# Linear bound on extremal functions of some forbidden patterns in 0-1 matrices

Radoslav Fulek

*School of Computing Science, Simon Fraser University, Burnaby, B.C., Canada V5A 1S6*

**Abstract**

In this paper by saying that a 0-1 matrix $A$ avoids a pattern $P$ given as a 0-1 matrix we mean that no submatrix of $A$ either equals $P$ or can be transformed into $P$ by replacing some 1 entries with 0 entries. We present a new method for estimating the maximal number of the 1 entries in a matrix that avoids certain pattern. Applying this method we give a linear bound on the maximal number $\mathrm{ex}(n, L_1)$ of the 1 entries in an $n$ by $n$ matrix avoiding pattern $L_1$ (Figure 1) and thereby we answer the question that was asked in [12]. Furthermore, we use our approach on patterns related to $L_1$.

*Key words:* forbidden submatrices, pattern avoidance, extremal problems, linear bounds

## 1 Introduction

We start with a presentation of our terminology. Since in this paper we consider the matrices whose elements are either ones or zeros, by matrix we will always mean 0-1 matrix. Let us call a *submatrix* of the matrix $A$ the matrix that is obtained from $A$ by deleting some rows and columns without permuting the remaining rows and columns. We say that a matrix $A'$ *represents* a *pattern* $P$ given as a not-all-zeros $k$ by $l$ matrix (i.e. a matrix with $k$ rows and $l$ columns), if $A'$ is a $k$ by $l$ matrix, which can be transformed to the matrix $P$ by replacing some (potentially none) 1 entries with 0 entries. If there exists a submatrix $A'$ representing $P$ in $A$ we say that $A$ *contains* $P$, otherwise $A$ *avoids* $P$. We define the *weight* $w(A)$ of the matrix as the number of the 1 entries in $A$.

*Email address:* `rfulek@sfu.ca` (Radoslav Fulek).

Then the *extremal function* $\text{ex}(n, P)$ of the forbidden pattern $P$ returns the maximal weight of an $n$ by $n$ matrix that avoids $P$.

The question of determining $\text{ex}(n, P)$ originated from the paper of Mitchell [9], which introduced an algorithm that finds the shortest rectilinear path between two points in the plane avoiding rectilinear obstacles. The complexity of this algorithm was hard to estimate, but its upper bound turned out to be the extremal function of certain collection of forbidden patterns, which were later treated by Bienstock and Győri in [1]. Other motivation comes from discrete geometry, as it was shown that some problems in this area can be reduced to the estimation of $\text{ex}(n, P)$ for appropriate pattern $P$, see e.g. [4], [10].

The problem raised by Hajnal and Füredi in [5] is to characterize all patterns with linear extremal function. Several considerable steps in this direction have been already made. The complete characterization of the forbidden patterns up to four 1 entries according to the asymptotic growth of their extremal function was done by Hajnal and Füredi in [5], and by Tardos in [12]. Tardos and Marcus in [8] proved a linear bound on $\text{ex}(n, P)$ if $P$ is a permutation matrix (permutation matrix has in each row and each column exactly one 1 entry) and thereby solved the open problem from [5]. Recently, based on the result of Klazar and Valtr in [7], Keszegh in his master's thesis [6] introduced a new type of reduction that preserves linearity. In connection with the characterization of all linear patterns Tardos asked in [12], whether $L_1$ (Figure 1) is a minimal (with respect to the number of 1 entries) pattern with non-linear extremal function. This note gives negative answer to that question and also rules out $L_2$ as a next natural candidate for a non-linear pattern.

In what follows we present some additional terminology. Let $G(V, E)$ denote an undirected graph with the set of vertices $V$ and a set of edges $E$. The *visibility representation* of the graph $G$ in the Euclidean plane $\mathbb{R}^2$ is constructed by mapping each vertex $u \in V$ to the horizontal line segment $h_u$ and each edge $(u, w) \in E$ to the vertical line segment $v_{uw}$ that joins horizontal segments $h_u$ and $h_w$. Moreover, the horizontal line segments are pairwise disjoint and the vertical segments are not allowed to meet horizontal segments besides two segments they join. $G$ admits a visibility representation if and only if $G$ is planar, see [11,3]. A planar embedding of $G$ could be obtained from visibility representation by contracting each horizontal segment into a single point.

## 2 Results

In this section we present bounds on the maximal number of the 1 entries in the matrix that avoids the pattern $L_1$ and in the matrix that avoids pattern $L_2$ (Figure 1). First, we prove a simple lower bound on $ex(n, P)$ that depends

$$L_1 = \begin{pmatrix} & \bullet & \bullet & \\ \bullet & & & \bullet \\ & \bullet & & \end{pmatrix} \qquad L_2 = \begin{pmatrix} & \bullet & \bullet & \bullet & \\ \bullet & & & & \bullet \\ & & \bullet & & \end{pmatrix}$$

Fig. 1. **Patterns $L_1$ and $L_2$**

only on the size of the matrix $P$.

**Proposition 1** *If $P = (p_{ij})$ is a $k$ by $l$ not-all-zero matrix then $ex(n, P) \geq n(k + l - 2) - (k - 1)(l - 1)$.*

**Proof.** We give the construction of a $n$ by $n$ matrix $A$ that avoids pattern $P$ with exactly $n(k + l - 2) - (k - 1)(l - 1)$ 1 entries. Let $p_{l'k'}$ be some 1 entry in $P$. Note that $P$ is not all zeros matrix. Then $A$ contains $(k - 1)n$ 1 entries in the first $(k' - 1)$ columns and in the last $(k - k')$ columns (i.e. there are no 0 entries in these columns), and additional $(n - k + 1)(l - 1)$ 1 entries in the first $(l' - 1)$ rows and the last $(l - l')$ rows (i.e. there are no 0 entries in these rows). $\square$

**Theorem 2** $5n - 6 \leq \mathrm{ex}(n, L_1) \leq 7n - 13$

**Proof.** Let $A = (a_{ij})$ denote the $n$ by $n$ matrix that avoids $L_1$. Let $A' = (a'_{ij})$ denote the matrix which we obtain from $A$, if we delete (i.e. replace by 0 entries) all leftmost and rightmost 1 entries in each row. In what follows we construct a visibility representation $\mathrm{VR}(G)$ of a graph $G(V, E)$, whose vertices correspond to non-empty rows (rows containing at least one 1 entry) of $A'$ and edges correspond to certain 1 entries in $A'$. We identify the element $a'_{ij}$ in $A'$ with the point $(j, -i)$ in $\mathbb{R}^2$. The minus sign before $i$ was introduced to preserve the first row of the matrix on the top.

We represent the $i$th row, that some vertex of $G$ corresponds to, with the horizontal line segment $h_i$ connecting the first and the last 1 entry of $A'$ in that row. If $i$th row contains only one 1 entry, $h_i$ consists only of one point.

We represent an 1 entry in $A'$, that is neither the bottommost nor the second bottommost one in its column, with a vertical line segment starting at this entry and extending downwards until it hits the next horizontal line segment.

So, for each 1 entry $a'_{ij}$ in $A'$ that is neither the bottommost nor the second bottommost 1 entry in its column we join the vertex $u$ by the edge $e$ with the vertex $v$ that corresponds to the $i'$th row, which is the row of $A'$ below $i$th row with the minimal index such that its leftmost and rightmost 1 entries in the $j_l$th and $j_r$th column satisfy $j_l \leq j \leq j_r$. We have the following simple observation about $G$.

**Observation 3** *If $G$ has multiple edges then $A$ contains $L_1$.*

**Proof.** Let $u$ and $v$ denote two vertices $u, v \in V(G)$ such that the $i'$th row corresponding to $v$ is below the $i$th row that corresponds to $u$ and such that they are joined by at least two edges $e, f \in E(G)$. We can assume that the 1 entries $a'_{ij}$, $a'_{ij'}$ that correspond to $e$,$f$ satisfy $j < j'$. Then the submatrix $B$ of $A$ that represents $L_1$ in $A$ consists of the $i$th, $i'$th row and the row that contains the bottommost 1 entry in the $j$th column. The columns of $B$ are those, which contain leftmost and rightmost 1 entries in the $i'$th row (deleted in $A'$), and the $j$th and $j'$th columns of $A$. $\square$

From Observation 3, we know that $G$ cannot contain multiple edges, as otherwise $A$ would contain $L_1$.

So, $G$ is the simple planar graph with at most $n-1$ vertices (the last row cannot correspond to any vertex) and there is one-one correspondence between the edges in $G$ and the 1 entries in $A'$ except for at most $2(n-2)$ entries (the first and the last columns are empty). So, we can conclude that the number of the 1 entries in $A'$ is at most $5n - 13$, as $3n - 9$ is the maximum number of edges in a simple planar graph on $n - 1$ vertices, see for instance [2]. Thus the number of the 1 entries in $A$ is at most $7n - 13$.

The lower bound follows from Proposition 1. $\square$

It turns out that our method still works for the pattern $L_2$ (Figure 1) obtained from $L_1$ by adding the 1 entry to the first row. Now, we only need to argue that the construction from the proof of Theorem 2 gives us a planar graph with the multiplicity of edges at most two when the matrix $A$ avoids $L_2$. Indeed, if we obtained some edge with multiplicity at least three, similar argument as in the proof of Theorem 2 would lead to the claim that $A$ contains forbidden pattern $L_2$. The maximal number of edges in a planar graph with $n - 1$ vertices and with the multiplicity of edges at most two is $6n - 18$. That gives us an upper bound $10n - 22$ on $\mathrm{ex}(n, L_2)$. For the lower bound we use Proposition 1. We have just proven the following theorem.

**Theorem 4** $6n - 8 \leq \mathrm{ex}(n, L_2) \leq 10n - 22$

## 3 Conclusions

It is easy to see that the presented approach can be applied to any pattern we obtain from $L_2$ by adding any number of 1 entries in the first row between existing 1 entries. Just consider a planar graph with the greater multiplicity of

$$L_3 = \begin{pmatrix} \bullet & & & \bullet & \\ & & & & \bullet \\ & \bullet & \bullet & & \end{pmatrix} \qquad L_4 = \begin{pmatrix} & \bullet & & & \bullet & \\ \bullet & & & & & \bullet \\ & & \bullet & \bullet & & \end{pmatrix}$$

Fig. 2. **Patterns $L_3$ and $L_4$**

$$\begin{pmatrix} \bullet & & & & \\ & & & \bullet & \\ & & & & \bullet \\ & \bullet & \bullet & & \end{pmatrix}$$

Fig. 3. **Pattern $L_5$**

edges. This bound also follows from the stated results through the reduction in [12].

Nevertheless, it would be interesting to figure out whether our method (with some modifications) can be applied to some other forbidden patterns. We propose two candidates $L_3$ and $L_4$ (Figure 2). Moreover, the linearity of $ex(n, L_3)$ using the reduction from [5] would give us a linear bound on $ex(n, L_5)$ and thereby solve an open problem that was asked in [6]. Note that the extremal functions of all mentioned patterns are in $O(n\alpha(n))$, where by $\alpha(n)$ we mean the inverse of the Ackerman function. That can be obtained using the upper bound on the maximal length of generalized Davenport-Schinzel sequences from [7], see e.g. [6].

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