-\pi}{2\lambda} - \frac{1}{2} \sum_{j \neq i} e^*_{j,\text{empty}} = \frac{1-\pi}{(N+1)\lambda}$$
$$U_i^*(\mathbf{e}, G_{\text{empty}}) = \frac{(1-\pi)^2}{(N+1)\lambda} - \lambda N \frac{(1-\pi)^2}{(N+1)^2\lambda^2} = \frac{(1-\pi)^2}{(N+1)^2\lambda} = \frac{1}{N} U^*_{empty}$$

  2. Planner's solution:

$$\tilde{e}_i = \frac{1-\pi}{2\lambda} - \sum_{j \neq i} \tilde{e}_j \quad \text{and} \quad \tilde{e}_i = \tilde{e}_j = \tilde{e}$$
$$\text{thus, } \tilde{e} = \frac{1-\pi}{2N\lambda} \quad \text{and} \quad \tilde{U}_{empty} = \frac{(1-\pi)^2}{4\lambda}$$

Comparing the above results it is easy to see that $e^*_{i,\text{empty}} > \tilde{e}_{i,\text{empty}}$ and $U^*_{empty} < \tilde{U}_{empty}$ for any $N > 1$ – individual agents over-supply crime effort in Nash equilibrium which makes them collectively worse off. Intuitively, in the empty network there are no positive externalities from effort provision, only the congestion costs which leads to over-supply of the action e_i similarly to as in the "tragedy of the commons" problem.

- **Full network** ($g_{ij} = 1$ for all $i \neq j$)

1. Nash solution – it is easy to show that

$$
\begin{aligned}
e^*_{full} &= \frac{1-\pi}{\lambda+\gamma+N(\lambda-\gamma)} \\
U^*_i(\mathbf{e},\mathbf{G}) &= \frac{\lambda(1-\pi)^2}{(\lambda+\gamma+N(\lambda-\gamma))^2} \\
\Rightarrow \qquad U^*_{full} &= \frac{N\lambda(1-\pi)^2}{(\lambda+\gamma+N(\lambda-\gamma))^2}
\end{aligned}
$$

2. Planner's solution:

$$
\begin{aligned}
\tilde{e}_{full} &= \frac{1-\pi}{2(\gamma+N(\lambda-\gamma))} \\
\Rightarrow \qquad \tilde{U}_{full} &= \frac{N(1-\pi)^2}{4(\gamma+N(\lambda-\gamma))}
\end{aligned}
$$

It is easy to show that $e^*_{full} > \tilde{e}_{full}$ if and only if $\lambda \geq \gamma$, which from Lemma 1 is the necessary and sufficient condition for existence of Nash equilibrium. Also it is easy to see that $U^*_{full} < \tilde{U}_{full}$. In the full network there are both positive and negative externalities from the effort of others. Lemma 1 suggests that to have an equilibrium the negative externalities must outweigh the positive externalities. Thus, the intuition from the empty network case carries over – effort in NE is over-supplied relative to the first-best.