Optimal trading strategies with limit orders

Rossella Agliardi* and Ramazan Gençay**

* Department of Mathematics, University of Bologna
viale Filopanti n.5, 40126 Bologna, Italy (e-mail: rossella.agliardi@unibo.it).
** Department of Economics, Simon Fraser University
8888 University Drive, Burnaby, British Columbia, V5A 1S6, Canada (e-mail: r.gencay@sfu.ca)

Abstract: A model is proposed to study optimal trading strategies in a limit order book, as typically arise when a trader has a block of shares to liquidate and she submits limit orders. The execution of limit orders is uncertain, which leads to a stochastic control problem. In contrast to previous literature, we allow the trader to choose both the quotes and the sizes of her submitted orders. Great attention is paid to how the trading strategy is affected by an order book’s characteristics, market volatility and the trader’s risk attitude. We prescribe an optimal splitting of the order size, which is new in the literature on optimal trade execution where this problem is solved only in the case of market orders, and at the same time we offer guidelines to optimally place orders further behind the best price or to (re)position them more aggressively. Thus this paper is the first attempt towards a more realistic modeling of optimal liquidation throughout limit orders.

Keywords: Optimal trade execution; Limit orders; Stochastic dynamic programming; Martingales.

1. INTRODUCTION

In most of the modern financial markets, trades occur throughout electronic technologies and orders to buy or sell are collected in 'limit order books' (LOBs). In LOBs there are two basic order types: market orders and limit orders. While market orders expose the trader to the uncertainty of the execution price, limit orders are buy/sell orders that are to be executed at their specified limit price or at a better one. A limit order sits in a LOB until it is either executed against a market order or it is canceled. Therefore when dealing with limit orders the main question at hand is to model the risk of no (or partial) execution. Some models for optimal order submission strategies in a market microstructure framework have been proposed in Harris (1998), Parlour (1998), Foucault (1999). On the other hand, there exists a stream of financial research explicitly addressing the problem of optimal trade execution. Optimal trading strategies with market orders have been studied extensively in the literature on optimal trade execution growing out of the seminal works of Almgren and Chriss (1999, 2001) and still expanding. There are fewer works on optimal trading strategies with limit orders or with the combined use of market and limit orders (Bayraktar and Ludkovski (2011), Cont and Kukanov (2012), Guéant and Lehalle (2012), Guéant, Lehalle and Fernandez-Tapia (2012), Guibaud and Pham (2013), Huitema (2011)).

All the above-quoted models adopt several simplifying assumptions, but they result in stochastic control problems which are rarely analytically tractable. A severe restriction in these model is that traders are allowed only to control for the posted price, but not for their order size. The model we propose here is able to capture the main features of the execution risk in a limit order book (i.e. the dependence of the execution rate of a limit order on the order price and on the trade size) and, at the same time, traders are allowed to optimally choose both the price quote and the trade size at each point in time.

We focus on the problem of an uninformed trader who has a sell order to liquidate before a deadline and submits sell limit orders by specifying a limit price and an order size at any point in time. Limit order submission is free of charge. Quotes and sizes are chosen in order to optimize the trader’s profit over the fixed time horizon, while keeping control of her risk inventory and of the cost of
getting rid of the remaining assets by the terminal time. The use of limit order contributes to profit
maximization, thanks to the favorable execution price, but, at the same time, exposes the trader to a
non-execution risk which increases the inventory risk and the cost due to alternative strategies (e.g.
market orders) to achieve asset liquidation within the time horizon. Thus the trader has to optimally
select the distance of her limit price from the market price of the stock, which is observed before
any order submission. This trader is not assumed to have private information about future changes
in the stock value, but the observed stock price reflects common knowledge. On the other hand, this
trader optimally posts her order sizes as well. Large orders should in principle speed up liquidation
and increase wealth (when combined with favorable quotes), but they make order execution
unlikely, thus resulting in an ineffective – and eventually harmful – tactic. Therefore an optimal
strategy with limit orders is the result of the balancing act between several conflicting effects. The
trader calibrates her ask price and order size to the limit order book, by considering the probability
of order fill as a function of the quote and order dimension. Such a function can be viewed as a
reduced form representation of the LOB’s liquidity characteristics, as explained in Section 2.

In this paper, a discrete time framework is adopted and the resulting problem is formulated as a
stochastic dynamic programming one. Although we provide an explicit analytic solution for few
points in time, this is enough to offer insight on the behavior of the traders pursuing optimal
strategies with limit orders. For example, an optimal combination of aggressive and non-aggressive
limit orders is obtained as a by-product of the whole optimizing strategy. Limit orders are classified
as aggressive (marketable) and non-aggressive (non-marketable). Marketable sell (buy) orders have
the limit price at or below (above) the best bid (ask). They are like price-protected market orders
and can be executed immediately. They are the most aggressively priced limit orders. Non-
marketable ones, either sit at, behind or improve the spread but are not executed right away until a
market order comes. We find that aggressive orders are most likely placed near the close of the
trading period, especially by strongly risk averse traders, when a LOB does not easily execute limit
orders, when the market price volatility is high and whenever an alternative use of market orders is
costly.

Overall, an optimal combination of sizes and quotes depends on the LOB’s speed of execution, the
trader’s urgency and risk attitude, the market volatility, and the relative cost of liquidation
throughout market orders.

The base model for the limit order book is presented in Section 2; Section 3 introduces the
optimization problem and Section 4 discusses the results and performs a sensitivity analysis.
Section 5 concludes and presents some directions of future research.

2. MODEL SETUP

Consider an agent who has to liquidate \( X \) shares and trades through a limit order book by
submitting limit orders. Denote the reference price of the stock at time \( t \) by \( S_t \), which represents the
common knowledge about the asset value: changes in this value reflect new information about the
firm or the economy or the effect of noise trades. \( S_t \) is assumed to be a martingale with respect to a
given filtration \((F_t)_{t \geq 0}\). Throughout the paper it is implicitly assumed that all random variables are
deﬁned on a given probability space equipped with the ﬁltration \((F_t)_{t \geq 0}\) and the conditional
expectations \( E_t \) are taken with respect to \( F_t \). In the sequel, we will mainly focus on the specific
case of a Brownian motion, \( S_t = S_0 + \sigma W_t \), where \( W_t \) is a Wiener process with respect to the given
ﬁltration, although alternative stochastic processes can be incorporated in this setting. In the base
model the effect on \( S_t \) of the price impact caused by one’s own trades is neglected.

A discrete time setting is adopted throughout the paper. The agent can submit sell limit orders at
the times \( t_0, \ldots, t_{N-1} \) and the block of shares needs to be liquidated within the time \( t_N \). Let \( x_0, \ldots,
x_{n-1} denote the order sizes and let \( \hat{S}_t = S_t + \delta_t \) denote the posted quote at time \( t_n \), that is, \( \delta_t \) denotes the distance of the proposed quote from the reference price, \( S_t \). In our framework, a limit sell order set at time \( t_n \) is represented as \((x_n, \delta_n)\), where we assume that both \((x_0, \delta_0)\) and \((\delta_0)_{n=0}^{N-1}\) are adapted to the reference filtration. Thus, in contrast with previous literature on optimal trade execution with limit orders, we model both order size and limit price, that is, the main parameters defining a limit order.

As the main risk under consideration is the risk of non-execution, we need to model the process whereby limit orders are filled. Our model shares some simplifying assumptions with previous literature in this field: no partial fill of orders is allowed and a non-executed order is canceled after a short time from its posting, as specified in Section 3 below. A special case is that of immediate-or-cancel orders (IOC) or fill-or-kill orders (FOK) which are valid only when they are presented to the market (See Harris (2003) for this nomenclature). Let \( \Lambda(x, \delta) \) denote the probability that an order \((x, \delta)\) is hit. It is well known that the fill rate crucially depends on the order size and limit price and it decreases with the order size and the distance of the limit price from the reference price. (See Harris and Hasbrouck (1996)). In Guéant et al. (2012) the dependence of the fill intensity on the quotes is modeled throughout an exponential function, \( \Lambda(\delta) = A \exp(-h \delta) \), which captures the fact that the closer to the reference price an order is posted, the faster it is executed. Such a shape for the dependence of the fill rate on the main explanatory factor is supported by the empirical literature (see Lo et al. (2002), for example).

Furthermore, one can assume that the parameter modeling the execution intensity, \( h \), is increasing (e.g. proportional) in the LOB's depth, because ‘when a book is thicker, limit orders take longer to execute’ (Goettler, Parlour, Rajan (2004)).

In this paper we adopt an exponential shape of the form \( \Lambda(x, \delta) = A \exp(-kx - h \delta) \), which makes the model compatible with the above quoted literature. From a theoretical perspective, this assumption is consistent with some simple models for market orders arrival in a LOB. For example, if the distribution of incoming market orders is exponential in the order size (i.e. the cumulative distribution function for order sizes \( y \) is of the form \( \exp(-h_q y) \)) and if the LOB has a rectangular shape of height \( q \), then the probability that a limit order \((x, \delta)\) is hit, in a given time span \( \Delta \tau \), is roughly computed as the probability that the number of incoming buy orders is larger than \( x + \delta q \), that is, it is of the form: \( A \exp(-h_q \Delta \tau (x + \delta q)) \), where \( A \) depends on the intensity of order arrival and on \( \Delta \tau \). This shows that the parameters governing \( \Lambda \) can be related to the intensity of market buy orders and on the LOB's shape.

3. THE OPTIMIZATION PROBLEM

This section introduces the optimization problem of an agent who has to liquidate a block of \( \bar{X} \) shares and submits limit orders \((x_n, \delta_n)\) at the times \( t_n, n=0,\ldots,N-1 \). Although the \( t_n \)'s are exogenous items in the model, we do not pose any restrictions on their number and succession, for example, they are not necessarily equally spaced.

Let us denote the number of shares in the portfolio at any time in \([t_n, t_{n+1}[\), \( n=0,\ldots,N-1 \), by

\[
X_n = \bar{X} - \sum_{j=0}^{\infty} I_j x_j
\]

where the indicator operator \( I_j \) takes value 1 if the order posted at time \( t_j \) is executed and 0 in the opposite case. We consider the case where an order submitted at time \( t_j \) is either executed or
cancelled within the time $t_{j+1}$ and assume that $I_j$ is independent of the future movements of the stock price. The wealth associated with the portfolio at the terminal time is

$$R_N = \bar{X}S_{t_N} + \sum_{n=0}^{N-1} x_n I_n (\delta_n + S_{t_n} - S_{t_{n+1}})$$

and in view of the martingale property of the stock price the expected value of the terminal wealth is:

$$E_0[R_N] = \bar{X}S_0 + E_0[\sum_{n=0}^{N-1} x_n \delta_n \Lambda_n]$$

(1)

where $\Lambda_n$ is a short notation for $\Lambda(x_n, \delta_n)$. If the final stock position is not 0, then the agent has to sell/buy the remaining inventory at the market price and she also incurs a quadratic transaction cost described by a parameter, $\ell > 0$. Such a shape for the terminal cost is consistent with a cost arising due to price impact when market orders are submitted in a LOB (See Almgren and Chriss (2001)).

In particular, here it can be computed as $\frac{\ell}{2} X_{N-1}^2$, as $X_{N-1}$ is the number of shares left in the portfolio near the deadline and the price impact is $\ell X_{N-1}$ when these shares are sold through market orders. The parameter $\ell$ can be related to the LOB’s characteristics, for example, it is the reciprocal of the depth for a block-shaped LOB in some classical models (Obizhaeva and Wang (2005)). Therefore, an interpretation for this cost function is that market orders are used as a fall-back should there be shares left at the end of the trading period. However, other cost functions might be accommodated in this framework.

As the trader is risk averse, she is also concerned about the risk of stock price fluctuation which is modeled throughout the variance of $\bar{X}S_{t_N} + \sum_{n=0}^{N-1} x_n I_n (S_{t_n} - S_{t_{n+1}})$. This term can be rewritten as:

$$\sum_{n=0}^{N-1} X_n (S_{t_n} - S_{t_{n+1}}) = \sum_{n=0}^{N-1} X_n \Delta S_n$$

(2)

In view of the law of total variance and arguing recursively, the variance of (2) at time 0 can be written as:

$$E_0[\sum_{n=0}^{N-1} \text{var}_n(X_n \Delta S_n)]$$

(3)

This expression is useful in order to set the optimization problem in a familiar form, at least from the perspective of the stochastic dynamic programming.

The trader’s optimal strategy consists in choosing the limit orders $(x_n, \delta_n)$, $x_n \geq 0$, so that the terminal expected wealth, (1), net of the final execution cost is maximized and, at the same time, the risk of changes in the fundamental stock price, (3), times a coefficient representing the trader’s risk aversion, $\gamma$, is minimized.

In summary, the trader’s optimization problem is as follows:

$$\sup_{x_n, \delta_n} E_0[\sum_{n=0}^{N-1} (x_n \delta_n \Lambda_n) - \frac{\ell}{2} (X_{n-1}^2 - 2x_n X_{n-1} \Lambda_n + x_n^2 \Lambda_n) \text{var}_n(\Delta S_n))$$

$$- \frac{\ell}{2} (X_{N-2}^2 + (X_{N-1} - 2x_{N-1} X_{N-2}) \Lambda_{N-1})]$$

(4)

where $X_n = X_{n-1} - x_n I_n$, $n=0,…,N-1$, and $X_{-1} = \bar{X}$.

By standard argument in stochastic dynamic programming, the optimal trading strategy with limit orders, i.e. the solution to (4), is obtained as the last step of the following problems.
\[ J_n(X) = 0 \]

\[ J_{N-1}(X) = \sup_{x_{N-1}, \delta_{N-1}} \left[ x_{N-1} \delta_{N-1} \Lambda_{N-1} - \frac{\ell + \gamma V_{N-1}}{2} \left( X^2 - 2x_{N-1}XX_{N-1} + x_{N-1}^2 \Lambda_{N-1} \right) \right] \]

\[ J_n(X) = \sup_{x_n, \delta_n} \left[ x_n \delta_n \Lambda_n - \frac{\gamma V_n}{2} \left( X^2 - 2x_nXX_n + x_n^2 \Lambda_n \right) \right] + E_n[J_{n+1}(X - x_n \delta_n)] \]

\[ J_n(X) = 0 \]

\[ J_{N-1}(X) = \sup_{x_{N-1}, \delta_{N-1}} \left[ x_{N-1} \delta_{N-1} \Lambda_{N-1} - \frac{\ell + \gamma V_{N-1}}{2} \left( X^2 - 2x_{N-1}XX_{N-1} + x_{N-1}^2 \Lambda_{N-1} \right) \right] \]

n=0,…,N-2. Here \( V_n \) is a short notation for \( \text{var}_n(\Delta S_n) \). In the base model, where the stock price follows a Brownian motion, \( V_n \) is equal to \( \sigma^2(t_n - t_{n-1}) \) where \( \sigma \) is the stock volatility. This case yields deterministic coefficients in the formulas for the optimal strategies \((x_n, \delta_n)\), and therefore it is adopted in the discussion in section 4. However, the model can be extended to incorporate jumps, that is, one can assume that the reference price follows:

\[ dS_t = \sigma dW_t + dJ_t \]

where \( J_t \) is the jump component. In this case, \( V_n \) represents the quadratic variation which is the sum of \( \sigma^2(t_n - t_{n-1}) \) and \( \sum_{t_{n-1} \leq t \leq t_n} \Delta S_t \). The jump part can be described by a Lévy measure and can in turn be decomposed into a sum of big jumps and small jumps (Lévy-Ito decomposition). The small jump component is relevant in high frequency finance, where an interpretation is in term of ‘the limited ability of the marketplace to absorb large transaction without market impact’ (Aït-Sahalia and Jacod (2012)). We point out that if the log of the stock price is assumed to follow a Brownian motion or, more generally, an Itô semimartingale, then this would make the optimal solution dependent on the path of the stock price. That is why most models on optimal trade execution adopt an arithmetic Brownian motion for the stock price and a similar simplification should be adopted if one wants to incorporate jumps. In view of the short time taken by order execution such a simplification turns out to be legitimate.

4. OPTIMAL POSTING OF LIMIT ORDERS

In this section Problem (4) is solved. Then some numerical examples are presented, to show how the optimal strategy affects a trader’s behavior.

**Proposition.** The solution \((x_n^*, \delta_n^*)\), n=0,…,N-1, of (4) is such that:

\[ x_{N-1}^* = 2/(2k + \tilde{v}_{N-1}) \quad \text{with} \quad \tilde{v}_{N-1} = v_{N-1} + \ell h \quad \text{and} \quad v_j \text{ is a short notation for } \gamma h V_j \text{ for any } j; \]

\[ x_n^*(X), \quad n=0,…,N-2 \text{ is a solution to the following equation:} \]

\[ (E_n) \quad 1 - k \frac{x_n^*}{h} \left( \sum_{j=n}^{N-2} v_j + \tilde{v}_{N-1} \right) \frac{x_n^*}{h} + \sum_{j=n+1}^{N-2} \left( \frac{x_j^* \Lambda^*_j(X) - x_j^* \Lambda^*_j(X - x_n^*)}{h x_n^*} + (x_j^* \Lambda^*_j)^*(X) \right) = 0 \]

where \( \Lambda^*_j(X) = \Lambda(x_j^*(X), \delta_j^*(X)) \);

\[ \delta_n^*(X) = \frac{1}{h} + \left( \sum_{j=n}^{N-2} v_j + \tilde{v}_{N-1} \right) \frac{2X}{2h} + \sum_{j=n+1}^{N-2} \left( \frac{x_j^* \Lambda^*_j(X) - x_j^* \Lambda^*_j(X - x_n^*)}{h x_n^*} \right), \quad n=0,…,N-1. \]

**Proof.** Problem (4) is written in the form (5) by the methods of stochastic dynamic programming. Then \( x_{N-1}^* \) and \( \delta_{N-1}^* \) are found by solving the equations: \( \partial_{\delta_{N-1}} J_{N-1} = 0 \) and \( \partial_{x_{N-1}} J_{N-1} = 0 \). The
solution \((x^*_n, \delta^*_n)\), \(n=0,\ldots,N-2\), are found by recursion. If one assumes that the result holds for 
\((x^*_{n+1}, \delta^*_{n+1})\), then one can compute that \(J_{n+1}(X)\) is 
\[
J_{n+1}(X) = \sum_{j=n+1}^{N-1} \left( x^*_j \Lambda^*_j(X) \right) \frac{h}{\Lambda_n} - \left( \sum_{j=n}^{N-2} v_j + \bar{v}_{N-1} \right) \frac{X^2}{2h}.
\]

Therefore 
\[
E_n[J_{n+1}(X - x_n I_n)] = \sum_{j=n+1}^{N-1} \left( x^*_j \Lambda^*_j(X - x_n) \Lambda_n + x^*_j \Lambda^*_j(X)(1 - \Lambda_n) \right) \frac{h}{\Lambda_n} - \left( \sum_{j=n}^{N-2} v_j + \bar{v}_{N-1} \right) \frac{X^2 - 2x_n X \Lambda_n + x^2 \Lambda_n}{2h}.
\]

Plugging this expression into \(J_n(X)\) one gets that \(J_n(X)\) is:
\[
J_n(X) = \sup_{x_n, \delta_n} \left[ (x_n \delta_n - (\sum_{j=n}^{N-2} v_j + \bar{v}_{N-1}) \frac{X^2}{2h} + \sum_{j=n+1}^{N-1} \left( x^*_j \Lambda^*_j(X - x_n) + x^*_j \Lambda^*_j(X) \right) \Lambda_n \right.
\]
\[\left. - (\sum_{j=n}^{N-2} v_j + \bar{v}_{N-1}) \frac{X^2}{2h} + \sum_{j=n+1}^{N-1} \left( x^*_j \Lambda^*_j(X) \right) \right] \Lambda_n.
\]

Thus \((x^*_n, \delta^*_n)\) are obtained from \(\partial_{x_n} J_n = 0\) and \(\partial_{\delta_n} J_n = 0\).

Let us now compute some numerical values for the optimal strategy in order to study the role of the several parameters. We explicitly solve the problem (5) in the case \(N=3\), that is, the block order is split into three limit orders and a final market order that is used in the case where there are shares left to liquidate. Consider a trader who has 200 shares to liquidate and whose degree of risk aversion is \(\gamma =1\). Let \(\sigma =0.01\) and let \(h/k=100\). We point out that the discrepancy in value for the two parameters governing the speed of order execution, i.e. \(h/k\), finds motivation in the empirical literature. For example, Lo et al. (2002) estimate the execution times for limit orders and incorporate the effect of several explanatory variables: they find that the distance between the limit price and the current quote midpoint has a significant effect on the execution time, while ‘the size of a limit order has relatively little impact on its time-to-completion’.

By changing some values for \(h\) and \(\ell\) we obtain the following results which are summarized in Table 1. Certain columns in Table 1 are labeled with E (executed) and N (non-executed) because the values for the sizes/quotes at any stage may depend on whether the previous order has been filled or not, that is, they represent the trader’s different behavior in the two scenarios. The label EE means that both previous orders have been executed, EN means that the first order has been executed while the last one has not, and so on.

\[
\begin{array}{cccccccccccc}
<table>
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<tr>
<th>h</th>
<th>\ell</th>
<th>x_0</th>
<th>\delta^*_0</th>
<th>x^*_0 (E)</th>
<th>\delta^*_1 (E)</th>
<th>x^*_1 (N)</th>
<th>\delta^*_1 (N)</th>
<th>x^*_2</th>
<th>\delta^*_2 (EE)</th>
<th>\delta^*_2 (EN)</th>
<th>\delta^*_2 (NE)</th>
<th>\delta^*_2 (NN)</th>
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<td>0.02</td>
<td>139.5</td>
<td>1.16</td>
<td>89.3</td>
<td>1.78</td>
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<td>3.08</td>
<td>1.29</td>
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<td>1.24</td>
<td>15.9</td>
<td>1.88</td>
<td>118.9</td>
<td>0.94</td>
<td>110.7</td>
<td>2.68</td>
<td>1.53</td>
<td>1.77</td>
<td>0.21</td>
</tr>
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<td>1</td>
<td>0.02</td>
<td>151.1</td>
<td>0.99</td>
<td>54.7</td>
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<td>149.9</td>
<td>0.98</td>
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<td>0.88</td>
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<tr>
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<td>101.1</td>
<td>0.47</td>
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<td>0.70</td>
<td>76.9</td>
<td>0.23</td>
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<td>1.04</td>
<td>0.34</td>
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</tr>
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<td>0.84</td>
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<td>33.2</td>
<td>-0.09</td>
<td>-0.50</td>
<td>0.10</td>
<td>-1.35</td>
</tr>
</tbody>
</table>
\end{array}
\]

Table 1

The lesson is that the agent tries to benefit from the posted quote \((\delta >0)\) mainly at the inception or, at a later stage, whenever she is not pressed by urgency because previous sell orders have been
already filled. She tends to place aggressive orders ($\delta < 0$) mainly when she is closer to the end of the trading period and previous orders have expired unfilled. Quotes with negative $\delta$ can be interpreted as related to marketable orders (see also Guéant and Lehalle, 2012), because, for a sell order of this type, the limit price is below the best bid. They are the most aggressively priced limit orders.

Ceteris paribus, the total size of limit orders is usually smaller for larger values of $h$ and $k$, that is, when the execution is more unlikely. Furthermore, the trader tends to oversell her inventory when limit orders are easy to fill and alternative order types incur a high execution cost. When the terminal liquidation at market price incurs a lower transaction cost the agents submits larger orders and sets less aggressive limit prices – or even behind the best price - at the end of the trading period, because she is less concerned about the liquidation cost of the inventory left and tries to profit by favorable quotes.

Let us now study the effect of risk aversion and volatility. Keeping all other parameters unchanged (with $h=1$ and $\ell=0.01$) we let $\gamma$ and the volatility vary. Some values are presented in Table 2 below.

<table>
<thead>
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<th>$\gamma$</th>
<th>$\sigma$</th>
<th>$x_0^*$</th>
<th>$\delta_0^*$</th>
<th>$x_1^*(E)$</th>
<th>$\delta_1^*(E)$</th>
<th>$x_1^*(N)$</th>
<th>$\delta_1^*(N)$</th>
<th>$x_2^*$</th>
<th>$\delta_2^*(EE)$</th>
<th>$\delta_2^*(EN)$</th>
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<td>0.70</td>
<td>76.9</td>
<td>0.23</td>
<td>66.4</td>
<td>1.04</td>
<td>0.34</td>
<td>0.54</td>
<td>-0.68</td>
</tr>
<tr>
<td>1</td>
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<td>75.9</td>
<td>-0.09</td>
<td>61.5</td>
<td>1.03</td>
<td>0.26</td>
<td>0.23</td>
<td>-1.12</td>
</tr>
<tr>
<td>0.1</td>
<td>1</td>
<td>112.5</td>
<td>-1.12</td>
<td>48.3</td>
<td>-0.11</td>
<td>124.4</td>
<td>-0.16</td>
<td>50</td>
<td>0.70</td>
<td>-0.27</td>
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<td>-2.50</td>
</tr>
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<td>0.30</td>
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<td>0.62</td>
<td>75.7</td>
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<td>0.29</td>
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<td>0.01</td>
<td>121.5</td>
<td>-0.12</td>
<td>54.6</td>
<td>0.44</td>
<td>83.1</td>
<td>-0.30</td>
<td>57.1</td>
<td>1.07</td>
<td>0.25</td>
<td>0.01</td>
<td>-1.57</td>
</tr>
</tbody>
</table>

Table 2

One finding is that negative quotes may appear even in the first stages whenever the price volatility is high or risk aversion is strong, because price risk is then an important consideration. In Table 3 below the execution probability is higher ($h=0.5$), which leads to larger order sizes and less aggressive orders at the beginning by even pushing them away from the market, while a larger value for $\ell$ ($\ell=0.02$) demands aggressive orders in the end, in the worst scenario, in order to speed up liquidation and avoid unfavorable liquidation costs. In this case, the agent is concerned about avoiding the non-execution risk when she is towards the end of the trading period.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\sigma$</th>
<th>$x_0^*$</th>
<th>$\delta_0^*$</th>
<th>$x_1^*(E)$</th>
<th>$\delta_1^*(E)$</th>
<th>$x_1^*(N)$</th>
<th>$\delta_1^*(N)$</th>
<th>$x_2^*$</th>
<th>$\delta_2^*(EE)$</th>
<th>$\delta_2^*(EN)$</th>
<th>$\delta_2^*(NE)$</th>
<th>$\delta_2^*(NN)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.01</td>
<td>146.6</td>
<td>-1.50</td>
<td>105.7</td>
<td>2.43</td>
<td>128.4</td>
<td>0.83</td>
<td>99.8</td>
<td>4.05</td>
<td>1.93</td>
<td>2.15</td>
<td>-1.02</td>
</tr>
<tr>
<td>0.1</td>
<td>143.4</td>
<td>0.21</td>
<td>74.7</td>
<td>1.64</td>
<td>131.3</td>
<td>-0.06</td>
<td>80.0</td>
<td>3.46</td>
<td>1.22</td>
<td>1.56</td>
<td>-2.80</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>149.6</td>
<td>-0.86</td>
<td>54.8</td>
<td>1.55</td>
<td>172.6</td>
<td>0.89</td>
<td>66.7</td>
<td>3.59</td>
<td>1.32</td>
<td>2.56</td>
<td>-4.67</td>
</tr>
</tbody>
</table>

Table 3

Finally we study the effect of the LOB’s depth on the optimal trading strategy. In the Tables 4 below we fix $h$ at 1 and let $k$ vary in order to represent several degrees of the easiness to execute large orders, which are in turn related to different depth’s levels of a limit order book. In both tables $\gamma=1$ and $\ell$ is chosen to model the price impact when market orders are used as a last resort in order to meet the target, while $\sigma=0.01$ in Table 4(a) and $\sigma=0.1$ in Table 4(b).
Table 4 (a)

<table>
<thead>
<tr>
<th>h/k</th>
<th>ℓ</th>
<th>x₀⁺</th>
<th>δ₀⁺</th>
<th>x₁⁺(E)</th>
<th>δ₁⁺(E)</th>
<th>x₁⁺(N)</th>
<th>δ₁⁺(N)</th>
<th>x₂⁺</th>
<th>δ₂⁺(EE)</th>
<th>δ₂⁺(EN)</th>
<th>δ₂⁺(NE)</th>
<th>δ₂⁺(NN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.01</td>
<td>101.1</td>
<td>0.47</td>
<td>69.9</td>
<td>0.70</td>
<td>76.9</td>
<td>0.23</td>
<td>66.4</td>
<td>0.04</td>
<td>0.34</td>
<td>0.54</td>
<td>-0.68</td>
</tr>
<tr>
<td>200</td>
<td>0.005</td>
<td>160.1</td>
<td>0.84</td>
<td>134.8</td>
<td>1.31</td>
<td>139.1</td>
<td>0.68</td>
<td>132.5</td>
<td>1.82</td>
<td>1.13</td>
<td>1.22</td>
<td>0.32</td>
</tr>
<tr>
<td>50</td>
<td>0.02</td>
<td>83.4</td>
<td>0.73</td>
<td>40.8</td>
<td>0.21</td>
<td>83.1</td>
<td>0.82</td>
<td>33.3</td>
<td>-0.19</td>
<td>-1.01</td>
<td>0.21</td>
<td>-2.69</td>
</tr>
</tbody>
</table>

Table 4 (b)

<table>
<thead>
<tr>
<th>h/k</th>
<th>ℓ</th>
<th>x₀⁺</th>
<th>δ₀⁺</th>
<th>x₁⁺(E)</th>
<th>δ₁⁺(E)</th>
<th>x₁⁺(N)</th>
<th>δ₁⁺(N)</th>
<th>x₂⁺</th>
<th>δ₂⁺(EE)</th>
<th>δ₂⁺(EN)</th>
<th>δ₂⁺(NE)</th>
<th>δ₂⁺(NN)</th>
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<td>48.3</td>
<td>-0.11</td>
<td>124.4</td>
<td>-0.16</td>
<td>50</td>
<td>0.70</td>
<td>-0.27</td>
<td>0.09</td>
<td>-2.50</td>
</tr>
<tr>
<td>200</td>
<td>0.005</td>
<td>165.2</td>
<td>0.05</td>
<td>61.1</td>
<td>1.19</td>
<td>112.7</td>
<td>-0.66</td>
<td>80</td>
<td>2.00</td>
<td>1.08</td>
<td>0.25</td>
<td>-1.40</td>
</tr>
<tr>
<td>50</td>
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<td>195.5</td>
<td>3.85</td>
<td>25.8</td>
<td>1.51</td>
<td>209.0</td>
<td>4.64</td>
<td>28.6</td>
<td>2.07</td>
<td>1.29</td>
<td>1.99</td>
<td>-4.57</td>
</tr>
</tbody>
</table>

In a thicker book, orders of larger size can be submitted and generally with more favorable quotes. This is in keeping with a puzzling outcome in Goettler, Parlour and Rajan (2005), Subsection 5.4, where it is found that increased depth beyond the best price on the same side of the book results in a lower frequency of aggressive limit orders and a higher frequency of limit orders placed on that side of the book behind the quotes, which is counterintuitive as ‘increased competition should result in more, not less, aggressive orders’¹. However, in a very thin book, limit orders are set far behind the best price in the early trades, while a very aggressive strategy needs to be adopted at the close of the trading period. We conclude that the way in which traders’ decisions are influenced by the state of the limit order book strongly depends on a trader’s urgency and on the time left to meet her target.

5. CONCLUSION

This paper studies the optimal trading strategy of an agent who trades through limit orders. Although we adopt a stylized framework, this is the first attempt to offer a realistic model in that both the order size and the quote are the main components of a limit order. Some of our findings are in keeping with previous literature, but we offer a more comprehensive framework where several factors are at work simultaneously. It turns out that the use of limit orders is optimal when the execution parameters, h and k, are small (easy execution), when the volatility is low and there is a low degree of risk aversion. The larger h and k, the higher the volatility and the risk aversion, the more beneficial is the use of marketable order (i.e. negative δ). Overall, it is beneficial to post LOs away from the market price at the beginning of trade execution, while the agent tries to increase the probability of order fill towards the end of the trading period. This paper provides some implications for the algorithmic trading in limit order markets and contributes to understanding how decisions on order submission are made. While our findings about traders’ strategies mostly confirm some previous literature, many of the prescriptive rules we find in Section 4 are new and can be used as ‘actions’ in algorithm mechanisms modelling traders’ behaviour (see Chiarella et al., 2013, for example).

A possible extension of the paper is to consider the case of more persistent limit orders or to allow for an optimal combination of market and limit order during the whole execution. However this would introduce a lot of mathematical complication and might eventually result in intractability.

¹ Actually, the result in Goettler et al. (2005) is presented for limit buys instead of limit sell orders.
Chiarella C., He X., Wei L., 2013. Learning and evolution of trading strategies in limit order markets, working paper